Quenched Large Deviations for one dimensional Nonlinear Filtering

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Abstract

Consider the standard, one dimensional, nonlinear filtering problem for diffusion processes observed in small additive white noise: $d\Xi_t = \beta(\Xi_t)dt + \sigma(\Xi_t)dB_t\,, dY_t^\varepsilon = \gamma(\Xi_t)dt + \varepsilon dV_t\,,$ where $B_., V_.$ are standard independent Brownian motions. Denote by $q_1^\varepsilon(\cdot)$ the density of the law of Ξ_1 conditioned on $\sigma(Y_t^\varepsilon : 0 \le t \le 1)$. We provide "quenched" large deviation estimates for the random family of measures $q_1^\varepsilon(x)dx$: there exists a continuous, explicit mapping $\bar{\mathcal{J}} : \mathbb{R}^2 \to \mathbb{R}$ such that for almost all $B_., V_.$, $\bar{\mathcal{J}}(\cdot, X_1)$ is a good rate function and for any measurable $G \subset \mathbb{R}$,

$$-\inf_{x\in G^o} \bar{\mathcal{J}}(x, X_1) \le \liminf_{\varepsilon\to 0} \epsilon \log \int_G q_1^\varepsilon(x)dx \le \limsup_{\varepsilon\to 0} \epsilon \log \int_G q_1^\varepsilon(x)dx \le -\inf_{x\in \bar{G}} \bar{\mathcal{J}}(x, X_1)\,.$$

1 Introduction and statement of results

Consider the one dimensional standard filtering problem: the signal process $\Xi_.$ is the solution of the stochastic differential equation (SDE)

$$d\Xi_t = \beta(\Xi_t)dt + \sigma(\Xi_t)dB_t\,, \quad \Xi_0 \sim q_0(x)\,, \tag{1.1}$$

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and the observation process is given by

$$dY_t^\varepsilon = \gamma(\Xi_t)dt + \varepsilon dV_t\,. \tag{1.2}$$

Here, $B_\cdot, V_\cdot$ are independent standard one dimensional Brownian motions, b, σ, h are continuous functions on $\mathbb{R}$, and $q_0(\cdot)$ is a density with respect to Lebesgue's measure, satisfying the following assumptions:

$$\begin{array}{ll}
(A-1') & \beta, \gamma, \beta', \gamma' \text{ are Lipschitz functions}\,. \\
(A-2') & \sigma(\cdot) \geq \sigma_0 > 0,\ \gamma'(\cdot) \geq \gamma_0 > 0\,. \\
(A-3') & |\log q_0(x) - \log q_0(y)| \leq c(1+|x-y|^2)\,, x, y \in \mathbb{R}\,. \\
(A-4') & \sigma, \sigma', \sigma'' \text{ are bounded Lipschitz functions}\,.
\end{array}$$

The filtering problem consists of evaluating the law of Ξ_t conditioned on the observation sigma-algebra $\mathcal{Y}_t^\varepsilon = \sigma\{Y_s^\varepsilon : 0 \leq s \leq t\}$. By classical results, see e.g. [5], under the above hypotheses this conditional law possesses a density with respect to the Lebesgue measure which we denote by $q_t^\varepsilon(x)$.

In order to state our results compactly, we make the observation that if one defines $\bar{G}(x) = \int_0^x (1/\sigma)(u)du$, and $X_t = \bar{G}(\Xi_t)$, then (X_t, Y_t^ε) satisfy a pair of SDE's of the form (1.1), (1.2) with $\sigma = 1$. Moreover, assumptions $(A-1') - (A-4')$ still apply for the coefficients of the SDE's defined by (X_t, Y_t^ε), viz

$$\begin{aligned}
dX_t &= b(X_t)dt + dB_t\,, \quad X_0 \sim p_0(\cdot) \\
dY_t^\varepsilon &= h(X_t)dt + \varepsilon dV_t\,,
\end{aligned} \tag{1.3}$$

satisfy the assumptions

$$\begin{array}{ll}
(A-1) & b, h, b', h' \text{ are Lipschitz functions} \\
(A-2) & h'(\cdot) \geq h > 0 \\
(A-3) & |\log p_0(x) - \log p_0(y)| \leq c(1+|x-y|^2)\,, x, y \in \mathbb{R}\,.
\end{array}$$

Thus, in what follows, we consider only the filtering problem (1.3) satisfying (A-1)–(A-3).

Let $\Omega_1 = \Omega_2 = C([0,1]; \mathbb{R})$, $\Omega = \Omega_1 \times \Omega_2$, $\mathcal{F}_i$ the Borel σ-algebra on Ω_i, $i = 1, 2$, $\mathcal{F}$ the Borel σ-algebra on Ω; let P_1, P_2 denote the Wiener measure on Ω_1, Ω_2, and $P = P_1 \otimes P_2$. (B, V) are then distributed according to P. The solution (X, Y) of the SDE (1.3) is then an $\mathcal{F}$ measurable, $C([0,1]; \mathbb{R}^2)$–valued, random variable.

Let $q_t^\varepsilon(\cdot)$ denote the conditional density of X_t conditioned on $\mathcal{Y}_t^\varepsilon$, which we consider as an $\mathcal{F}$-measurable map from Ω to $M_1(\mathbb{R})$, the space of probability measures on $\mathbb{R}$. Note that q_t^ε is in fact measurable with respect to the ε-dependent σ-algebra $\mathcal{Y}_t^\varepsilon \subset \mathcal{F}$.

Define the following quantities:

$$\begin{aligned}
F(x,m) &= \int_0^x (h(u) - h(m))du\,, \\
f_2(x,m) &= mh'(x) - h(m), \\
H(x,m) &= (h(x) - h(m))(h(m) - mh'(x))\,,
\end{aligned}$$

and let

$$\mathcal{J}(x,T,m) = \sup_{\phi \in L^2[0,T]} \mathcal{I}(x,\phi,T)\,, \tag{1.4}$$

where

$$\mathcal{I}(x,\phi,T) = \int_0^T [H(\psi_s^x,m) - H(\psi_s^m,m)]ds - \frac{1}{2}\int_0^T \phi_s^2 ds + \int_0^T [f_2(\psi_s^x,m) - f_2(\psi_s^m,m)]\phi_s ds\,,$$

with

$$\dot{\psi}_s^y = -(h(\psi_s^y) - h(m)) + \phi_s\,,\ \psi_0^y = y\,. \tag{1.5}$$

Let $\mathcal{J}(x,m) = \lim_{T\to\infty} \mathcal{J}(x,T,m)$,

$$\bar{\mathcal{J}}(x,m) = F(x,m) - \mathcal{J}(x,m) - \inf_{x\in\mathbb{R}} (F(x,m) - \mathcal{J}(x,m))\,,$$

(the existence of the limit defining $\mathcal{J}(x,m)$ and the finiteness of $\bar{\mathcal{J}}(x,m)$ are ensured by Lemma 2.2 below). Our main result is the following theorem. For standard definitions concerning the LDP, see [2].

Theorem 1.1 *Assume (A-1)–(A-3). Then, the family of (random) probability measures $q_1^\varepsilon(x)dx$ satisfies a quenched LDP with continuous, good rate function $\bar{\mathcal{J}}(x,X_1)$. That is, for any measurable set $G \subset \mathbb{R}$,*

$$\begin{aligned} -\inf_{x\in G^o} \bar{\mathcal{J}}(x,X_1) &\leq \liminf_{\varepsilon\to 0} \epsilon \log \int_G q_1^\varepsilon(x)dx \\ &\leq \limsup_{\varepsilon\to 0} \epsilon \log \int_G q_1^\varepsilon(x)dx \\ &\leq -\inf_{x\in\bar{G}} \bar{\mathcal{J}}(x,X_1)\,, \quad P-a.s. \end{aligned} \tag{1.6}$$

In fact, we have the estimate, valid for any fixed compact set $K_0 \subset \mathbb{R}$,

$$\lim_{\varepsilon\to 0} \sup_{x\in K_0} |\varepsilon \log q_1^\varepsilon(x) + \bar{\mathcal{J}}(x,X_1)| = 0\,, \quad P-a.s. \tag{1.7}$$

(It will be obvious from the proof that the fixed time 1 can be replaced by any fixed time $t \in (0,1]$, that is the statement of Theorem 1.6 remains true with q_t^ε and X_t replacing q_1^ε and X_1).

Remark *In the particular case $h(x) = x$, Theorem 1.1 can be deduced from the results of [9].*

We conclude this introduction with some comments. Our motivation for the study of the large deviations of the optimal filter is their need in a variety of applications such as tracking (see [8]) or the study of the filter memory length (see [1]). In the one dimensional linear observation case studied in [9], precise pointwise estimates can be derived by comparison with the linear filtering problem, whose (Gaussian) solution is known explicitly. In contrast, here, the main tool used in the proof of Theorem 1.1 is the representation, due to Picard [6], of the density q_1^ε in terms of an auxiliary sub-optimal filter, and the availability of good estimates on the performance of this suboptimal filter. These results are not available in the general multi-dimensional case. When they are, e.g. in the setup discussed in [7], we believe our analysis can be carried through.

We also note that a natural question to consider is the question of "annealed" large deviations, that is the existence of limits of the form

$$\lim_{\varepsilon\to 0}\varepsilon\log E\int_G q_1^\varepsilon(x+X_1)dx\,.$$

This question requires the analysis of large deviations for Picard's filters, and will hopefully be treated elsewhere.

2 Definitions, properties of the rate function, and an auxiliary LDP

The filtering problem we are going to analyse is (1.3), and the assumptions $(A-1)$, $(A-2)$ and $(A-3)$ will be assumed to hold throughout the paper.

It is known from the results of Picard [6] that the conditional law $q_1^\varepsilon(x)dx$ has a small variance, and that there exist finite dimensional filters that provide good approximations of the unknown state. We shall now recall the formula derived by Picard [6] for $q_1^\varepsilon(x)$, which was used there to study approximate filters. It will be an essential tool for our large deviation results.

Define the approximate filter

$$dM_t^\varepsilon = b(M_t^\varepsilon)dt + \frac{1}{\varepsilon}(dY_t^\varepsilon - h(M_t^\varepsilon)dt)\,,$$

with M_0^ε chosen arbitrarily and independently of ε, and let $\bar{m}_s = M_{1-s}^\varepsilon$. Then,

$$d\bar{m}_s = -\frac{1}{\varepsilon}(d\bar{y}_s^\varepsilon - h(\bar{m}_s)ds) - b(\bar{m}_s)ds\,,$$

with initial conditions $\bar{m}_0 = M_1^\varepsilon$, where $\bar{y}_s^\varepsilon = Y_{1-s}^\varepsilon$.

Let

$$d\bar{X}_s^x = -\frac{1}{\varepsilon}(h(\bar{X}_s^x) - h(\bar{m}_s))ds - b(\bar{X}_s^x)ds + d\widetilde{W}_s\,,\ \bar{X}_0^x = x\,, \tag{2.1}$$

with $\widetilde{W}$ a standard Brownian motion, independent of $B., V.$. Throughout, we let $\mathbb{E}$ and $\mathbb{P}$ denote expectations and probabilities with respect to the law of the Brownian motion $\widetilde{W}.$.

One of the main contributions in [6] was to express the conditional density $q_1^\varepsilon(x)$ in terms of the law of the auxiliary process $\{\bar{X}_{1-t}^x, 0\le t\le 1\}$, which fluctuates backward in time, starting at time 1 from the position x, around the trajectory of the approximate filter $M_\cdot^\varepsilon$. That formula of Picard will be given below, see (2.3), (2.4). We mention here the following estimate (see Picard [6]) : There exists a deterministic constant $c>0$ such that

$$|\bar{X}_s^x - \bar{X}_s^{\bar{m}_0}| \le |x-\bar{m}_0|e^{-cs/\varepsilon}. \tag{2.2}$$

For $0 < a < b \leq 1$, and a, b which may depend on ε, define the functions

$$\begin{aligned}
I_\varepsilon &= \log p_0(\bar{X}_1^x) - \log p_0(\bar{X}_1^{\bar{m}_0}) + \frac{F(\bar{X}_1^x, \bar{m}_1) - F(\bar{X}_1^{\bar{m}_0}, \bar{m}_1)}{\varepsilon} \\
&+ h(\bar{m}_1)\frac{\bar{X}_1^x - \bar{X}_1^{\bar{m}_0}}{\varepsilon} - \bar{m}_1 \frac{h(\bar{X}_1^x) - h(\bar{X}_1^{\bar{m}_0})}{\varepsilon}, \\
J_\varepsilon(a,b) &= \tfrac{1}{\varepsilon}\int_a^b [f_1(\bar{X}_s^x, \bar{m}_s) - f_1(\bar{X}_s^{\bar{m}_0}, \bar{m}_s)]ds, \\
K_\varepsilon(a,b) &= \tfrac{1}{\varepsilon}\int_a^b [f_2(\bar{X}_s^x, \bar{m}_s) - f_2(\bar{X}_s^{\bar{m}_0}, \bar{m}_s)]d\widetilde{W}_s, \\
L_\varepsilon(a,b) &= \tfrac{1}{\varepsilon^2}\int_a^b [H(\bar{X}_s^x, \bar{m}_s) - H(\bar{X}_s^{\bar{m}_0}, \bar{m}_s)]ds,
\end{aligned}$$

and

$$f_1(x,m) = -mh'(x)b(x) + \frac{1}{2}h''(x) + h(x)b(x) - \frac{1}{2}h'(x) - \varepsilon b'(x)\,.$$

When $a = 0$ and $b = 1$, we omit the variables (a, b) and write simply $J_\varepsilon = J_\varepsilon(0, 1)$, etc.

Define

$$A^\varepsilon(x) = I_\varepsilon + J_\varepsilon + K_\varepsilon(\varepsilon(\log\varepsilon)^2, 1) + L_\varepsilon(\varepsilon(\log\varepsilon)^2, 1)\,,$$

and

$$B^\varepsilon(x) = K_\varepsilon(0, \varepsilon(\log\varepsilon)^2) + L_\varepsilon(0, \varepsilon(\log\varepsilon)^2)\,.$$

Thus,

$$A^\varepsilon(x) + B^\varepsilon(x) = I_\varepsilon + J_\varepsilon + K_\varepsilon + L_\varepsilon\,.$$

Now Picard's formula [6] for a version of the conditional density of X_1 given $\mathcal{Y}_1^\varepsilon$ reads

$$q_1^\varepsilon(x) := \frac{\rho_1^\varepsilon(x)}{\int_{\mathbb{R}} \rho_1^\varepsilon(x)dx}\,, \tag{2.3}$$

where

$$\rho_1^\varepsilon(x) := e^{-F(x,\bar{m}_0)/\varepsilon}\mathbb{E}\left[\exp\left(A^\varepsilon(x) + B^\varepsilon(x)\right)\right]\,. \tag{2.4}$$

We begin by stating an auxiliary LDP needed in the sequel. Let

$$d\hat{X}_s^{x,\varepsilon} = -(h(\hat{X}_s^{x,\varepsilon}) - h(\xi_s^\varepsilon))ds - \varepsilon b(\hat{X}_s^{x,\varepsilon})ds + \sqrt{\varepsilon}d\hat{W}_s\,,\ \hat{X}_0^{x,\varepsilon} = x\,, \tag{2.5}$$

where $\hat{W}_s = \widetilde{W}_{\varepsilon s}/\sqrt{\varepsilon}$ is a standard Brownian motion and ξ_s^ε is independent of $\{\hat{W}_\cdot\}$. Further, define the random variables

$$\hat{K}_\varepsilon^x(t) = \sqrt{\varepsilon}\int_0^t [f_2(\hat{X}_s^{x,\varepsilon}, m) - f_2(\hat{X}_s^{m,\varepsilon}, m)]d\hat{W}_s, \tag{2.6}$$

and

$$\hat{L}_\varepsilon^x(t) = \int_0^t [H(\hat{X}_s^{x,\varepsilon}, m) - H(\hat{X}_s^{m,\varepsilon}, m)]ds. \tag{2.7}$$

Here, m is a (possibly random) constant independent of $\{\hat{W}_\cdot\}$. To each $x \in \mathbb{R}$ and $\phi \in L^2[0,T]$, we associate $\psi \in H^1[0,T]$, the solution of the ODE

$$\dot{\psi}_s = -(h(\psi_s) - h(m)) + \phi_s\,,\ \psi_0 = x\,.$$

We now have:

Lemma 2.1 *Fix $T > 0$ and assume that $\sup_{0\le t\le T} |\xi^\varepsilon_t - m| \to_{\varepsilon\to 0} 0$. For each compact set K_0, the law of the process $\{\hat{X}^{x,\varepsilon}_s, 0 \le s \le T\}$, conditioned on $\{\xi^\varepsilon_s, 0 \le s \le T\}$, satisfies the LDP in $C([0,T];\mathbb{R})$, uniformly in $x \in K_0$, with good rate function*

$$\mathcal{I}_{x,T}(\psi) = \frac{1}{2}\int_0^T \left(\dot{\psi}_s + h(\psi_s) - h(m)\right)^2 ds$$

if $\psi_0 = x$ and $\dot{\psi} \in L^2[0,T]$, and $\mathcal{I}_{x,T}(\psi) = \infty$ otherwise. Further, the law of the random processes $\{\hat{X}^{x,\varepsilon}_s, \hat{K}^x_\varepsilon(s), \hat{L}^x_\varepsilon(s), 0 \le s \le T\}$, conditioned on $\{\xi^\varepsilon_s, 0 \le s \le T\}$, satisfies the LDP in $C([0,T];\mathbb{R}^3)$, uniformly in $x \in K_0$, with good rate function

$$\hat{\mathcal{I}}_{x,T}(\psi_1,\psi_2,\psi_3) = \inf_{\phi\in\mathcal{M}(\psi_1,\psi_2,\psi_3)} \frac{1}{2}\int_0^T \phi_s^2 ds\,.$$

Here,

$$\begin{aligned}\mathcal{M}(\psi_1,\psi_2,\psi_3) \;=\; \{\phi \in L_2[0,T] : \psi_1 = M(\phi, x)\,,&\\ \psi_2(t) = \int_0^t [f_2(\psi_1(s), m) - f_2(M(\phi,m)(s), m)]\phi(s)ds\,,&\\ \psi_3(t) = \int_0^t [H(\psi_1(s), m) - H(M(\phi,m)(s), m)]ds\}\,,&\end{aligned}$$

and we use the convention that the infimum over an empty set is $+\infty$. Finally, for any $c > 0$,

$$\limsup_{\varepsilon\to 0} \sup_{x\in K_0} \varepsilon \log \mathbb{E}\left(\exp\left(\frac{c}{\varepsilon}(|\hat{K}^x_\varepsilon(T)| + |\hat{L}^x_\varepsilon(T)|)\right)\right) < \infty\,. \tag{2.8}$$

Recall that a rate function is good if its level sets are compact.

Proof of Lemma 2.1 The proof of the LDP is standard, see e.g. the proof of Theorem 1.1 in Millet-Nualart-Sanz [4], noting that the additional drift term $-\varepsilon b - h(m) + h(\xi^\varepsilon_s)$ does not modify the argument. The exponential integrability (2.8) holds by noting that

$$\begin{aligned}\mathbb{E}\left(\exp\left(\frac{c}{\varepsilon}(|\hat{K}^x_\varepsilon(T)| + |\hat{L}^x_\varepsilon(T)|)\right)\right) &\le \mathbb{E}^{1/2}\left(\exp\left(\frac{2c}{\varepsilon}|\hat{K}^x_\varepsilon(T)|\right)\right)\mathbb{E}^{1/2}\left(\exp\left(\frac{2c}{\varepsilon}|\hat{L}^x_\varepsilon(T)|\right)\right)\\ &\le \exp(c'/\varepsilon)\mathbb{E}^{1/2}\left(\exp\left(\frac{c'}{\varepsilon}\int_0^T (1 + |\hat{X}^{x,\varepsilon}_s|)ds\right)\right)\end{aligned}$$

where the second inequality is due to the boundedness of f_2 and to the fact that $H(x,m) \le c(m)(1+|x|)$. The estimate (2.8) follows by an application of Hölder's inequality and Lemma 3.4 below. ◊

We end this section by collecting some estimates and then establishing some properties of the rate function $\mathcal{J}(x,m)$. Throughout the paper we use c to denote a generic constant, whose value may change from line to line, but which always does not depend on ε. We will make repeated use of the following easy estimates: first, because of (A-1) and (A-2), there exists a constant $c(m)$ such that for all $x, y \in \mathbb{R}$,

$$|f_1(x,m) - f_1(y,m)| \leq c(m)[(1+|y|)|x-y| + |x-y|^2]. \tag{2.9}$$

Further, since h is Lipschitz, and h' is bounded and Lipschitz, it is not hard to see that there exists a constant $c(m)$ such that for all $x, y \in \mathbb{R}$,

$$|H(x,m) - H(y,m)| \leq c(m)[|x-y| + |y|(1 \wedge |x-y|)] \tag{2.10}$$

Lemma 2.2 *For each fixed m and x, the limit $\mathcal{J}(x,m) = \lim_{T\to\infty} \mathcal{J}(x,T,m)$ exists. $\bar{\mathcal{J}}(x,m)$ is finite for all $x \in \mathbb{R}$, is continuous, and is a good rate function.*

Proof of Lemma 2.2 One has

$$\psi_t^x - \psi_t^m = x - m - \int_0^t (h(\psi_s^x) - h(\psi_s^m))ds.$$

Because $h'(\cdot) \geq h \geq 0$, we deduce that

$$\frac{d}{dt}|\psi_t^x - \psi_t^m|^2 = -2(\psi_t^x - \psi_t^m)(h(\psi_t^x) - h(\psi_t^m)) \leq -2h|\psi_t^x - \psi_t^m|^2,$$

hence,

$$\frac{d}{dt}\log|\psi_t^x - \psi_t^m|^2 \leq -2h,$$

implying that

$$|\psi_t^x - \psi_t^m| \leq |x-m|\exp(-ht). \tag{2.11}$$

We also need an estimate on $|\psi_s^m - m|$. Indeed, by (1.5),

$$\begin{aligned} \frac{d}{ds}(\psi_s^m - m)^2 &\leq -2h(\psi_s^m - m)^2 + 2\phi_s(\psi_s^m - m) \\ &\leq -h(\psi_s^m - m)^2 + \frac{1}{h}\phi_s^2, \end{aligned}$$

where we used (A-2) in the first step and the inequality $2ab \leq ha^2 + b^2/h$ in the last. Therefore,

$$\frac{d}{ds}(e^{hs}(\psi_s^m - m)^2) \leq \frac{e^{hs}}{h}\phi_s^2,$$

implying that

$$|\psi_s^m - m| \leq \sqrt{h^{-1}\int_0^s e^{-h(s-r)}\phi_r^2 dr}. \tag{2.12}$$

We next show that there exists a constant $R := R(x,m) < \infty$ independent of ε, T, uniformly bounded for x in compacts, such that the supremum in the definition of $\mathcal{J}(x,T,m)$ can be taken on $\phi \in B_{R,T} := \{\phi \in L^2[0,T] : ||\phi||_2 \le R\}$. Indeed, note that using (2.11) and (A-1) in the first inequality, and $2ab \le \delta a^2 + b^2/\delta$ in the second ($\delta > 0$ arbitrary),

$$\begin{aligned}
|\int_0^T [f_2(\psi_s^x, m) - f_2(\psi_s^m, m)]\phi_s ds| &\le c\int_0^T \min(1, |x-m|e^{-hs})|\phi_s| ds \\
&\le \frac{c}{\delta}\int_0^T \min(1, |x-m|e^{-hs})^2 ds + \delta c \int_0^T \phi_s^2 ds \\
&\le \frac{c}{\delta}(\log^+ |x-m| + 1) + \delta c \int_0^T \phi_s^2 ds\,,
\end{aligned}$$

where c depends on m but does not depend on T, δ or $|x-m|$. Similarly, using (2.10) in the first inequality,

$$\begin{aligned}
|\int_0^T [H(\psi_s^x, m) - H(\psi_s^m, m)] ds| &\le c\int_0^T |\psi_s^x - \psi_s^m| ds + c\int_0^T |\psi_s^m| \min(1, |\psi_s^x - \psi_s^m|) ds \\
&\le \frac{c(1+|x-m|)}{\delta} + c\delta \int_0^T \phi_s^2 ds\,,
\end{aligned}$$

where we used (2.11) and (2.12) in the last inequality. Choosing δ small enough, we see that for some $c = c(m) > 0$ independent of T,

$$-c(1+|x-m|) - \frac{1}{4}\int_0^T \phi_s^2 ds \le \mathcal{I}(x,\phi,T) \le c(1+|x-m|) - \frac{1}{4}\int_0^T \phi_s^2 ds\,. \tag{2.13}$$

One concludes the existence of an $R = R(|x-m|)$ independent of T and continuous in $|x-m|$ such that

$$\sup_{\phi \in L^2[0,T]} \mathcal{I}(x,\phi,T) = \sup_{\phi \in B_{R,T}} \mathcal{I}(x,\phi,T)\,.$$

Next, for $\phi \in B_{R,T}$, it follows from (2.12) that for some constant $c = c(R)$, $\sup_{s \le T} |\psi_s^m - m| \le c$. Thus, since $H(\cdot, m)$ is locally Lipschitz, for any $t \le T$,

$$\int_t^T |H(\psi_s^x, m) - H(\psi_s^m, m)| ds \le ce^{-ht}\,,$$

and

$$|\int_t^T [f_2(\psi_s^x, m) - f_2(\psi_s^m, m)]\phi_s ds| \le \int_t^T ce^{-hs}|\phi_s| ds \le ce^{-ht}\,.$$

Here, the constant c depends on m, x, is bounded uniformly on x in compacts, and does not depend on T. For $t < T$, let $\mathcal{J}(x,T,m)_t$ denote the solution to the variational problem (1.4) where the sup is taken over those $\phi \in L^2[0,T]$ satisfying $\phi_s = 0$ for $t \le s \le T$. We get

$$\begin{aligned}
\mathcal{J}(x,T,m) \le\ & \mathcal{J}(x,T,m)_t + 2\sup_{\phi \in B_{R,T}} \int_t^T |H(\psi_s^x, m) - H(\psi_s^m, m)| ds \\
& + \sup_{\phi \in B_{R,T}} \int_t^T |[f_2(\psi_s^x, m) - f_2(\psi_s^m, m)]\phi_s| ds \le \mathcal{J}(x,T,m)_t + g_t(x)\,,
\end{aligned}$$

where $g_t(x) = 3ce^{-ht}$. Similarly we obtain

$$\mathcal{J}(x,t,m) - g_t(x) \le \mathcal{J}(x,T,m)_t \le \mathcal{J}(x,T,m) \le g_t(x) + \mathcal{J}(x,T,m)_t \le 2g_t(x) + \mathcal{J}(x,t,m)\,.$$

Since $g_t(x) \to 0$ as $t \to \infty$ uniformly on compacts, it follows that the limit $\mathcal{J}(x,m) = \lim_{T\to\infty} \mathcal{J}(x,T,m)$ exists, and moreover the convergence is uniform on compacts. Next, by (2.13), there exists a constant c, independent of T and x, such that

$$-c(1+|x-m|) \le \mathcal{J}(x,T,m) \le c(1+|x-m|)\,.$$

Hence, $|\mathcal{J}(x,m)| \le c(1+|x-m|)$. On the other hand,

$$\liminf_{|x|\to\infty} \frac{F(x,m)}{|x-m|^2} \ge c' > 0\,.$$

Thus, there exists a compact K_1 such that

$$\sup_{x\in\mathbb{R}} (\mathcal{J}(x,m) - F(x,m)) = \sup_{x\in K_1} (\mathcal{J}(x,m) - F(x,m)) < \infty\,,$$

implying the finiteness of $\bar{\mathcal{J}}(\cdot,m)$. To see that $\bar{\mathcal{J}}(\cdot,m)$ is continuous, note that for any $\delta > 0$ there exists a $\phi := \phi^x_\delta \in L^2[0,T]$ such that

$$\begin{aligned}\mathcal{J}(x,T,m) \;&\ge\; -\delta + \int_0^T [H(\psi^x_s,m) - H(\psi^m_s,m)]ds \\ &\quad + \int_0^T [f_2(\psi^x_s,m) - f_2(\psi^m_s,m)]\phi_s ds - \frac{1}{2}\int_0^T \phi_s^2 ds\,,\end{aligned}$$

and hence

$$\mathcal{J}(x,T,m) - \mathcal{J}(x',T,m) \ge -\delta + \int_0^T [H(\psi^x_s,m) - H(\psi^{x'}_s,m)]ds + \int_0^T [f_2(\psi^x_s,m) - f_2(\psi^{x'}_s,m)]\phi_s ds\,.$$

Interchanging the role of x and x', we thus get that

$$\limsup_{|x-x'|\to 0} |\mathcal{J}(x,T,m) - \mathcal{J}(x',T,m)| \le \delta\,.$$

The arbitrariness of δ proves then that for each $T > 0$, $\mathcal{J}(\cdot,T,m)$ are continuous functions. The uniform convergence on compacts of $\mathcal{J}(\cdot,T,m)$ to $\mathcal{J}(\cdot,m)$ then implies the continuity of the latter, allowing one to conclude the continuity of $\bar{\mathcal{J}}(\cdot,m)$. The level sets of $\bar{\mathcal{J}}(\cdot,m)$ are compact because

$$\liminf_{|x|\to\infty} \bar{\mathcal{J}}(x,m) \ge -C_1 + \liminf_{|x|\to\infty} (C_2|x-m|^2 - C_3|x-m|) = \infty\,, \tag{2.14}$$

for some $C_1, C_2, C_3 > 0$ depending on m only. $\diamondsuit$

Remark *As in (1.4), define, for* $|\lambda - 1| < 1/2$,

$$\mathcal{J}^\lambda(x,T,m) = \sup_{\phi\in L^2[0,T]} \left\{ \lambda \int_0^T [H(\psi_s^x, m) - H(\psi_s^m, m)] ds - \frac{1}{2}\int_0^T \phi_s^2 ds \right. \\ \left. + \lambda \int_0^T [f_2(\psi_s^x, m) - f_2(\psi_s^m, m)]\phi_s ds \right\}, \tag{2.15}$$

with ψ_s^y *as in (1.5). Define similarly* $\mathcal{J}^\lambda(x,m)$ *and* $\bar{\mathcal{J}}^\lambda(x,m)$*. The same argument as above shows that* $\bar{\mathcal{J}}^\lambda(x,m)$ *is a continuous good rate function , and that*

$$\mathcal{J}^\lambda(x,T,m) \to_{T\to\infty} \mathcal{J}^\lambda(x,m)\,, \quad \bar{\mathcal{J}}^\lambda(x,m) \to_{\lambda\to 1} \bar{\mathcal{J}}(x,m)\,, \quad \textit{uniformly on compacts.} \tag{2.16}$$

3 Estimates

Lemma 3.1 *Assume (A-1)–(A-3). Then for each* $\beta > 0$ *and* $\theta \in \mathbb{R}$,

$$\limsup_{\varepsilon\to 0} \varepsilon \log \sup_{x\in\mathbb{R}} e^{-\beta|x-\bar{m}_0|^2/\varepsilon} \mathbb{E}\left[e^{\theta(A^\varepsilon(x)+B^\varepsilon(x))}\right] < \infty,, \quad P-a.s$$

and for each $\alpha > 0$, $\beta > 0$ *and* $\theta \in \mathbb{R}$*, there exists a* $M_\alpha = M_\alpha(\beta,\theta) > 0$ *such that*

$$\limsup_{\varepsilon\to 0} \varepsilon \log \sup_{|x-\bar{m}_0|>M_\alpha} e^{-\beta|x-\bar{m}_0|^2/\varepsilon} \mathbb{E}\left[e^{\theta(A^\varepsilon(x)+B^\varepsilon(x))}\right] < -\alpha\,, \quad P-a.s.$$

Lemma 3.2 *Assume (A-1)–(A-3), and fix* K_0 *a compact set. Then, for each* $\delta > 0$,

$$\limsup_{\varepsilon\to 0} \sup_{x\in K_0} \varepsilon \log \mathbb{P}(\varepsilon|A^\varepsilon(x)| > \delta) = -\infty\,, \quad P-a.s., \tag{3.1}$$

and, for any $\lambda \in \mathbb{R}$,

$$\limsup_{\varepsilon\to 0} \sup_{x\in K_0} \varepsilon \log \mathbb{E}(\exp \lambda |A^\varepsilon(x)|) = 0\,, \quad P-a.s. \tag{3.2}$$

Lemma 3.3 *Assume (A-1)–(A-3). Then,*

$$\varepsilon \sup_{x\in K_0} |\log \mathbb{E}\Big[\exp(A^\varepsilon(x) + B^\varepsilon(x))\Big] \exp(-\mathcal{J}(x,X_1)/\varepsilon)| \to_{\varepsilon\to 0} 0\,, \quad P-a.s. \tag{3.3}$$

Before proving Lemma 3.1, we state and prove an auxiliary result, which provides a precise estimate of the exponential moments of $|\bar{X}_s^{\varepsilon,\bar{m}_0}|$, uniformly in $s \in [0,1]$.

Lemma 3.4 *For each* $\varepsilon_0 > 0$*, there exists a constant* $c > 0$ *such that for all* $\lambda > 0$, $0 < \varepsilon \leq \varepsilon_0$, $s \in [0,1]$,

$$\mathbb{E}\left(e^{\lambda|\bar{X}_s^{\varepsilon,\bar{m}_0}|}\right) \leq 4e^{\lambda\|\bar{m}\|_\infty + c\varepsilon(\lambda+\lambda^2)} \\ \leq 4e^{c(\bar{m})\lambda + c\varepsilon\lambda^2}$$

Proof : We write from now on X_s for $\bar{X}_s^{\varepsilon,\bar{m}_0}$. Note that X_s solves the SDE

$$\begin{cases} dX_s = -\frac{1}{\varepsilon}(h(X_s) - h(\bar{m}_s))ds - b(X_s)ds + d\widetilde{W}_s, \\ X_0 = \bar{m}_0, \end{cases}$$

where $\{\widetilde{W}_s;\ s \geq 0\}$ is a standard one dimensional Brownian motion under $\mathbb{P}$.

It follows easily from the monotonicity of h and a well–known comparison theorem for one dimensional SDEs (see e.g. [3, Chap. 6]), that

$$\underline{X}_s \leq X_s \leq \overline{X}_s.$$

where $\overline{X}$ (resp. $\underline{X}$) solves the same SDE as X, but with $h(\bar{m}_s)$ replaced by $h(\sup_{0\leq s\leq 1} \bar{m}_s)$ (resp. $h(\inf_{0\leq s\leq 1} \bar{m}_s)$). Since

$$\mathbb{E}e^{\lambda|X_s|} \leq \mathbb{E}e^{\lambda\overline{X}_s} + \mathbb{E}e^{\lambda(-\underline{X}_s)},$$

it suffices to prove that each of the two terms on the right is dominated by

$$2e^{\lambda||\bar{m}||_\infty + c\varepsilon(\lambda+\lambda^2)}.$$

We shall do the calculations for the first term, the second one being estimated exactly in the same way. In other words, we need to estimate $\mathbb{E}\, e^{\lambda X_s}$, where $X.$ solves

$$\begin{cases} dX_s = -\frac{1}{\varepsilon}(h(X_s) - h(\alpha))ds - b(X_s)ds + d\widetilde{W}_s, \\ X_0 = \bar{m}_0, \end{cases}$$

and $\alpha = \sup_{0\leq s\leq 1} \bar{m}_s$.

We now let $X_t^\alpha = X_t - \alpha$ and deduce from assumptions (A-1)–(A-2) that

$$\begin{cases} dX_s^\alpha = -\frac{c_\varepsilon(X_s^\alpha)}{\varepsilon} X_s^\alpha ds - b(\alpha)ds + d\widetilde{W}_s, \\ X_0^\alpha = \bar{m}_0 - \alpha, \end{cases}$$

where $c_\varepsilon(\cdot)$ is such that for $\varepsilon_0 > 0$ small enough, $0 < \underline{c} \leq c_\varepsilon(x) \leq \bar{c}$ whenever $0 < \varepsilon \leq \varepsilon_0$ and $x \in \mathbb{R}$. We assume from now on that $0 < \varepsilon \leq \varepsilon_0$ and delete the superscript α. Since $m_0 - \alpha \leq 0$ and we are looking for an upper bound, we may as well replace $m_0 - \alpha$ by 0. Now X_s equals

$$-b(\alpha)\int_0^s \bar{e}^{\int_r^s \frac{c_\varepsilon(X_r)}{\varepsilon}du} dr \leq \varepsilon|b(\alpha)|/\underline{c} \leq c\varepsilon,$$

plus the solution of

$$\begin{cases} dX_s = -\frac{c_\varepsilon(X_s)}{\varepsilon} X_s ds + d\widetilde{W}_s, \\ X_0 = 0. \end{cases}$$

It remains to show that the solution of this last equation satisfies

$$\mathbb{E}e^{\lambda X_s} \leq 2e^{c\varepsilon\lambda^2}.$$

But X_t is increased if we both replace $c_\varepsilon(x)$ by $\underline{c}$, and reflect the solution at 0 (so that it remains nonnegative). Hence it remains to estimate $\mathbb{E}e^{\lambda X_s}$, where X solves

$$\begin{cases} dX_s = -\frac{\underline{c}}{\varepsilon}X_s ds + d\widetilde{W}_s + dL_s, & X_0 = 0, \\ X_s \geq 0, \{L_s\} \text{ continuous and increasing}, & \int_0^t X_s dL_s = 0. \end{cases} \tag{3.4}$$

But we have the :

Lemma 3.5 *The law of $\{X_s; s \geq 0\}$, solution of (3.4), coïncides with the law of $\{|Y_s|; s \geq 0\}$, where Y is the Ornstein–Uhlenbeck process satisfying*

$$dY_s = -\frac{\underline{c}}{\varepsilon}Y_s ds + d\widetilde{W}_s, \;\; Y_0 = 0.$$

Let us admit for a moment that result. We have

$$\mathbb{E}e^{\lambda X_s} \leq \mathbb{E}e^{\lambda|Y_s|} \leq 2\mathbb{E}e^{\lambda Y_s} \leq 2e^{c\varepsilon\lambda^2},$$

since $Y_s \simeq N(0, \sigma_\varepsilon^2)$, with

$$\sigma_\varepsilon^2 = \int_0^s e^{-\frac{2\underline{c}}{\varepsilon}(s-r)} dr \leq \frac{1}{2\underline{c}}\varepsilon\,,$$

which completes the proof of Lemma 3.4, modulo establishing Lemma 3.5, to which we next proceed.

Proof of Lemma 3.5

Tanaka's formula yields,

$$\begin{cases} d|Y_s| = -\frac{\underline{c}}{\varepsilon}|Y_s| ds + d\overline{W}_s + dK_s, & |Y_0| = 0, \\ \{K_s\} \text{ continuous and increasing}, & \int_0^t |Y_s| dK_s = 0, \end{cases}$$

where $\overline{W}_s = \int_0^s \text{sign}(Y_r) d\widetilde{W}_r$ is a standard Brownian motion. Hence X and $|Y|$ solve the same reflected SDE with different driving Brownian motions. Since uniqueness in law holds for that equation, the result follows. ◊

We now proceed to the

Proof of Lemma 3.1 It follows from Cauchy-Schwarz that

$$\mathbb{E}e^{\theta(I_\varepsilon + J_\varepsilon + K_\varepsilon + L_\varepsilon)} \leq \left(\mathbb{E}e^{4\theta I_\varepsilon}\mathbb{E}e^{4\theta J_\varepsilon}\mathbb{E}e^{4\theta K_\varepsilon}\mathbb{E}e^{4\theta L_\varepsilon}\right)^{1/4}$$

We will now bound each term of the right–hand side. Clearly

$$\begin{aligned} I_\varepsilon &\leq c\Big(1+\varepsilon^{-1}[(1+|\bar X_1^{\bar m_0}|)|\bar X_1^x-\bar X_1^{\bar m_0}|+|\bar X_1^x-\bar X_1^{\bar m_0}|^2]\Big) \\ &\leq c\Big(1+\varepsilon^{-1}[(1+|\bar X_1^{\bar m_0}|)e^{-c/\varepsilon}|x-\bar m_0|+e^{-2c/\varepsilon}|x-\bar m_0|^2]\Big)\,, \end{aligned}$$

and with the help of Lemma 3.4 we deduce that

$$\mathbb{E}e^{4\theta I_\varepsilon}\leq ce^{c(|x-\bar m_0|\frac{e^{-c/\varepsilon}}{\varepsilon}+|x-\bar m_0|^2\frac{e^{-2c/\varepsilon}}{\varepsilon})}\leq ce^{\frac{c|x-\bar m_0|^2e^{-c/\varepsilon}}{\varepsilon}}. \tag{3.5}$$

Finally

$$\mathbb{E}\;e^{4\theta I_\varepsilon}=c\,e^{c(\varepsilon)|x-\bar m_0|^2}, \tag{3.6}$$

where $c(\varepsilon)\to 0$ as $\varepsilon\to 0$.

Similarly, we obtain that

$$\begin{aligned} J_\varepsilon &\leq \frac{c}{\varepsilon}\int_0^1[(1+|\bar X_s^{\bar m_0}|)|x-\bar m_0|e^{-cs/\varepsilon}+|x-\bar m_0|^2e^{-2cs/\varepsilon}]ds \\ &\leq c(1+|x-\bar m_0|^2)+\frac{c}{\varepsilon}|x-\bar m_0|\int_0^1|\bar X_s^{\bar m_0}|e^{-cs/\varepsilon}ds. \end{aligned}$$

Hence

$$\begin{aligned} \mathbb{E}e^{4\theta J_\varepsilon} &\leq ce^{c|x-\bar m_0|^2}\mathbb{E}e^{\frac{c}{\varepsilon}|x-\bar m_0|\int_0^1|\bar X_s^{\bar m_0}|e^{-cs/\varepsilon}ds} \\ &\leq ce^{c|x-\bar m_0|^2}c'(\varepsilon)\int_0^1\frac{e^{-c\,s/\varepsilon}}{\varepsilon}\mathbb{E}\left(e^{c|x-\bar m_0|\cdot|\bar X_s^{\bar m_0}|}\right)ds, \end{aligned}$$

where $c'(\varepsilon)\to 1$ as $\varepsilon\to 0$, and we have used Jensen's inequality to interchange the exponential and the integral over $[0,1]$ with respect to the probability measure $c'(\varepsilon)\dfrac{e^{-c\,s/\varepsilon}}{\varepsilon}ds$. Now, using Lemma 3.4, we deduce that

$$\begin{aligned} \mathbb{E}e^{4\theta J_\varepsilon} &\leq ce^{c|x-\bar m_0|^2}\sup_{0\leq s\leq 1}\mathbb{E}\left(e^{c|x-\bar m_0|\cdot|\bar X_s^{\bar m_0}|}\right) \\ &\leq c\,e^{c(1+\varepsilon)|x-\bar m_0|^2} \\ &\leq c\,e^{c|x-\bar m_0|^2}. \end{aligned} \tag{3.7}$$

We now consider the third term. First note that

$$\begin{aligned} |f_2(x,m)-f_2(y,m)|^2 &= m^2|h'(x)-h'(y)|^2 \\ &\leq m^2(|h'(x)|+|h'(y)|)|h'(x)-h'(y)| \\ &\leq c\,m^2|x-y|. \end{aligned}$$

Consequently, for any $0 < a < b$,

$$\begin{aligned}
&\mathbb{E} \quad e^{\frac{4\theta}{\varepsilon}\int_a^b [f_2(\bar{X}_s^x,\bar{m}_s)-f_2(\bar{X}_s^{\bar{m}_0},\bar{m}_s)]d\tilde{W}_s} \\
\leq\ & \left(\mathbb{E}\ e^{\frac{8\theta}{\varepsilon}\int_a^b [f_2(\bar{X}_s^x,\bar{m}_s)-f_2(\bar{X}_s^{\bar{m}_0},\bar{m}_s)]d\tilde{W}_s - \frac{32\theta^2}{\varepsilon^2}\int_a^b |f_2(\bar{X}_s^x,\bar{m}_s)-f_2(\bar{X}_s^{\bar{m}_0},\bar{m}_s)|^2 ds}\right)^{1/2} \\
& \times \left(\mathbb{E}\ e^{\frac{32\theta^2}{\varepsilon^2}\int_a^b |f_2(\bar{X}_s^x,\bar{m}_s)-f_2(\bar{X}_s^{\bar{m}_0},\bar{m}_s)|^2 ds}\right)^{1/2} \\
\leq\ & \left(\mathbb{E}\ e^{\frac{32\theta^2}{\varepsilon^2}\int_a^b |f_2(\bar{X}_s^x,\bar{m}_s)-f_2(\bar{X}_s^{\bar{m}_0},\bar{m}_s)|^2 ds}\right)^{1/2} \qquad (3.8)
\end{aligned}$$

But, using the previous inequality we deduce that

$$\begin{aligned}
\frac{32\theta^2}{\varepsilon^2}\int_a^b |f_2(\bar{X}_s^x,\bar{m}_s)-f_2(\bar{X}_s^{\bar{m}_0},\bar{m}_s)|^2 ds &\leq \frac{c(m)}{\varepsilon^2}|x-\bar{m}_0|\int_a^b e^{-cs/\varepsilon} ds \\
&\leq \frac{c(m)}{c\varepsilon}|x-\bar{m}_0|[e^{-ac/\varepsilon}-e^{-bc/\varepsilon}]. \qquad (3.9)
\end{aligned}$$

Consequently

$$\mathbb{E}\ e^{4\theta K_\varepsilon} \leq e^{\frac{c(m)}{\varepsilon}|x-\bar{m}_0|}$$

and

$$\mathbb{E}\ e^{4\theta K_\varepsilon(\varepsilon(\log\varepsilon)^2,1)} \leq e^{c(m)|x-\bar{m}_0|}. \qquad (3.10)$$

We finally estimate the fourth and last term. It follows from (2.10) that

$$|H(\bar{X}_s^x,\bar{m}_s) - H(\bar{X}_s^{\bar{m}_0},\bar{m}_s)| \leq c(m)\left[e^{-cs/\varepsilon}|x-\bar{m}_0| + |\bar{X}_s^{\bar{m}_0}|(1\wedge e^{-cs/\varepsilon}|x-\bar{m}_0|)\right]$$

Let $\tau_\varepsilon \triangleq \frac{\varepsilon}{c}\log^+|x-\bar{m}_0| \vee a$. We have

$$L_\varepsilon(a,t) \leq \frac{c(m)e^{-ca/\varepsilon}}{c\varepsilon}|x-\bar{m}_0| + \frac{c(m)}{\varepsilon^2}\int_a^{\tau_\varepsilon\wedge t}|\bar{X}_s^{\bar{m}_0}|ds + \frac{c(m)}{\varepsilon^2}\int_{\tau_\varepsilon\wedge t}^t |\bar{X}_s^{\bar{m}_0}||x-\bar{m}_0|e^{-cs/\varepsilon}ds. \qquad (3.11)$$

(note that the second term vanishes if $\tau_\varepsilon = a$).

We first estimate, for $\tau_\varepsilon > a$,

$$\begin{aligned}
\mathbb{E}\ e^{\frac{c(m)}{\varepsilon^2}\int_0^{\tau_\varepsilon\wedge t}|\bar{X}_s^{\bar{m}_0}|ds} &= \mathbb{E}\ e^{\int_0^{\tau_\varepsilon\wedge t}\frac{c(m)(\tau_\varepsilon\wedge t)}{\varepsilon^2}|\bar{X}_s^{\bar{m}_0}|\frac{ds}{\tau_\varepsilon\wedge t}} \\
&\leq \sup_{0\leq s\leq t}\mathbb{E}\ e^{\frac{c(m)\tau_\varepsilon}{\varepsilon^2}|\bar{X}_s^{\bar{m}_0}|} \\
&\leq e^{\frac{c(m)}{\varepsilon}(1+(\log^+|x-\bar{m}_0|)^2))}.
\end{aligned}$$

We next consider the last term in (3.11),

$$\begin{aligned}
\int_{\tau_\varepsilon\wedge t}^t |\bar{X}_s^{\bar{m}_0}|e^{-cs/\varepsilon}|x-\bar{m}_0|ds &= |x-m_0|e^{-c\tau_\varepsilon/\varepsilon}\int_{\tau_\varepsilon\wedge t}^t|\bar{X}_s^{\bar{m}_0}|e^{-c(s-\tau_\varepsilon)/\varepsilon}ds \\
&\leq |x-m_0|e^{-c\tau_\varepsilon/\varepsilon}\int_0^t|\bar{X}_s^\varepsilon|e^{-cs/\varepsilon}ds,
\end{aligned}$$

where $\bar X^\varepsilon_s = \bar X^{\bar m_0}_{\tau_\varepsilon\wedge t+s}$. In fact, in case $|x-\bar m_0|\le 1$ and $a=0$, $\tau_\varepsilon = 0$ and the inequality should be replaced by an equality. Consequently

$$\begin{aligned}\mathbb{E}\, e^{\frac{c}{\varepsilon^2}\int_{\tau_\varepsilon\wedge t}^t |\bar X^{\bar m_0}_s| e^{-cs/\varepsilon}|x-\bar m_0| ds} &\le \mathbb{E}\, e^{\frac{c|x-m_0|e^{-c\tau_\varepsilon/\varepsilon}}{\varepsilon^2}\int_0^t |\bar X^\varepsilon_s| e^{-cs} ds} \\ &\le \mathbb{E}\, e^{\int_0^t \frac{c|x-m_0|e^{-c\tau_\varepsilon/\varepsilon}}{\varepsilon}|\bar X^\varepsilon_s|\frac{e^{-cs/\varepsilon}}{c'(\varepsilon)\varepsilon}ds} \end{aligned} \tag{3.12}$$

Hence, for $a=\varepsilon(\log\varepsilon)^2$,

$$\mathbb{E}\, e^{\frac{c}{\varepsilon^2}\int_{\varepsilon(\log\varepsilon)^2}^t |\bar X^{\bar m_0}_s| e^{-cs/\varepsilon}|x-\bar m_0| ds} \le c\,, \tag{3.13}$$

while, for $a=0$,

$$\begin{aligned}\mathbb{E}\, e^{\frac{c}{\varepsilon^2}\int_{\tau_\varepsilon\wedge t}^t |\bar X^{\bar m_0}_s| e^{-cs/\varepsilon}|x-\bar m_0| ds} &\le \sup_{0\le s\le 2t} \mathbb{E}\, e^{\frac{c}{\varepsilon}|\bar X^{\bar m_0}_s|} \\ &\le c\, e^{c/\varepsilon}, \end{aligned}$$

implying that

$$\mathbb{E} e^{4\theta L_\varepsilon} \le e^{c(m)(|x-m_0|+1)/\varepsilon}\,.$$

We can now conclude. Summing up all the previous estimates, we deduce that for some $\varepsilon_0>0$, $c(m)>0$, all $x\in\mathbb{R}$, $0<\varepsilon\le\varepsilon_0$,

$$\mathbb{E}\left[\exp(\theta(A^\varepsilon(x)+B^\varepsilon(x)))\right] \le e^{\frac{c(m)}{\varepsilon}(1+|x-\bar m_0|)+c(m)|x-\bar m_0|^2}$$

Consequently, if $\varepsilon\le\beta/2c(m)$,

$$\begin{aligned}&\varepsilon\log\left[e^{-\beta|x-\bar m_0|^2/\varepsilon}\mathbb{E}[\exp(\theta(A^\varepsilon(x)+B^\varepsilon(x)))\right] \\ &\quad\le -\frac{\beta}{2}|x-\bar m_0|^2 + c(m)|x-\bar m_0| + c(m).\end{aligned}$$

The sup over all x of the right hand side is a finite quantity independent of ε, and the same right hand side tends to $-\infty$ as $|x-\bar m_0|$ tend to infinity, uniformly with respect to ε. Lemma 3.1 is proved

◊

Proof of Lemma 3.2 (3.2) follows from

$$\mathbb{E}(\exp(\lambda|A^\varepsilon(x)|)) \le \mathbb{E}(\exp(\lambda A^\varepsilon(x))) + \mathbb{E}(\exp(-\lambda A^\varepsilon(x)))\,,$$

and (3.6), (3.7), (3.10), (3.13), while (3.2) implies (3.1) by Chebycheff's inequality. ◊

Proof of Lemma 3.3 We begin with the following result.

Lemma 3.6 *There exists a* $0 < p < 1/2$ *such that, for* P*-a.e.* ω*, there exist constants* $C(\omega), \varepsilon_0(\omega) > 0$ *such that for all* $\varepsilon \le \varepsilon_0$*,*

$$\sup_{0\le s\le \varepsilon(-\log\varepsilon)^2} |\bar{m}_s - \bar{m}_0| \le C\varepsilon^p \,.$$

Proof of Lemma 3.6 Recall that $\bar{m}_s = M^\varepsilon_{1-s}$ where $M^\varepsilon_\cdot$ is Picard's filter

$$dM^\varepsilon_t = b(M^\varepsilon_t)dt + \frac{1}{\varepsilon}(h(X_t) - h(M^\varepsilon_t))dt + dV_t \,. \tag{3.14}$$

We undo the time reversal, hence the statement reads: P-a.s., there exists a $C(\omega)$, $p > 0$, $\varepsilon_0(\omega) > 0$ such that

$$\sup_{0<\varepsilon\le\varepsilon_0(\omega)} \sup_{1-\varepsilon(\log\varepsilon)^2\le s\le 1} |M^\varepsilon_1 - M^\varepsilon_s|(\omega) \le C(\omega)\varepsilon^p \,.$$

It follows (either by direct computation similar to (3.17) below using

$$d(X_s - M^\varepsilon_s) = [b(X_s) - b(M^\varepsilon_s) + \frac{1}{\varepsilon}(h(M^\varepsilon_s) - h(X_s))]ds + dB_s - dV_s \,,$$

or from e.g. [6]) that for all $s \ge 1/2$,

$$E|h(X_s) - h(M^\varepsilon_s)| \le c\sqrt{\varepsilon}, E\int_0^s |h(X_r) - h(M^\varepsilon_r)|ds \le c\sqrt{\varepsilon} \,. \tag{3.15}$$

This implies, using (3.14) in the first inequality and (3.15) in the second, that for any $0 < \alpha < 1/2$,

$$\begin{aligned} E\sup_{1-\alpha\le s\le 1} |M^\varepsilon_1 - M^\varepsilon_s| &\le E\int_{1-\alpha}^1 |b(M^\varepsilon_s) + \frac{h(X_s) - h(M^\varepsilon_s)}{\varepsilon}|ds \quad + E\sup_{1-\alpha\le s\le 1}|V_1 - V_s| \\ &\le c(\sqrt{\alpha} + \alpha/\sqrt{\varepsilon}) \,, \end{aligned}$$

implying with the choice $\alpha = \varepsilon(\log\varepsilon)^2$ that

$$E\sup_{1-\varepsilon(\log\varepsilon)^2\le s\le 1} |M^\varepsilon_1 - M^\varepsilon_s| \le c\sqrt{\varepsilon}(\log\varepsilon)^2 \,. \tag{3.16}$$

For $n = 1, 2, \ldots$, let $\varepsilon_n = n^{-6}$, and define the event

$$\mathcal{A}_n = \{\sup_{1-2\varepsilon_n(\log\varepsilon_n)^2\le s\le 1} |M^{\varepsilon_n}_1 - M^{\varepsilon_n}_s| \le \varepsilon_n^{1/4}\} \,.$$

Then, by (3.16), $P(\mathcal{A}^c_n) \le \varepsilon_n^{1/5} = n^{-6/5}$, implying by the Borel-Cantelli lemma that there exists for almost all ω a $n_0 = n_0(\omega)$ such that for all $n > n_0$,

$$\sup_{1-2\varepsilon_n(\log\varepsilon_n)^2\le s\le 1} |M^{\varepsilon_n}_1 - M^{\varepsilon_n}_s| \le \varepsilon_n^{1/4} \,.$$

It thus remains to evaluate $|M_s^{\varepsilon'} - M_s^{\varepsilon_n}|$ for $\varepsilon_{n+1} < \varepsilon' < \varepsilon_n$, and $1 - \varepsilon_n(\log \varepsilon_n)^2 \leq s \leq 1$. But, from (3.14),

$$\begin{aligned} \frac{d}{ds}(M_s^{\varepsilon'} - M_s^{\varepsilon_n}) &= -\frac{c(\varepsilon_n, \varepsilon', s)}{\varepsilon'}(M_s^{\varepsilon'} - M_s^{\varepsilon_n}) + (\frac{1}{\varepsilon'} - \frac{1}{\varepsilon_n})(h(X_s) - h(M_s^{\varepsilon_n}))\,, \\ M_0^{\varepsilon'} - M_0^{\varepsilon_n} &= 0\,, \end{aligned} \tag{3.17}$$

where

$$c(\varepsilon_n, \varepsilon', s) = \frac{h(M_s^{\varepsilon'}) - h(M_s^{\varepsilon_n})}{M_s^{\varepsilon'} - M_s^{\varepsilon_n}} - \varepsilon' \frac{b(M_s^{\varepsilon'}) - b(M_s^{\varepsilon_n})}{M_s^{\varepsilon'} - M_s^{\varepsilon_n}} \geq c_1 > 0\,, \quad \text{for all } 0 < \varepsilon' \leq \varepsilon_0\,.$$

Moreover, for $\varepsilon_{n+1} \leq \varepsilon' \leq \varepsilon_n$, $1/\varepsilon' - 1/\varepsilon_n \leq cn^5$. Hence, with $t_n = 1 - 2\varepsilon_n(\log \varepsilon_n)^2$,

$$\begin{aligned} |M_s^{\varepsilon'} - M_s^{\varepsilon_n}| &\leq cn^5 e^{-(s-t_n)c_1/\varepsilon_n} \int_0^{t_n} |h(X_r) - h(M_r^{\varepsilon_n})| dr \\ &\quad + cn^5 n^{-6} \sup_{t_n \leq r \leq 1} |h(X_r) - h(M_r^{\varepsilon_n})| \\ &\leq cn^{-6}\zeta_n + cn^{-1}(\zeta_n^1 + \zeta_n^2 + \zeta_n^3)\,. \end{aligned}$$

Here,

$$\begin{aligned} \zeta_n = \int_0^{t_n} |h(X_r) - h(M_r^{\varepsilon_n})| dr\,, \zeta_n^1 = |h(X_1) - h(M_1^{\varepsilon_n})|\,, \\ \zeta_n^2 = \sup_{t_n \leq r \leq 1} |h(X_1) - h(X_r)|\,, \zeta_n^3 = \sup_{t_n \leq r \leq 1} |h(M_1^{\varepsilon_n}) - h(M_r^{\varepsilon_n})|\,. \end{aligned}$$

On $\mathcal{A}_n$, $\zeta_n^3 \leq cn^{-6/4}$, while since $E(\zeta_n) \leq c\sqrt{\varepsilon_n}$, we have that $P(\zeta_n \geq 1) \leq c\sqrt{\varepsilon_n} = cn^{-3}$, implying that for all $n > n_1(\omega) \geq n_0(\omega)$, $\zeta_n \leq 1$. Since also $E(\zeta_n^1) \leq c\sqrt{\varepsilon_n}$, and $E\zeta_n^2 \leq c\sqrt{\varepsilon_n}|\log \varepsilon_n|$, the same argument implies that for all $n > n_2(\omega) \geq n_1(\omega)$, $\max(\zeta_n^1, \zeta_n^2) \leq cn^{-6/4}\log n$. For such n, it follows that for all $\varepsilon_{n+1} \leq \varepsilon \leq \varepsilon_n$, and $1 - \varepsilon(\log \varepsilon)^2 \leq s \leq 1$,

$$|M_1^\varepsilon - M_s^\varepsilon| \leq cn^{-2} \leq c\varepsilon_n^{1/3} \leq \varepsilon^{1/4}\,.$$

◊

Remark *In fact the same proof shows that*

$$\sup_{0 \leq s \leq \varepsilon(-\log \varepsilon)^2} |\bar{m}_s - X_1| \leq C(\omega)\varepsilon^p\,, \quad P-a.s. \tag{3.18}$$

Because

$$|f_2(x, m) - f_2(x, m')| \leq C|m - m'|\,,$$

it holds that for any λ,

$$\begin{aligned} &\mathbb{E}\left(\exp\left(\frac{\lambda}{\varepsilon}\int_0^{\varepsilon(-\log\varepsilon)^2} [f_2(\bar{X}_s^x, \bar{m}_s) - f_2(\bar{X}_s^x, \bar{m}_0)] d\widetilde{W}_s\right)\right) \\ \leq\; &\mathbb{E}\left(\exp\left(\frac{\lambda^2 C^2}{2\varepsilon^2}\int_0^{\varepsilon(-\log\varepsilon)^2} |\bar{m}_s - \bar{m}_0|^2 ds\right)\right) \\ \leq\; &e^{\lambda^2 c \varepsilon^{p-1}}\,, \end{aligned}$$

where $c = c(\omega)$, and similarly

$$\mathbb{E}\left(\exp\left(\frac{\lambda}{\varepsilon}\int_0^{\varepsilon(-\log\varepsilon)^2}[f_2(\bar{X}_s^{\bar{m}_0},\bar{m}_s) - f_2(\bar{X}_s^{\bar{m}_0},\bar{m}_0)]d\widetilde{W}_s\right)\right) \le e^{\lambda^2 c\varepsilon^{p-1}}.$$

and hence

$$\mathbb{E}\left(\exp\left(\frac{\lambda}{\varepsilon}\int_0^{\varepsilon(-\log\varepsilon)^2}\left[\left[f_2(\bar{X}_s^x,\bar{m}_s) - f_2(\bar{X}_s^{\bar{m}_0},\bar{m}_s)\right] - \left[f_2(\bar{X}_s^x,\bar{m}_0) - f_2(\bar{X}_s^{\bar{m}_0},\bar{m}_0)\right]\right]d\widetilde{W}_s\right)\right) \le e^{\lambda^2 c\varepsilon^{p-1}}. \tag{3.19}$$

Further, because

$$|H(x,m) - H(x,m')| \le c(1+|m|+|x|)|m-m'|,$$

it holds that

$$\begin{aligned} &\mathbb{E}\left(\exp\left(\frac{\lambda}{\varepsilon^2}\int_0^{\varepsilon(-\log\varepsilon)^2}|H(\bar{X}_s^x,\bar{m}_s) - H(\bar{X}_s^x,\bar{m}_0)|ds\right)\right), \\ \le\ &\mathbb{E}\left(\exp\left(\frac{\lambda}{\varepsilon^2}\int_0^{\varepsilon(-\log\varepsilon)^2}(1+|\bar{m}_0|+|\bar{X}_s^x|)\ |\bar{m}_s-\bar{m}_0|ds\right)\right), \\ \le\ &e^{c\lambda\varepsilon^{p/2-1}}, \end{aligned}$$

where the last inequality is due to Lemma 3.4 and is uniform in $x \in K_0$. A similar inequality applies when replacing $\bar{X}_s^x$ by $\bar{X}_s^{\bar{m}_0}$, and hence

$$\sup_{x\in K_0}\mathbb{E}\left(\exp\left(\frac{\lambda}{\varepsilon^2}\int_0^{\varepsilon(-\log\varepsilon)^2}\left[\left[H(\bar{X}_s^x,\bar{m}_s) - H(\bar{X}_s^{\bar{m}_0},\bar{m}_s)\right] - \left[H(\bar{X}_s^x,\bar{m}_0) - H(\bar{X}_s^{\bar{m}_0},\bar{m}_0)\right]\right]ds\right)\right) \le e^{\lambda c\varepsilon^{p/2-1}}. \tag{3.20}$$

Define $\bar{B}^\varepsilon(x) = \bar{K}_\varepsilon + \bar{L}_\varepsilon$, where

$$\bar{K}_\varepsilon = \frac{1}{\varepsilon}\int_0^{\varepsilon(-\log\varepsilon)^2}[f_2(\bar{X}_s^x,\bar{m}_0) - f_2(\bar{X}_s^{\bar{m}_0},\bar{m}_0)]d\widetilde{W}_s,$$

and

$$\bar{L}_\varepsilon = \frac{1}{\varepsilon^2}\int_0^{\varepsilon(-\log\varepsilon)^2}[H(\bar{X}_s^x,\bar{m}_0) - H(\bar{X}_s^{\bar{m}_0},\bar{m}_0)]ds,$$

and introduce the quantities

$$\bar{K}_\varepsilon^T = \frac{1}{\varepsilon}\int_0^{T\varepsilon}[f_2(\bar{X}_s^x,\bar{m}_0) - f_2(\bar{X}_s^{\bar{m}_0},\bar{m}_0)]d\widetilde{W}_s,$$

and

$$\bar{L}_\varepsilon^T = \frac{1}{\varepsilon^2}\int_0^{T\varepsilon}[H(\bar{X}_s^x,\bar{m}_0) - H(\bar{X}_s^{\bar{m}_0},\bar{m}_0)]ds.$$

Defining $I_1(\varepsilon,T) := \bar{K}_\varepsilon - \bar{K}_\varepsilon^T + \bar{L}_\varepsilon - \bar{L}_\varepsilon^T$, we show below the

Lemma 3.7 *For any $c > 0$,*

$$\limsup_{T\to\infty}\limsup_{\varepsilon\to 0}\varepsilon \sup_{x\in K_0} \log \mathbb{E}(e^{c(|I_1(\varepsilon,T)|)}) = 0\,.$$

Admitting for the moment Lemma 3.7, we can now complete the proof of Lemma 3.3. Write

$$A^\varepsilon(x) + B^\varepsilon(x) = A^\varepsilon(x) + I_1(\varepsilon,T) + I_2(\varepsilon) + \bar{K}_\varepsilon^T + \bar{L}_\varepsilon^T\,,$$

where

$$I_2(\varepsilon) = K_\varepsilon(0,\varepsilon(-\log\varepsilon)^2) - \bar{K}_\varepsilon + L_\varepsilon(0,\varepsilon(-\log\varepsilon)^2) - \bar{L}_\varepsilon\,.$$

Here, for any constant c, we have by (3.2), (3.19), (3.20) that

$$\limsup_{\varepsilon\to 0}\sup_{x\in K_0}\varepsilon\log\mathbb{E}(\exp c(A^\varepsilon(x)+I_2(\varepsilon))) = 0\,. \tag{3.21}$$

Fix $\lambda > 1$ and $\lambda^* = \lambda/(\lambda-1)$. An application of Hölder's inequality yields

$$\begin{aligned}
\limsup_{\varepsilon\to 0}\sup_{x\in K_0}\varepsilon\log\mathbb{E}\exp(A^\varepsilon(x)+B^\varepsilon(x)) \;&\le\; \limsup_{\varepsilon\to 0}\sup_{x\in K_0}\varepsilon\lambda^{-1}\log\mathbb{E}\exp(\lambda(\bar{K}_\varepsilon^T+\bar{L}_\varepsilon^T)) \\
&\quad + \limsup_{\varepsilon\to 0}\sup_{x\in K_0}\varepsilon(2\lambda^*)^{-1}\log\mathbb{E}\exp(2\lambda^* I_1(\varepsilon,T)) \\
&\quad + \limsup_{\varepsilon\to 0}\sup_{x\in K_0}\varepsilon(2\lambda^*)^{-1}\log\mathbb{E}\exp(2\lambda^*(A^\varepsilon(x)+I_2(\varepsilon))) \\
&\le\; \limsup_{\varepsilon\to 0}\sup_{x\in K_0}\varepsilon\lambda^{-1}\log\mathbb{E}\exp(\lambda(\bar{K}_\varepsilon^T+\bar{L}_\varepsilon^T)) + q_T(\lambda)\,,
\end{aligned}$$

where $q_T(\lambda)\to_{T\to\infty} 0$. Here, the second inequality is due to (3.21), and the convergence of $q_T(\lambda)$ is due to Lemma 3.7. Next, take $m = \bar{m}_0$ and $\xi_s^\varepsilon = \bar{m}_{\varepsilon s}$ in (2.5), (2.6), and (2.7). Note that $(\bar{K}_\varepsilon^T, \bar{L}_\varepsilon^T)$ is identical in law to $(\varepsilon^{-1}\hat{K}_\varepsilon^x(T), \varepsilon^{-1}\hat{L}_\varepsilon^x(T))$ and, due to Lemma 3.6 and (3.18), $\sup_{0\le s\le T}|\xi_s^\varepsilon - X_1| \to_{\varepsilon\to 0} 0$ a.s. and ξ_s^ε is independent of $\hat{W}_\cdot$. Thus, applying Varadhan's lemma, see e.g. [2, Theorem 4.3.1], and noting the integrability condition (2.8), one concludes from Lemma 2.1 that

$$\limsup_{\varepsilon\to 0}\sup_{x\in K_0}\varepsilon\log\mathbb{E}\exp(\lambda(\bar{K}_\varepsilon^T+\bar{L}_\varepsilon^T) - \mathcal{J}^\lambda(x,T,X_1)/\varepsilon) \le 0\,.$$

Hence,

$$\limsup_{\varepsilon\to 0}\sup_{x\in K_0}\varepsilon\log\mathbb{E}\exp(A^\varepsilon(x)+B^\varepsilon(x))\exp(-\mathcal{J}^\lambda(x,T,X_1)/\varepsilon\lambda) \le q_T(\lambda)\,.$$

Taking the limit $T\to\infty$ followed by $\lambda\to 1$ and using (2.16) yields

$$\limsup_{\varepsilon\to 0}\sup_{x\in K_0}\varepsilon\log\mathbb{E}\exp(A^\varepsilon(x)+B^\varepsilon(x))\exp(-\mathcal{J}(x,X_1)/\varepsilon) \le 0\,. \tag{3.22}$$

The same argument applies for a complementary lower bound, using

$$\begin{aligned}
\liminf_{\varepsilon\to 0}\inf_{x\in K_0}\varepsilon\log\mathbb{E}\exp(A^\varepsilon(x)+B^\varepsilon(x)) \;&\ge\; \liminf_{\varepsilon\to 0}\inf_{x\in K_0}\varepsilon\lambda\log\mathbb{E}\exp(\lambda^{-1}(\bar{K}_\varepsilon^T+\bar{L}_\varepsilon^T)) \\
&\quad - \limsup_{\varepsilon\to 0}\sup_{x\in K_0}\varepsilon(2\lambda^*)\log\mathbb{E}\exp(((2\lambda^*)^{-1}I_1(\varepsilon,T)) \\
&\quad - \limsup_{\varepsilon\to 0}\sup_{x\in K_0}\varepsilon(2\lambda^*)\log\mathbb{E}\exp((2\lambda^*)^{-1}(A^\varepsilon(x)+I_2(\varepsilon)))
\end{aligned}$$

and following the same steps as in the proof of the upper bound. ◊

Proof of Lemma 3.7 Note first that, due to (2.2) and the fact that $h'(\cdot)$ is Lipschitz,

$$|f_2(\bar{X}_s^x,\bar{m}_0)-f_2(\bar{X}_s^{\bar{m}_0},\bar{m}_0)|\leq c(\bar{m}_0)|x-\bar{m}_0|e^{-cs/\varepsilon}.$$

Hence, for any $\theta\in\mathbb{R}$,

$$\begin{aligned}\mathbb{E}(e^{\theta(\bar{K}_\varepsilon-\bar{K}_\varepsilon^T)}) &\leq e^{\theta^2\bar{m}_0^2|x-\bar{m}_0|^2\int_{T\varepsilon}^{\varepsilon(-\log\varepsilon)^2}e^{-2cs/\varepsilon}ds/\varepsilon^2}\\ &\leq e^{\theta^2\bar{m}_0^2|x-\bar{m}_0|^2e^{-2cT}/\varepsilon},\end{aligned}\tag{3.23}$$

where $c>0$ depends on $\bar{m}_0$ only. On the other hand,

$$\begin{aligned}|H(\bar{X}_s^x,\bar{m}_0)-H(\bar{X}_s^{\bar{m}_0},\bar{m}_0)| &\leq c(\bar{m}_0)(|\bar{X}_s^x-\bar{X}_s^{\bar{m}_0}|+|\bar{X}_s^{\bar{m}_0}(\bar{X}_s^x-\bar{X}_s^{\bar{m}_0})|)\\ &\leq |x-\bar{m}_0|c(\bar{m}_0)e^{-cs/\varepsilon}(1+|\bar{X}_s^{\bar{m}_0}|).\end{aligned}$$

Hence, using Lemma 3.4 in the last inequality,

$$\begin{aligned}\mathbb{E}(e^{\theta(\bar{L}_\varepsilon-\bar{L}_\varepsilon^T)}) &\leq e^{|\theta|c(\bar{m}_0)|x-\bar{m}_0|e^{-cT}/\varepsilon}\mathbb{E}(e^{|\theta|c(\bar{m}_0)|x-\bar{m}_0|\int_{T\varepsilon}^{\varepsilon(-\log\varepsilon)^2}|\bar{X}_s^{\bar{m}_0}|e^{-cs/\varepsilon}ds/\varepsilon^2})\\ &\leq e^{|\theta|c(\bar{m}_0)|x-\bar{m}_0|e^{-cT}/\varepsilon}\frac{1}{\varepsilon}\int_{T\varepsilon}^{\varepsilon(-\log\varepsilon)^2}e^{-c(s-T\varepsilon)/\varepsilon}\mathbb{E}(e^{|\theta|c(\bar{m}_0)|x-\bar{m}_0||\bar{X}_s^{\bar{m}_0}|e^{-cT}/\varepsilon})ds\\ &\leq e^{|\theta|c(\bar{m}_0,K_0)e^{-cT}/\varepsilon}.\end{aligned}\tag{3.24}$$

The lemma follows by combining (3.23) and (3.24).

◊

Proof of Theorem 1.1 We note first that for any compact K_0

$$\begin{aligned}\int_{\mathbb{R}}\rho_1^\varepsilon(x)dx &= \int_{K_0}\rho_1^\varepsilon(x)dx+\int_{K_0^c}\rho_1^\varepsilon(x)dx\\ &\leq \int_{K_0}e^{-F(x,\bar{m}_0)/\varepsilon}\mathbb{E}(\exp(A^\varepsilon(x)+B^\varepsilon(x)))dx\\ &\quad+\int_{K_0^c}e^{-\beta|x-\bar{m}_0|^2/\varepsilon}\mathbb{E}(\exp(A^\varepsilon(x)+B^\varepsilon(x)))dx,\end{aligned}\tag{3.25}$$

where the quadratic growth of F at infinity was used. Applying Lemma 3.1 in order to get rid of the second integral for K_0 large enough and Lemma 3.3 and (3.18) to control the first, we find that

$$\varepsilon\log\int_{\mathbb{R}}\rho_1^\varepsilon(x)dx\to_{\varepsilon\to0}-\inf_{x\in\mathbb{R}}(F(x,X_1)-J(x,X_1)),\quad P-a.s.$$

The uniform estimate of Lemma 3.3 then completes the proof of the Theorem. ◊

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