April 23, 2003; Due May 7, 2003.

Math 8652: Homework set #6 (Spring 2003)

1. Let X_n be a time homogeneous Markov chain with countable state space S, and let $s \in S$ be a recurrent state. Let τ_i denote the successive return times to s. Prove that $\{tau_{i+1} - \tau_i\}_{i\geq 1}$ is an i.i.d. sequence.

2. Suppose X_n is a Markov chain. A function f is called superharmonic if $f(x) \geq \sum_y p(x,y)f(y)$. Prove that for a superharmonic function, $f(X_n)$ is a supermartingale. Assuming the Markov chain is irreducible, prove that it is recurrent if and only if every non-negative superharmonic function is a constant.

3. Let X_n be a simple random walk on $[-L, L]$, reflected at both ends (that is, $p(L, L-1) = 1, p(-L, -L+1) = 1$). With $T_0 = \min\{n \geq 1 : X_n = 0\}$, compute $E_0(T_0)$.

4. Consider the Markov chain you constructed in HW4, #2. Assuming $p_1 > p$, compute the expected return time to 0, that is the expected time between successive emptying of the queue.

5. Suppose X_n is an irreducible Markov chain with stationary probability distribution π. Let f be a bounded real valued function. Prove that $S_n := n^{-1}\sum_{i=1}^n f(X_i)$ converges almost surely, for any initial point, to $a = \sum \pi(i)f(i)$. What are the minimal assumptions on f that allow your proof to be carried?
(*) (optional) For $a = 0$, prove the CLT for $\sqrt{n}S_n$.

Assignment rules: Submitted work must be your own. You may, and in fact are encouraged to, collaborate on an assignment, provided that no more than 3 people are collaborating. In such case, you are requested to note the names of your collaborators on your submission. If collaboration is significant (more than two questions), you are requested to jointly submit your assignment.