

March 31, 2003; Due April 14, 2003.

Math 8652: Homework set #4 (Spring 2003)

1. Let X_n be a Markov processes (with respect to a filtration \mathcal{F}_n) taking values in R , with transition probabilities $p_n(x, dy)$, and define $d_n(x) = \int y p_n(x, dy) - x$. Prove that $M_n = X_n - \sum_{i=1}^n d(X_{i-1})$ is a martingale. Conclude that if $|X_n - X_{n-1}| \leq 1$ a.s. then $M_n/n \rightarrow 0$ a.s.
2. Construct a Markov process corresponding to the following “story”; prove that the process you constructed is Markov, and compute its transition probabilities:
A server serves a queue of customers. At each instant n , a customer joins the queue with probability $p < 1$, independently of the number of customers in the queue and of the server. The server serves the queue if it is not empty, and the length of each service time is geometric, of parameter p_1 , independent of the length of the queue.
(*)(optional) Give a condition on p, p_1 that ensures that the queue empties infinitely often, almost surely.
3. Let S_n denote a simple random walk on Z , that starts at 0 and is stopped when it reaches either L or $-L$ (that is, if $S_n = L$ then $S_{n+1} = L$). Prove that S_n is a Markov chain, and write the transition probabilities. In terms of these transition probabilities, write an equation for $f(k) = P(\tau < k)$, where $\tau = \inf\{n : |S_n| = L\}$. Prove that $\tau < \infty$, a.s. for this Markov chain.
Hint: It is useful to consider P^x and not only P^0 .
4. A model for the stock market is the following: at each trading day, a catastrophe occurs independently of the stock price, and the history of the process, with probability p . If a catastrophe occurs, the stock drops by 15%. Otherwise, the stock raises by 1%.

Describe the stock market as a Markov process, and write its transition probabilities.

Assignment rules: Submitted work must be your own. You may, and in fact are encouraged to, collaborate on an assignment, provided that no more than 3 people are collaborating. In such case, you are requested to note the names of your collaborators on your submission. If collaboration is significant (more than two questions), you are requested to jointly submit your assignment.