Date: September 19, 2003; Corrected September 29, 2003; Due: October 10, 2003

Math 8659: Homework set #1 (Fall 2003)

1. Let $\bar{\mathcal{I}}$ denote the completion of the invariant σ-field $\mathcal{I}$ for a measure preserving transformation T. Prove that $\bar{\mathcal{I}}$ is σ-field which is almost invariant, that is if $A \in \bar{\mathcal{I}}$ then $P(A \Delta T^{-1} A) = 0$. Here Δ is symmetric difference.
   Note: the completion is with respect to $\mathcal{F}$. That is, $A \in \bar{\mathcal{I}}$ if $A \in \mathcal{F}$ and there exists a set $B \in \mathcal{I}$ with $P(A \Delta B) = 0$.

2. Let α be irrational and let T_α be the α rotation of the circle, $T_\alpha x = (x+\alpha)_{\mathrm{mod}\,1}$ for $x \in [0,1)$. We proved in class that if $A \subset [0,1)$ is such that $\mathrm{Leb}(A) > 0$ then for almost every $x \in [0,1)$,
   $$n^{-1} \sum_{j=0}^{n-1} \mathbf{1}_{T_\alpha^j x \in A} \to_{n\to\infty} \mathrm{Leb}(A)\,.$$
   Prove that if $A = [a,b)$, $0 \le a < b < 1$, then the above holds for *every* $x \in [0,1)$.

3. Prove the following generalization of Kac' Lemma: let T be a measure preserving transformation, let A be such that $P(A) > 0$, and let $T_1 = \inf\{n \ge 1 : T^n \omega \in A\}$. Define $B = \cup_{j\ge 0} T^{-j} A$. Prove that
   $$E[T_1 | \omega \in A] = \frac{P(B)}{P(A)}\,.$$
   Hint: you can rework the proof given in class for the case $P(B) = 1$, but it is slicker and more instructive to give a proof based on the Kakutani towers!

4. Give an example of a measure preserving transformation and a set A with $P(A) > 0$ such that
   $$E[T_1 | \omega \in A] \neq \frac{1}{P(A)}\,.$$

5. Let $N(B,\omega)$ be a Poisson process indexed by the Borel subsets of R^2, with intensity measure μ. Prove that $N(B_1 \times [0,1], \omega)$ is a Poisson process indexed by the Borel subsets of R (i.e, with B_1 being a Borel subset of R), and compute its intensity measure.

6. Let N_t be a standard Poisson process (with $t \in R_+$). On the event $N_1 = k$, denote by $t_1, \ldots . t_k$ the jump times of the path $\{N_t\}_{1 \le t \le 1}$. Prove that the t_i are random variables, and compute the joint distribution of $t_1, \ldots, t_k$ *conditioned on* $N_1 = k$.

Assignment rules: Submitted work must be your own. You may, and in fact are encouraged to, collaborate on an assignment, provided that no more than 2 people are collaborating. In such case, you are requested to note the names of your collaborators on your submission. If collaboration is significant (more than two questions), you are requested to jointly submit your assignment.