January 31, 2003; Due February 14, 2003.

Math 8652: Homework set #1 (Spring 2003)

1. Suppose $\mathcal{H}$ is a Hilbert space and M a closed linear subspace of $\mathcal{H}$ with $M \neq \mathcal{H}$. For $x \in \mathcal{H} \setminus M$, write $x = Px + Qx$ where $Px \in M$ and Qx is in the orthogonal complement of M. Prove that $||Qx|| = \inf_{z \in M} ||x - z||$.

2. Give an example of a Hilbert space H and a convex subset $K \subset H$, such that there does not exist a $y \in K$ with $||y|| = \inf_{z \in K} ||z||$.

3. Define the space $L_\infty = L_\infty(\Omega, \mathcal{F}, \mu)$ as the space of equivalence classes of bounded random variables, with the norm $||X||_\infty = \operatorname{ess\,sup} |X|$, that is,

$$||X||_\infty = \inf\{x : \mu(X > x) = 0\}\,.$$

   Prove that $|| \cdot ||_\infty$ is a norm (i.e., a nonnegative linear function with $||X||_\infty = 0$ iff $X = 0$ μ-almost surely.) Prove that L_∞ is a complete space, i.e. that every Cauchy sequence converges in L_∞.

4. Let μ denote the Gaussian law on R and let ν denote the exponential law on R. Decide which of the following statements is true and which is false. Compute the Radon-Nikodym derivative of ν with respect to μ.

   - $\nu << \mu$.
   - $\mu << \nu$.
   - μ is singular with respect to ν.
   - ν and μ are mutually absolutely continuous.

5. Suppose μ_n is a sequence of probability measures on R converging weakly to μ.

   - Prove or find a counter example to the statement: μ_n is absolutely continuous with respect to μ.
   - Suppose $\mu_n << \nu$ for some measure ν. Prove or find a counter example to the statement $\mu << \nu$.

Assignment rules: Submitted work must be your own. You may, and in fact are encouraged to, collaborate on an assignment, provided that no more than 3 people are collaborating. In such case, you are requested to note the names of your collaborators on your submission. If collaboration is significant (more than two questions), you are requested to jointly submit your assignment.