February 14, 2003; Due February 28, 2003.

Math 8652: Homework set #2 (Spring 2003)

1. Let A be an event with $P(A) > 0$ and let $\mathcal{G}$ be a sub σ-field. Find a non-trivial collection of sets $\mathcal{C}$ such that $P(G|A) = E[1_G P(A|\mathcal{G})]/P(A)$ for every $G \in \mathcal{C}$.

2. Prove Hölder's inequality for conditional expectations:
$$E[|XY|\,|\mathcal{G}] \leq E[|X|^p\,|\mathcal{G}]^{1/p} E[|Y|^q\,|\mathcal{G}]^{1/q}\,,$$
where $1/p + 1/q = 1$, $p \geq 1$.

3. Let X_i be a sequence of i.i.d. Gaussian variables of mean 1 and variance 1. Let N denote a Poisson random variable, independent of the X_i's, of parameter 1. Set $S = \sum_{i=1}^{N} X_i$ and $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$. Compute:
   a) ES, ES^2.
   b) $E(S|\mathcal{F}_n)$.

4. A minimum mean square estimator for a random variable $X \in L^2$ given a σ-field $\mathcal{G}$ is the (L^2), $\mathcal{G}$-measurable random variable $\hat{X}$ that minimizes $E(|X - \hat{X}|^2)$.
   a) Prove that $\hat{X} = E[X|\mathcal{G}]$.
   b) Let X, Y be jointly Gaussian, i.e. such that any linear combination of X and Y is a Gaussian random variable. Evaluate $E[X|\sigma(Y)]$. (Hint: you may use the expression for the joint p.d.f. of X and Y, and prove that the r.c.p.d. of X given $\sigma(Y)$ is Gaussian.)

5. Prove that if $X \in L^p$, $p > 1$, and $\mathcal{F}_n$ is a filtration, then $E[X|\mathcal{F}_n]$ converges to X in L^p (you may use the fact, proved in class, that it does in L^1 and a.s.).

6. Suppose X, Y are independent Gaussian random variables, and let R, Θ be their polar coordinates transformation ($R \in (-\infty, \infty), \Theta \in [0, \pi)$). We asked Alice and Bob to give a sense to "$P[X|X = Y]$".

   Alice defined $Z_1 = X - Y, Z_2 = X + Y$ and $\mathcal{G} = \sigma(Z_1)$. She computed the r.c.p.d. of Z_2 given $\mathcal{G}$, and found it has a density $f_{Z_2|Z_1}(z_2|z_1)$, jointly continuous in (z_1, z_2). Retrace her computation and find the density (as a function of z_2) when $z_1 = 0$.

   Bob defined $\mathcal{G}' = \sigma(\theta)$. He computed the r.c.p.d. of X given $\mathcal{G}'$, and found it has a density $f_{R|\Theta}(r|\theta)$, which is jointly continuous as a function of (r, θ). Retrace his computation and find the density (as a function of r) when $\theta = \pi/4$.

   Note now that $\{Z_1 = 0\} = \{\Theta = \pi/4\}$ and that, on this event, $Z_2 = \sqrt{2}R$. Explain why nevertheless, the densities you computed above differ functionally.