Quenched, Annealed and Functional Large Deviations for One-Dimensional Random Walk in Random Environment

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Abstract: Suppose that the integers are assigned random variables \( \{\omega_i\} \) (taking values in the unit interval), which serve as an environment. This environment defines a random walk \( \{X_n\} \) (called a RWRE) which, when at \( i \), moves one step to the right with probability \( \omega_i \), and one step to the left with probability \( 1 - \omega_i \). When the \( \{\omega_i\} \) sequence is i.i.d., Greven and den Hollander (1994) proved a large deviation principle for \( X_n/n \) conditional upon the environment, with deterministic rate function. We consider in this paper large deviations, both conditioned on the environment (quenched) and averaged on the environment (annealed), for the RWRE, which forces us to consider also the ergodic environment case. The annealed rate function is the solution of a variational problem involving the quenched rate function and specific relative entropy. We also give detailed qualitative description of the resulting rate functions. Our techniques differ from those of Greven and den Hollander, and allow us to present also a trajectorial (quenched) large deviation principle.

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1 Introduction and Statement of Results

Let $\Sigma = [0,1]^Z$, let $\omega = (\omega_i)_{i \in Z} \in \Sigma$ be a collection of random variables which serve as an environment, and define $\rho_i = \rho_i(\omega) = \frac{1}{\omega_i}, \ i \in Z$. Here and throughout, we omit $\omega$ from the argument of functions if no confusion occurs. We denote by $\theta : \Sigma \rightarrow \Sigma$ the shift on $\Sigma$, given by $(\theta \omega)(i) = \omega(i + 1)$.

For each $\omega \in \Sigma$, we denote by $P_\omega$ the distribution of the random walk $(X_n)_{n=0,1,2,\ldots}$ which, when at location $i$, moves to $i + 1$ with probability $\omega_i$ and to $i - 1$ with probability $1 - \omega_i$. We sometimes use also $X^n_0$ to denote the same random walk initialized at $i$. We will often look at functions of $\omega$ which are given in terms of the random walk, e.g. $\log E_\omega[e^{-X_n}]$. Let $T_k = \inf\{n : X_n = k\}, \ k = 0, \pm 1, \pm 2, \cdots$ and 

$$
\tau_k = T_k - T_{k-1} \quad k > 0,
$$

$$
\tau_k = T_k - T_{k+1} \quad k < 0,
$$

with the convention that $\infty - \infty = \infty$ in this definition.

Let now $M_1(\Sigma), M^f_1(\Sigma)$ and $M^c_1(\Sigma)$ be the spaces of probability measures, stationary probability measures and ergodic probability measures, on $\Sigma$. All spaces of probability measures in this paper are equipped with the topology induced from weak convergence. Let $K \subset (0,1)$ be some fixed compact subset of $(0,1)$. We denote $M^f_1(\Sigma)^+ := \{\eta \in M^f_1(\Sigma) : \int \log \rho_0(\omega)\eta(d\omega) \leq 0\}$, and, for any set $M \subset M_1(\Sigma)$, we let $M^K = M \cap \{\eta : \text{supp}(\eta_0) \subset K \subset (0,1)\}$. We also denote by $\omega_{\min} = \omega_{\min}(\eta) := \min\{z : z \in \text{supp}\eta_0\}$ where $\eta_0$ denotes the marginal of $\eta$, $\omega_{\max} = \omega_{\max}(\eta) := \max\{z : z \in \text{supp}\eta_0\}$, and let $\rho_{\max} = \rho_{\max}(\eta) := (1 - \omega_{\min})/\omega_{\min}$. Finally, we use also the notation 

$$
M^{1/2} := \{\eta \in M_1^f(\Sigma) : \omega_{\min}(\eta) \leq 1/2, \omega_{\max}(\eta) \geq 1/2\}. \quad (1)
$$

For $\eta \in M^c_1(\Sigma)^+$, the random walk $(X_n)$ is either recurrent (if $\int \log \rho_0(\omega)\eta(d\omega) = 0$) or $X_n \rightarrow +\infty$ (if $\int \log \rho_0(\omega)\eta(d\omega) < 0$), see [13, Chap. IV, Theorem 2.3 and Corollary 2.4] or [1].

Let $Z^-_\eta := \rho_i + \rho_{i-1} + \rho_{i-1}\rho_{i-2} + \cdots$. Note that if the walk is transient to the right, i.e. if $\eta \in M^c_1(\Sigma)^+ \setminus M^c_1(\Sigma)^-$, then $Z^-_\eta < \infty$, $\eta$-a.s.. It can be shown that if 

$$
v_{\eta}^{-1} := \int (1 + 2Z^-_\eta)\eta(d\omega) = \int (1 + Z^-_0 + Z^-_0)\eta(d\omega) < \infty, \quad (2)
$$

then the random walk has the positive speed $v_{\eta}$, i.e. for $\eta$-a.e. $\omega$, we have $X_n/\sqrt{n} \rightarrow v_{\eta}$, $P_\omega$-a.s., cf [1]. In fact, we will see, c.f. the proof of Lemma 1 below, that $E_\omega[v_{\eta}] = 1 + 2Z^-_\eta$, and one could use this to rerun the argument of Solomon [18] yielding the speed of the RWRE. In particular, if $\eta$ is a product measure, one recovers Solomon’s formula for the speed: $\int (1 + 2Z^-_0)\eta(d\omega) < \infty$ if $\langle \rho \rangle := \int \rho_0(\omega)\eta(d\omega) < 1$ and, in this case,

$$
v_{\eta} = \frac{1 - \langle \rho \rangle}{1 + \langle \rho \rangle}.
$$

For a product measure $\eta$, the study of large deviations for the law of $X_n/n$ was initiated in [10], where a large deviation principle for the distributions of $X_n/n$ under $P_\omega$ was derived, for $\eta$-a.e. $\omega$. We refer to such results in the sequel as quenched statements. Tail estimates, in the subexponential regime, both in the quenched and in the annealed regime, i.e. under the law $P = \eta(d\omega) \otimes P_\omega$, are presented in [3], [9], [15], [14]. We refer to the introduction sections of [10] and [3], as well as to [11] and [16], for more about the history of the problem and a description of limit laws not mentioned above.
The approach of [10] to large deviation statements involves looking at the RWRE as a Markov chain in the space of environments, and the quenched Large Deviation Principle (LDP) is obtained by an appropriate contraction. More precisely, the rate function is the solution of a variational problem and is shown to be the Legendre transform of certain Lyapunov exponents. Our goal in this paper is to suggest a different point of view for obtaining large deviation theorems, both annealed and quenched, for the general ergodic \( \eta \). We do so by building on recursion ideas which formed the key to [3], leading to rather simple proofs of the LDP’s. As an application of our methods, we show how functional LDP’s can be obtained by essentially the same methods.

Turning to the description of our results, we present first quenched LDP’s for the distribution of \( T_n/n \) or \( T_{-n}/n \), respectively. LDP’s in the annealed case follow. The LDP’s for the distribution of \( X_n/n \) follow then by “time inversion”. The linear pieces of the quenched rate function of \( X_n/n \) near 0 in the transient case, first discovered by Greven and den Hollander, can be explained in terms of the linear pieces of the quenched rate function for \( T_n/n \) or \( T_{-n}/n \) at infinity. We finally use the quenched LDP’s obtained and derive a function space LDP (quenched) for the trajectories of the random walk \( X_n \).

After the bulk of this work was completed, we learnt of a recent very interesting preprint of Zerner [21], where he uses similar recursion ideas to analyze certain multi-dimensional RWRE’s. Among other results, Zerner shows how to re-derive some of Greven and den Hollander’s results using a hitting time decomposition similar to ours.

A crucial role in our results is played by the function

\[
\varphi(\lambda, \omega) := E_\omega[e^{\lambda \tau_1} \mathbf{1}_{\tau_1 < \infty}].
\]  

(3)

A characterization of \( \varphi(\lambda, \omega) \) in terms of continued fraction expansions is provided in Section 2, Lemma 1.

Let next

\[
\tau_\omega := E_\omega[\tau_1 | \tau_1 < \infty]
\]

(4)

(with the value +\( \infty \) allowed), and define

\[
I^\tau_\eta(\omega) = \sup_{\lambda \in \mathbb{R}} \{ \lambda u - \int \log \varphi(\lambda, \omega) \eta(d\omega) \}.
\]

(5)

**Theorem 1** Assume \( \eta \in M^\tau_1(\Sigma)^K \). Then, for \( \eta \)-a.e. \( \omega \), the distributions of \( T_n/n \) under \( P_\omega \) satisfy a weak LDP with deterministic, convex rate function \( I^\tau_\eta \). Further, \( I^\tau_\eta(\cdot) \) is decreasing on \([1, \int \tau_\omega \eta(d\omega)]\) and increasing on \([\int \tau_\omega \eta(d\omega), \infty)\).

For a formal definition of a LDP and weak LDP, we refer to [4, Section 1.2]. A discussion of different possible shapes for the rate function \( I^\tau_\eta \), as well as graphs of such functions, are provided in Section 5.

Theorem 1 obviously implies also a LDP for \( T_{-n}/n \), simply by symmetry (i.e., space reversal of the measure \( \eta \)). An alternative expression for the rate function in this latter LDP, which is an intermediate step in our proof of Theorem 1, is provided by the following:

**Proposition 1** Assume \( \eta \in M^\tau_1(\Sigma)^K \). Then,

\[
\int \log E_\omega[e^{\lambda \tau_{-1}} \mathbf{1}_{\tau_{-1} < \infty} \eta(d\omega)] = \int \log E_\omega[e^{\lambda \tau_1} \mathbf{1}_{\tau_1 < \infty} \eta(d\omega)] + \int \log \rho_0(\omega) \eta(d\omega).
\]

(6)
Further, if $\eta \in M_{\mathcal{F}}(\Sigma)^{+,K}$, then the distributions of $T_{\omega}/n$ under $P_\omega$ satisfy, for $\eta$-a.e. $\omega$, a weak LDP with deterministic rate function

$$I^\tau_{\eta}(u) := I^\tau_{\eta}(u) - \int \log \rho_0(\omega) \eta(d\omega), \quad 1 \leq u < \infty. \quad (7)$$

We note that in both Theorem 1 and Proposition 1, the LDP’s are weak due to possible positive probability mass at $+\infty$. The LDP of Theorem 1 can be strengthened to a full LDP if $\eta \in M_{\mathcal{F}}(\Sigma)^{+,K}$.

We next turn our attention to the annealed situation. Denote by $h(\cdot|\alpha)$ the specific relative entropy with respect to any $\alpha \in M_1(\Sigma)$. We say that $\nu \in M_\mathcal{F}(\Sigma)$ is locally equivalent to the product of its marginals if its restrictions to $M_1([0, 1]^n)$ is equivalent to $\prod_{i=1}^n \nu_i$ for arbitrary $n$, i.e. if for any measurable $A \subset [0, 1]^n$, $\nu(A) = 0$ if and only if $\prod \nu_i(A) = 0$. We say that it satisfies the process level LDP if, denoting $R_n = \frac{1}{n} \sum_{j=0}^{n-1} \delta_{\omega_j \omega} \in M_1(\Sigma)$, the random variables $R_n$ satisfy the LDP in $M_1(\Sigma)$ with rate function $h(\cdot|\alpha)$. Let $F_n := \sigma(\{\omega_0, \ldots, \omega_n\})$. We will use the following Assumption (A) on $\alpha$:

**Assumption (A):** $\alpha$ is locally equivalent to the product of its marginals and, for each $\eta \in M_{\mathcal{F}}(\Sigma)^K$, there is a sequence $\{\eta^n\}$ of ergodic measures with $\eta^n \rightarrow \eta$ weakly and $h(\eta^n|\alpha) \rightarrow h(\eta|\alpha)$.

Note that product measures and Markov processes with bounded transition kernels satisfy the process level LDP, c.f. [7], as well as Assumption (A), c.f. [8], Lemma 4.8.

For $u \geq 1$, let

$$I^{\tau,\alpha}_{\eta}(u) = \inf_{\eta \in M_{\mathcal{F}}(\Sigma)} [I^\tau_{\eta}(u) + h(\eta|\alpha)]. \quad (8)$$

**Theorem 2** Let $\alpha \in M_{\mathcal{F}}(\Sigma)^K$ satisfy the process level LDP and Assumption (A). Then the distributions of $T_{\omega}/n$ under $P$ satisfy a (weak) LDP with convex rate function $I^{\tau,\alpha}_{\eta}$.

We may now turn our attention to LDP’s for the $X_n$ process. Let, for $\eta \in M_{\mathcal{F}}(\Sigma)^{+,K}$,

$$I^\eta_{\mathcal{F}}(v) = \begin{cases} v I^\tau_{\eta}(\frac{1}{v}), & 0 \leq v \leq 1 \\ |v| I^{-\tau}_{\eta}(\frac{1}{|v|}), & -1 \leq v \leq 0, \end{cases} \quad (9)$$

where $I^\tau_{\eta}$ and $I^{-\tau}_{\eta}$ were defined in (5) and (7), and the value at $v = 0$ is taken as

$$I^\eta_{\mathcal{F}}(0) = \lim_{v \to 0} v I^\tau_{\eta}(\frac{1}{v}).$$

Let $\text{Inv} : \Sigma \rightarrow \Sigma$ denote the map satisfying $(\text{Inv}\omega)_i = 1 - \omega_{-i}$, and let $\eta^{\text{inv}} = \eta \circ \text{Inv}^{-1}$. For $\eta \in M_{\mathcal{F}}(\Sigma)^K \setminus M_{\mathcal{F}}(\Sigma)^{+,K}$, define $I^\eta_{\mathcal{F}}(v) = I^{\eta^{\text{inv}}}_{\mathcal{F}}(-v)$, and note that $\eta^{\text{inv}} \in M_{\mathcal{F}}(\Sigma)^{+,K}$ while $I^{-\tau}_{\eta}(\cdot) = I^{-\tau}_{\eta^{\text{inv}}}(\cdot)$.

**Theorem 3** Assume $\eta \in M_{\mathcal{F}}(\Sigma)^K$. For $\eta$-a.e. $\omega$, the distributions of $X_{\omega}/n$ under $P_\omega$ satisfy a large deviation principle with convex rate function $I^\eta_{\mathcal{F}}(\cdot)$. This rate function was derived in [10] for the i.i.d. case, i.e. the case where $\eta$ is a product measure. For some properties of the rate function $I^\eta_{\mathcal{F}}(\cdot)$, see Section 5.
The annealed LDP for the distributions of \( X_n/n \) will be proved in the same way, using the annealed LDP for the averaged hitting times. Let

\[
I^a_\alpha(v) = \begin{cases} 
v I^a_\alpha \left( \frac{1}{v} \right), & 0 \leq v \leq 1 \\
|v| I^a_\alpha^{\text{inv}} \left( \frac{1}{|v|} \right), & -1 \leq v \leq 0. 
\end{cases}
\]  (10)

We have the following large deviation principle.

**Theorem 4** Assume \( \alpha \in M^*_f(\Sigma)^K \) satisfies the process level LDP and Assumption (A). Then, the distributions of \( X_n/n \) under \( P \) satisfy a LDP with convex rate function \( I^a_\alpha \).

We note that the quenched rate function \( I^q_\eta \) and the annealed rate function \( I^a_\alpha \) are related by the following variational formula:

\[
I^a_\alpha(v) = \inf_{\eta \in M^*_f(\Sigma)} \left[ I^q_\eta(v) + |v|h(\eta|\alpha) \right].
\]  (11)

where \( vh(\eta|\alpha) = \infty \) if \( h(\eta|\alpha) = \infty \). In particular, we always have \( I^a_\alpha \leq I^q_\eta \).

We conclude this introduction by a functional LDP. Let \( S_n(t) \triangleq n^{-1} \sum_{i=1}^{\lfloor nt \rfloor} X_i \), \( t = 0,1/n,2/n, \ldots, 1 \), linearly interpolated elsewhere. Throughout, we use the symbol \( \mathcal{L} \) to denote the class of Lipschitz functions of Lipschitz constant bounded by 1, equipped with the supremum topology. Define the functional \( I^{q, \text{raj}}_\eta \) : \( \mathcal{L} \to [0, \infty] \) by

\[
I^{q, \text{raj}}_\eta(\phi) \triangleq \int_0^1 I^q_\eta(\phi(t))dt.
\]

**Theorem 5** Let \( \eta \in M^*_f(\Sigma)^K \).
1. \( I^{q, \text{raj}}_\eta \) is a good rate function on \( \mathcal{L} \).
2. For \( \eta \) -a.e. \( \omega \), the distributions of \( S_n(\cdot) \) under \( P_\omega \) satisfy in \( \mathcal{L} \) a LDP with rate function \( I^{q, \text{raj}}_\eta \).

The structure of the article is as follows: In Section 2 we provide the proofs of the quenched LDP for hitting times. Section 3 is devoted to the corresponding annealed results. Section 4 is devoted to LDP’s for the position \( X_n \) and to the quenched trajectoryal LDP. In Section 5 we study the various rate functions in this paper. Finally, Section 6 describes some questions and open problems.

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## 2 Properties of \( \phi(\lambda, \omega) \) and proofs of the quenched LDP for hitting times.

We begin by deriving a representation of \( \phi(\lambda, \omega) \):

**Lemma 1** For any \( \lambda \in \mathbb{R} \), we have that whenever \( \phi(\lambda, \omega) < \infty \) a.s., then

\[
\phi(\lambda, \omega) = \left| \frac{1}{e^{\lambda}(1+\rho_0(\omega))} - \frac{\rho_0(\omega)}{e^{\lambda}(1+\rho_0(\omega))} - \frac{\rho_{-1}(\omega)}{e^{\lambda}(1+\rho_{-1}(\omega))} - \cdots \right|.
\]  (12)
Further, for \( \eta \in M^r_1(\Sigma)^{+,K} \), and \(1 < u < E[\tau_1] \leq \infty \), there exists a unique \( \lambda_0 = \lambda_0(u, \eta) \) such that \( \lambda_0 < 0 \) and
\[
u = \int \frac{d}{d\lambda} \log \varphi(\lambda, \omega) \big|_{\lambda=\lambda_0} \eta(d\omega).
\]
(13)

Finally, for \( \nu \) as above
\[
\inf_{\eta \in M^r_1(\Sigma)^{+,K}} \lambda_0(u, \eta) > -\infty.
\]
(14)

**Proof of Lemma 1.** Pathwise decomposition yields the following formula for \( \tau_1 \):
\[
\tau_1 = \mathbb{1}_{X_1 = 1} + \mathbb{1}_{X_1 = -1} (\tau'_1 + \tau''_1 + 1)
\]
(15)
where \( \tau'_1 + 1 \) is the first hitting time of 0 after time 1 (possibly infinite) and \( \tau'_1 + \tau''_1 + 1 \) is the first hitting time of +1 after time \( \tau'_1 + 1 \). Note that, under \( P_\omega \), the law of \( \tau'_1 \) conditioned on the event \( X_1 = -1 \) is \( P_{\theta^{-1} \omega}(\tau_1 \in \cdot) \)
and, conditioned on the event \( \tau'_1 < \infty \), \( \tau''_1 \) is independent of \( \tau'_1 \) and has law \( P_\omega(\tau_1 \in \cdot) \). Therefore, we have
\[
\varphi(\lambda, \omega) = E_\omega[e^{\lambda \tau_1} \mathbb{1}_{\tau_1 \in \infty}]
\]
\[
= P_\omega[X_1 = 1] E_\omega[e^{\lambda \tau_1} \mathbb{1}_{\tau_1 \in \infty} | X_1 = 1] + P_\omega[X_1 = -1] E_\omega[e^{\lambda \tau_1} \mathbb{1}_{\tau_1 \in \infty} | X_1 = -1]
\]
\[
= \omega_0 e^\lambda + (1 - \omega_0) E_\omega[e^{\lambda (\tau_1 - \theta^{-1})} \mathbb{1}_{\tau_1 \in \infty} | \tau_1, \theta^{-1} \in \infty] E_\omega[e^{\lambda \tau_1} \mathbb{1}_{\tau_1 \in \infty}] e^\lambda
\]
\[
= \omega_0 e^\lambda + (1 - \omega_0) e^{\lambda \varphi(\lambda, \theta^{-1} \omega)} \varphi(\lambda, \omega).
\]
Hence, if \( \varphi(\lambda, \omega) < \infty \) then \( \varphi(\lambda, \theta^{-1} \omega) < \infty \), and
\[
\varphi(\lambda, \omega) = \frac{\omega_0 e^\lambda}{1 - (1 - \omega_0) e^{\lambda \varphi(\lambda, \theta^{-1} \omega)}} = \frac{1}{(1 + \rho_0(\omega)) e^{-\lambda} - \rho_0(\omega) \varphi(\lambda, \theta^{-1} \omega)}.
\]
(16)

In the same way,
\[
\varphi(\lambda, \theta^{-1} \omega) = \frac{1}{(1 + \rho_0 \circ \theta^{-1}) e^{-\lambda} - \rho_0 \circ \theta^{-1} \varphi(\lambda, \theta^{-2} \omega)} = \frac{1}{(1 + \rho_{-1}) e^{-\lambda} - \rho_{-1} \varphi(\lambda, \theta^{-2} \omega)}.
\]
By iteration, we get the representation of \( \varphi \) as a continued fraction, i.e., (12). (For a reference on continued fractions, see [12], [20]).

Let now \( \eta \in M^r_1(\Sigma)^{+,K} \). Then, the indicator can be dropped in the definition of \( \varphi(\lambda, \omega) \), and, with \( \lambda < 0 \)
and
\[
g(\lambda) := \int \frac{d}{d\lambda} \log \varphi(\lambda, \omega) \eta(d\omega) = \int \frac{E_\omega[\tau_1 e^{\lambda \tau_1}]}{E_\omega[e^{\lambda \tau_1}]} \eta(d\omega),
\]
we have
\[
g(0) = \int E_\omega[\tau_1] \eta(d\omega) = E_\eta[\tau_1],
\]
and the strictly increasing, continuous function \( g(\cdot) \) satisfies \( g(\lambda) \geq 1 \) and \( g(\lambda) \xrightarrow{\lambda \to \infty} 1 \).
To complete the proof of (14), note that
\[
1 \leq \frac{E_\omega[\tau_1 e^{\lambda \tau_1}]}{E_\omega[e^{\lambda \tau_1}]} = \frac{P_\omega[\tau_1 = 1]e^\lambda + E_\omega[\tau_1 e^{\lambda \tau_1}1_{\tau_1 \geq 2}]}{P_\omega[\tau_1 = 1]e^\lambda + E_\omega[e^{\lambda \tau_1}1_{\tau_1 \geq 2}]} \leq \frac{\omega_0 e^\lambda + e^{3\lambda/2}E_\omega[\tau_1 e^{\lambda \tau_1-3\lambda/2}1_{\tau_1 \geq 2}]}{\omega_0 e^\lambda} \leq 1 + \frac{ce^{\lambda/3}}{\omega_0},
\]
for some constant $c$ independent of $\omega$ or $\lambda$. Taking $\lambda \to -\infty$ yields the uniform convergence of the right hand side of (17) to 1, and hence (14).

\textbf{Remark}: In the same way, taking expectations in (15) and iterating yields $E_\omega[\tau_1] = 1 + 2Z_0^-$, cf (2).

We may now deal in more details with the situation when positive $\lambda$ is needed in $\varphi(\lambda, \omega)$:

\textbf{Lemma 2} Let $\eta \in M^*_i(\Sigma)^{+,K}$. Then

(i) There is a deterministic $\infty > \lambda_{\text{crit}} \geq 0$, depending only on $\eta$, such that for $\lambda < \lambda_{\text{crit}}$, $E_\omega[e^{\lambda \tau_1}] < \infty$ for $\eta$-a.e. $\omega$, $E_\omega[e^{\lambda \tau_1}]$ has the form (12) and for $\lambda > \lambda_{\text{crit}}$, $E_\omega[e^{\lambda \tau_1}] = \infty$ for $\eta$-a.e. $\omega$.

(ii) Let $u_{\text{crit}} = \infty$ if $\int E_\omega[\tau_1 e^{\lambda \tau_1}] \eta(d\omega) = \infty$ and $u_{\text{crit}} := \int \frac{d}{d\lambda} \log \varphi(\lambda, \omega) \big|_{\lambda = \lambda_{\text{crit}}} \eta(d\omega)$ else. For $E[\tau_1] \leq u < u_{\text{crit}}$, there exists a unique $\lambda_0 = \lambda_0(u, \eta)$ such that $\lambda_0 \geq 0$ and

\[
u = \int \frac{d}{d\lambda} \log \varphi(\lambda, \omega) \big|_{\lambda = \lambda_0} \eta(d\omega).
\]

\textbf{Remark}: $u_{\text{crit}}$ can be infinite in the general ergodic case, for instance if $\eta = \frac{1}{2}\delta_{\omega_1} + \frac{1}{2}\delta_{\omega_2}$.

\textbf{Proof of Lemma 2}.

(i) Let $\lambda_c(\omega) := \sup\{\lambda : E_\omega[e^{\lambda \tau_1}] < \infty\}$. Since, using (15), $E_{\theta\omega}[e^{\lambda \tau_1}] \geq (1 - \omega_1)E_\omega[e^{\lambda \tau_1}]$, we have $\lambda_c(\theta\omega) \leq \lambda_c(\omega)$. But $\lambda_c(\theta\omega)$ and $\lambda_c(\omega)$ have the same distribution, hence $\lambda_c(\theta\omega) = \lambda_c(\omega)$ for $\eta$-a.e. $\omega$, i.e. $\lambda_c$ is shift-invariant. Since $\eta$ is ergodic, this implies that $\lambda_c(\omega) = \int \lambda_c(\omega) \eta(d\omega) := \lambda_{\text{crit}}$ for $\eta$-a.e. $\omega$. Lemma 1 implies the continued fraction expansion stated in (12).

(ii) As in the proof of Lemma 1, let

\[
g(\lambda) := \int \frac{E_\omega[\tau_1 e^{\lambda \tau_1}]}{E_\omega[e^{\lambda \tau_1}]} \eta(d\omega).
\]

Then $g$ is strictly increasing and continuous in $\lambda$ for $\lambda < \lambda_{\text{crit}}$, $g(0) = E_\eta[\tau_1] < \infty$, and $g(\lambda) \xrightarrow{\lambda \to \lambda_{\text{crit}}} u_{\text{crit}}$.

\textbf{Proof of Theorem 1}.

It is clearer to consider first the case $\eta \in M^*_i(\Sigma)^{+,K}$. The claims on the convexity and monotonicity of $I^q_\eta(\cdot)$ are a direct consequence of the definition and Lemmas 1 and 2. Considering the bounds themselves, we start by showing that for $1 < u \leq E[\tau_1]$, and $\eta \in M^*_i(\Sigma)^{+,K}$,

\[
\lim_{n \to \infty} \sup \frac{1}{n} \log P_\omega \left[ \frac{1}{n} \sum_{j=1}^{n} \tau_j \leq u \right] \leq \sup_{\lambda \leq 0} \left[ \lambda u - \int \log \varphi(\lambda, \omega) \eta(d\omega) \right] = -I^q_\eta(u).
\]
Indeed, Chebyshev’s inequality implies that, for \( \lambda \leq 0 \),

\[
P_\omega \left[ \frac{1}{n} \sum_{j=1}^{n} \tau_j \leq u \right] \leq E_\omega \left[ e^{\lambda \sum_{j=1}^{n} \tau_j} \right] e^{-\lambda n u}.
\]

Note that, because for \( \eta \)-a.e \( \omega \), \( \tau_1, \ldots, \tau_n \) are finite and therefore independent under \( P_\omega \),

\[
\frac{1}{n} \log E_\omega \left[ e^{\lambda \sum_{j=1}^{n} \tau_j} \right] = \frac{1}{n} \sum_{j=1}^{n} \log E_\omega [e^{\lambda \tau_j}] = \frac{1}{n} \sum_{j=0}^{n-1} \log E_{\omega \circ \theta_j} [e^{\lambda \tau_j}]
\]

\[
= \frac{1}{n} \sum_{j=0}^{n-1} \log \varphi(\lambda, \theta^j \omega) \xrightarrow{n \to \infty} \int \log \varphi(\lambda, \omega) \eta(d\omega) \quad \eta \text{-a.e. } \omega,
\]

due to the ergodic theorem. This proves the inequality in (20). Now, Lemma 1 implies that the sup in (20) is attained for \( \lambda = \lambda_0(u) \), and further is equal to the supremum over \( \lambda \in \mathbb{R} \).

Still with \( \eta \in M^+_f(\Sigma)+^K \), let \( u \geq E[\tau_1] \), and note that for \( \lambda \geq 0 \), we have

\[
P_\omega \left[ \frac{1}{n} \sum_{j=1}^{n} \tau_j \geq u \right] \leq E_\omega \left[ e^{\lambda \sum_{j=1}^{n} \tau_j} \right] e^{-\lambda n u}
\]

and

\[
\frac{1}{n} \log P_\omega \left[ \frac{1}{n} \sum_{j=1}^{n} \tau_j \geq u \right] \leq \frac{1}{n} \sum_{j=0}^{n-1} \log \varphi(\lambda, \theta^j \omega) - \lambda u \xrightarrow{n \to \infty} \int \log \varphi(\lambda, \omega) \eta(d\omega) - \lambda u \text{ for } \eta \text{-a.e. } \omega,
\]

due to the ergodic theorem. Since \( \lambda \geq 0 \) was arbitrary, we have

\[
\limsup_{n \to \infty} \frac{1}{n} \log P_\omega \left[ \frac{1}{n} \sum_{j=1}^{n} \tau_j \geq u \right] \leq - \sup_{\lambda \geq 0} \left[ \lambda u - \int \log \varphi(\lambda, \omega) \eta(d\omega) \right]. \tag{21}
\]

Now Lemma 2 implies that the sup in (21) is attained (with \( u \leq u_{\text{crit}} \)) for \( \lambda = \lambda_0(u) \), and at \( \lambda = \lambda_{\text{crit}} \) otherwise, and further is equal to the supremum over all \( \lambda \in \mathbb{R} \). Hence,

\[
\limsup_{n \to \infty} \frac{1}{n} \log P_\omega \left[ \frac{1}{n} \sum_{j=1}^{n} \tau_j \geq u \right] \leq - I^\tau_\eta^q(u). \tag{22}
\]

The upper bound in the LDP for \( \eta \in M^+_f(\Sigma)+^K \) follows from (20), (22) and the convexity of \( I^\tau_\eta^q(\cdot) \).

To prove the lower bounds in Theorem 1 for \( \eta \in M^c(\Sigma)+^K \), we will follow a standard change of measure, using the one dimensional structure and independence. See [4, Pg. 31–33] for a similar argument. Fix \( u \in (1, \infty) \), \( M \in (u + 1, \infty) \) (eventually, we will take \( M \to \infty \), and in fact for \( u < E[\tau] \) we could take \( M = \infty \) throughout). Let

\[
A_{n,M} = \{ \tau_j \leq M, j = 1, \ldots, n \},
\]

and let \( \tilde{P}_{\omega,n}(\cdot) \) denote the law of \( \{ \tau_i \}_{i=1}^{n} \), conditioned on \( A_{n,M} \). Note that \( \{ \tau_i \}_{i=1}^{n} \) are still independent, although not identically distributed, under the law \( \tilde{P}_{\omega,n} \). We let

\[
\varphi_M(\lambda, \omega) := E_\omega [1_{\{\tau_j \leq M\}} \exp(\lambda \tau_j)], \varphi_M(\lambda, \omega) := E_\omega [\exp(\lambda \tau_j)] = \frac{\varphi_M(\lambda, \omega)}{P_\omega(\tau_1 \leq M)}.
\]
\[
\log \varphi(\lambda) = \int \log(\varphi(\lambda, \omega)) \eta(d\omega), \quad \log \varphi_M(\lambda) = \int \log(\varphi_M(\lambda, \omega)) \eta(d\omega), \quad C_M = \int \log P_\omega[\tau_1 \leq M] \eta(d\omega),
\]

and
\[
\tilde{\Lambda}_M(\lambda) := \log \varphi_M(\lambda) := \log \varphi_M(\lambda) - C_M.
\]

Note that \(0 \geq C_M \geq \int \log \omega_0 \eta(d\omega) > -\infty\) and \(C_M \to M \to \infty 0\) because \(\tau_1 < \infty\), \(\eta\)-a.s. We have
\[
\frac{1}{n} \log \tilde{P}_\omega \left[ \frac{1}{n} \sum_{j=1}^n \tau_j \in (u - \delta, u + \delta) \right] \geq \frac{1}{n} \log \tilde{P}_\omega \left[ \frac{1}{n} \sum_{j=1}^n \tau_j \in (u - \delta, u + \delta) \right] + \frac{1}{n} \log P_\omega[A_{n,M}]
\]

where \(\tilde{P}_\omega\) is the standard extension of \((\tilde{P}_\omega, \eta)\). Using now the fact that \(\varphi_M(\lambda, \omega)\) is smooth and convex, we define, for \(M\) large enough, \(\lambda_M(u)\) such that
\[
u = \int \frac{d \log \varphi_M(\lambda, \omega)}{d\lambda} \bigg|_{\lambda = \lambda_M(u)} \eta(d\omega).
\]

Define \(\tilde{Q}_\omega\) such that, for each \(n\),
\[
\frac{d \tilde{Q}_\omega}{d \tilde{P}_\omega, \eta} = \frac{1}{Z_{n,\omega}} \exp(\lambda_M(u) \sum_{j=1}^n \tau_j)
\]

with \(Z_{n,\omega} = \prod_{j=1}^n \varphi_M(\lambda_M(u), \theta^j \omega)\). Then we have
\[
\tilde{P}_\omega \left[ \frac{1}{n} \sum_{j=1}^n \tau_j \in (u - \delta, u + \delta) \right] \\
\geq \exp \left( -n \lambda_M(u) (u + \delta) + \sum_{j=1}^n \log \varphi_M(\lambda_M(u), \theta^j \omega) \right) \tilde{Q}_\omega \left[ \frac{1}{n} \sum_{j=1}^n \tau_j - u \leq \delta \right].
\]}

(23)

By the ergodic theorem, with the last equality due to our choice of \(\lambda_M(u)\),
\[
\int \frac{1}{n} \sum_{j=1}^n \tau_j d \tilde{Q}_\omega \xrightarrow{n \to \infty} \int d\eta \int \tau_1 d \tilde{Q}_\omega = u.
\]}

(24)

The independence of the \(\tau_j\) under \(\tilde{Q}_\omega\) implies that
\[
\int \left( \frac{1}{n} \sum_{j=1}^n (\tau_j - E_{\tilde{Q}_\omega}[\tau_j]) \right)^4 d \tilde{Q}_\omega \leq 3M^4 \frac{n^2}{n^2}.
\]

Hence, using (24) and the Borel-Cantelli lemma,
\[
\tilde{Q}_\omega \left[ \frac{1}{n} \sum_{j=1}^n \tau_j - u \geq \delta \right] \xrightarrow{n \to \infty 0}, \eta - a.s.
\]

Substituting in (23), taking logarithms, dividing by \(n\), and letting \(n \to \infty\) and then \(\delta \to 0\), we conclude that
\[
\lim_{\delta \to 0} \liminf_{n \to \infty} \frac{1}{n} \log \tilde{P}_\omega \left[ \frac{1}{n} \sum_{j=1}^n \tau_j \in (u - \delta, u + \delta) \right] \geq - \left( \lambda_M(u) u - \liminf_{n \to \infty} \frac{1}{n} \sum_{j=1}^n \log \varphi_M(\lambda_M(u), \theta^j \omega) \right)
\]

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Let $I^*(u) = \limsup_{M \to \infty} I_M(u)$. Because $\varphi_M(\cdot)$ is non-decreasing in $M$, so is $-I_M(\cdot)$, implying that $I^*(u) \geq 0$ and, because $I_M(u) < \infty$ for large $M$, also $I^*(u) < \infty$. Hence, the level sets \( \{ \lambda : \lambda u - \log \varphi_M(\lambda) \geq I^*(u) \} \) are non-empty, compact, nested sets and hence contain some $\lambda^* < \infty$ in their intersection. By Lebesgue's monotone convergence, we get

$$\log \varphi(\lambda^*) = \lim_{M \to \infty} \log\varphi_M(\lambda^*) \leq \lambda^* u - I^*(u),$$

implying that $I^*_\eta(u) = \sup_{\lambda \in \mathbb{R}} (\lambda u - \log \varphi(\lambda)) \geq I^*(u)$ and hence, in conjunction with (25), the lower bound with rate function $I^*_\eta(u)$.

We will see below how to prove Proposition 1. Assuming it has been proved, we show how to complete the proof of Theorem 1 for $\eta \in M^+_\tau(\Sigma)^K \setminus M^+_\tau(\Sigma)^+,K$. Clearly, $\int \log \rho_\theta(\omega_\eta)(d\omega) = -\int \log \rho_\theta(\omega_\eta)(d\omega)$, and further the law of $\sum_{j=1}^n \tau_j$ under $\eta$ is the same as the law of $\sum_{j=1}^n \tau_j$ under $\eta_{\text{inv}} \in M^+_\tau(\Sigma)^+,K$. Hence, the distributions of $\frac{1}{n} \sum_{j=1}^n \tau_j$ under $P_\omega$ satisfy, $\eta$-a.s., the LDP with rate function

$$I^*_\eta(u) + \int \log \rho_\theta(\omega_\eta)(d\omega).$$

The conclusion follows by equation (6).

Define

$$\tilde{\varphi}(\lambda, \omega) = E_\omega \left[ e^{\lambda \tau_{\eta_{\text{inv}}}} \mathbf{1}_{\{\tau_{\eta_{\text{inv}}} < 1}\} \right].$$

Note that for $\lambda \leq 0$, $\tilde{\varphi}(\lambda, \omega) \leq 1$ (with strict inequality even at $\lambda = 0$ if $X_\omega \to \infty$, $\eta$-a.s.), while it is easy to check that because $\eta_{\theta}(1) = 0$, one has also that $\tilde{\varphi}(\lambda, \omega) > 0$. It is not hard to see that the same type of recursion as in (15) (using the indicators in the definition of $\tilde{\varphi}$) leads to the formula

$$\tilde{\varphi}(\lambda, \omega) = (1 - \omega_0)e^\lambda + \omega_\eta e^\lambda \tilde{\varphi}(\lambda, \theta\omega) \tilde{\varphi}(\lambda, \omega).$$

(27)

Note that (27) implies that $\tilde{\varphi}(\lambda, \omega) = \varphi(\lambda, \text{Inv } \omega)$. Although one could use this observation and a rerun of the proof of Theorem 1 to derive a weak LDP for $S_\tau^-$, we take a slightly shorter route by considering an auxiliary Markov chain. Let $\tilde{\tau}_1, \tilde{\tau}_2, \tilde{\tau}_3, ..., \tilde{\tau}_N$ have the distribution of $\tau_1, \tau_2, \tau_3, ..., \tau_N$, conditioned on $T_{\tilde{\tau}} < \infty$. In fact the law of $\tilde{\tau}_1$ does not depend on $N$: the distributions of $(X_0, X_1, X_2, ..., X_{T_{\tilde{\tau}}})$ under $P_\omega$, conditioned on $T_{\tilde{\tau}} < \infty$, $N = 1, 2, ..., N$ form a consistent family whose extension is again a Markov chain. To see this, let $P_{\omega,N} := P_{\omega}[\tau_{\tilde{\tau}} < \infty]$, restricted to $(X_0, X_1, X_2, ..., X_{T_{\tilde{\tau}}})$, and compute (with $x_i > -N$),

$$P_{\omega,N}[X_{n+1} = x_{n+1}|X_1 = x_1, ..., X_n = x_n] = \frac{P_{\omega,N}[X_{n+1} = x_{n+1}, X_1 = x_1, ..., X_n = x_n]}{P_{\omega,N}[X_1 = x_1, ..., X_n = x_n]} = \frac{P_{\omega}[X_{n+1} = x_{n+1}, X_1 = x_1, ..., X_n = x_n, T_{\tilde{\tau}} < \infty]}{P_{\omega}[X_1 = x_1, ..., X_n = x_n, T_{\tilde{\tau}} < \infty]}$$

$$= \frac{P_{\omega}[X_{n+1} = x_{n+1}, X_1 = x_1, ..., X_n = x_n]P_{\theta^*+1}\omega[T_{\tilde{\tau}} < \infty]}{P_{\omega}[X_1 = x_1, ..., X_n = x_n]P_{\theta^*+1}\omega[T_{\tilde{\tau}} < \infty]}$$

$$= \frac{P_{\omega}[X_{n+1} = x_{n+1}|X_1 = x_1, ..., X_n = x_n]P_{\theta^*+1}\omega[T_{\tilde{\tau}} < \infty]}{P_{\omega}[X_1 = x_1, ..., X_n = x_n]P_{\theta^*+1}\omega[T_{\tilde{\tau}} < \infty]}$$

$$= \omega_{x_n}P_{\theta^*+1}\omega[T_{\tilde{\tau}} < \infty].$$

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where we used the Markov property in the third and in the fourth equality. The last term depends neither on $N$ nor on $x_1, \ldots, x_{n-1}$. Therefore, the extension of $(P_{\sigma, N})_{N \geq 1}$ is the distribution of the Markov chain with transition probabilities $\tilde{\omega}_i = \omega_i P_{\sigma + 1, \omega} [T_{-1} < \infty], i \in \mathbb{Z}$. In particular, $\tilde{\tau}_1, \tilde{\tau}_2, \tilde{\tau}_3, \ldots$ are independent under $P_{\omega}$ and, with a slight abuse of notations, form a stationary sequence under $P$. Let

$$\bar{\varphi}(\lambda, \omega) := E_{\omega} [e^{\lambda \bar{\tau} - 1}] = \frac{\varphi(\lambda, \omega)}{P_{\omega} [T_{-1} < \infty]}$$  \hspace{1cm} (28)$$

We will show the following LDP.

**Theorem 6** With $\eta \in M_1(\Sigma)^{+ \cdot K}$, the distributions of $\frac{1}{n} \sum_{j=1}^{n} \tilde{\tau}_j$ under $P_{\omega}$ satisfy for $\eta$-a.e. $\omega$, a LDP with deterministic rate function $I_{\eta}^\rho_\omega$.

Equipped with Theorem 6, we can now complete the:

**Proof of Proposition 1** Note that for $A \subseteq [1, \infty)$, we have

$$P_{\omega} \left[ \frac{1}{n} \sum_{j=1}^{n} \tilde{\tau}_j \in A, T_{-n} < \infty \right] = P_{\omega} \left[ \frac{1}{n} \sum_{j=1}^{n} \tilde{\tau}_j \in A \right] P_{\omega} [T_{-n} < \infty]$$

Recall the following result, whose proof can be found, e.g., in [2, Pg. 65-71].

**Lemma 3** Assume $(\omega_x)_{x \in \mathbb{Z}}$ is such that $X_n \to +\infty$ $P_{\omega}$-a.s. Then

$$P_{\omega} [\min_k X_k \leq -n] = \frac{\sum_{j=0}^{\infty} \prod_{i=-n+1}^{j} \rho_i}{1 + \sum_{j=-n+1}^{\infty} \prod_{i=-n+1}^{j} \rho_i}.$$  \hspace{1cm} (29)

Due to Lemma 3, we have (\eta-a.s.) that

$$\int \log P_{\omega} [\tilde{\tau} < \infty] \eta(d\omega) = \lim_{n \to \infty} \frac{1}{n} \log P_{\omega} [T_{-n} < \infty] = \int \log \rho_0(\omega) \eta(d\omega),$$  \hspace{1cm} (29)

and one invokes Theorem 6, (28) and (7) to complete the proof of Proposition 1. \hfill \Box

**Proof of Theorem 6.** We will rerun the proof of Theorem 1, after showing the following.

**Lemma 4** For any $\eta \in M_1(\Sigma)^{+ \cdot \cdot}$, we have, with $\varphi(\lambda, \omega) = E_{\omega} [e^{\lambda \tau} - 1]$, $\tilde{\varphi}(\lambda, \omega) := E_{\omega} [e^{\lambda \bar{\tau} - 1}]$, $\int \log \tilde{\varphi}(\lambda, \omega) \eta(d\omega) = \int \log \varphi(\lambda, \omega) \eta(d\omega).$  \hspace{1cm} (30)

**Proof of Lemma 4.** Considering (28) and (29), (30) is equivalent to

$$\int \log \tilde{\varphi}(\lambda, \omega) \eta(d\omega) = \int \log \varphi(\lambda, \omega) \eta(d\omega) + \int \log \rho_0(\omega) \eta(d\omega)$$  \hspace{1cm} (31)

To prove (31), define $\Lambda = \{\lambda : \int \log \varphi(\lambda, \omega) \eta(d\omega) < \infty\}$. Fix $M < \infty$, consider the event $D_M := \{\tau_1 < T_M\}$, and define

$$\tilde{\varphi}^M(\lambda, \omega) = E_{\omega} [e^{\lambda \bar{\tau} - 1}; D_M]$$

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for $\lambda \in \Lambda$. Note that on $D_M$, $T_M = \tau^{-1}_- + \tau^{-1}_+ + T^0_M$ where $\tau^{-1}_- + \tau^{-1}_+$ is the first hitting time of $0$ after $\tau^{-1}_-$. This path decomposition now yields, similarly to (15), that

$$E_\omega[e^{\lambda T_M}; D_M] = E_\omega[e^{\lambda T_M}] \hat{\varphi}^M(\lambda, \omega) \varphi(\lambda, \theta^{-1}\omega),$$

(32)

implying, for $\lambda \in \Lambda$ and all $\omega$ with $P_\omega[T_M < \infty] = 1$

$$1 \geq \frac{E_\omega[e^{\lambda T_M}; D_M]}{E_\omega[e^{\lambda T_M}]} = \hat{\varphi}^M(\lambda, \omega) \varphi(\lambda, \theta^{-1}\omega),$$

(33)

and hence, for $\lambda \in \Lambda$,

$$-\log \varphi(\lambda, \theta^{-1}\omega) \geq \log \hat{\varphi}^M(\lambda, \omega),$$

implying by monotonicity also that $-\log \varphi(\lambda, \theta^{-1}\omega) \geq \log \varphi(\lambda, \omega)$. Since $\log \varphi(\lambda, \omega) \geq \log \hat{\varphi}^M(\lambda, \omega) \geq \lambda + \log(1 - \omega_0)$, it follows that both $\log \varphi(\lambda, \omega)$ and $\log \hat{\varphi}^M(\lambda, \omega)$ are integrable for $\lambda \in \Lambda$.

Next, using again path decomposition one finds that, $\eta$-a.s.,

$$E_\omega\left[e^{\lambda \tau^{-1}_-}; D_M\right] = (1 - \omega_0)e^{\lambda} + \omega_0e^{\lambda}E_\theta_\omega\left[e^{\lambda \tau^{-1}_1}; D_{M-1}\right] E_\omega\left[e^{\lambda \tau^{-1}_1}; D_M\right].$$

Hence, $\eta$-a.s.,

$$\hat{\varphi}^M(\lambda, \omega) \hat{\varphi}^{M-1}(\lambda, \theta\omega) = \frac{e^{-\lambda} \hat{\varphi}^M(\lambda, \omega)}{\omega_0} - \rho_0(\omega),$$

and similarly, by (16),

$$\rho_0(\omega) \varphi(\lambda, \omega) \varphi(\lambda, \theta^{-1}\omega) = \frac{e^{-\lambda} \varphi(\lambda, \omega)}{\omega_0} - 1.$$

Hence, $\eta$-a.s.,

$$\rho_0(\omega) \left(1 - \hat{\varphi}^M(\lambda, \omega) \varphi(\lambda, \theta^{-1}\omega)\right) \varphi(\lambda, \omega) = \rho_0(\omega) \varphi(\lambda, \omega) - \hat{\varphi}^M(\lambda, \omega) \frac{e^{-\lambda} \varphi(\lambda, \omega)}{\omega_0} + \hat{\varphi}^M(\lambda, \omega)$$

$$= \left(1 - \varphi(\lambda, \omega) \hat{\varphi}^{M-1}(\lambda, \theta\omega) \right) \hat{\varphi}^M(\lambda, \omega).$$

Therefore, $\eta$-a.s.,

$$\log \rho_0(\omega) + \log \varphi(\lambda, \omega) - \log \hat{\varphi}^M(\lambda, \omega) = \log(1 - \hat{\varphi}^{M-1}(\lambda, \theta\omega) \varphi(\lambda, \omega)) - \log(1 - \hat{\varphi}^M(\lambda, \omega) \varphi(\lambda, \theta^{-1}\omega)), $$

and averaging over $M = 2$ to $K$ and taking expectations (using stationarity and (32))! yields, for $\lambda \in \Lambda$,

$$E[\log \rho_0] + E[\log \varphi(\lambda, \omega)] - (K - 1)^{-1} \sum_{M=2}^{K} E[\log \hat{\varphi}^M(\lambda, \omega)]$$

$$= -(K - 1)^{-1} \left(E \left[ \log(1 - \hat{\varphi}^K(\lambda, \omega) \varphi(\lambda, \theta^{-1}\omega)) - \log(1 - \hat{\varphi}^M(\lambda, \omega) \varphi(\lambda, \theta^{-1}\omega)) \right]\right)$$

$$= -(K - 1)^{-1} \left(E \left[ \log \frac{E_\omega[e^{\lambda T_K}; D^\perp_K]}{E_\omega[e^{\lambda T_K}]} \right] + \text{const} \right).$$

(34)

But, using again the Markov property and stationarity of $\eta$,

$$(K - 1)^{-1} E \left[ \log E_\omega[e^{\lambda T_K}; D^\perp_K] \right] = (K - 1)^{-1} \sum_{M=1}^{K} E \left[ \log E_\omega[e^{\lambda \tau_1} \mathbb{1}_{\tau_M > \tau_1}] \right] \xrightarrow[K \rightarrow \infty]{} E[\log \varphi(\lambda, \omega)],$$

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due to monotone convergence, implying that the right hand side of (34) vanishes for $K \to \infty$. Substituting in (34) and using monotone convergence again, we get (31) for $\lambda \in \Lambda$. To get that the left hand side of (31) is $+\infty$ for $\lambda \in \Lambda^c$, assume otherwise, and reverse the role of $\bar{\varphi}$ and $\varphi$ in the above proof, while replacing $D_M$ by $\bar{D}_M = \{ \tau_1 < T_{-M} < \infty \}$. 

To complete the proof of Theorem 6, note that all that is needed in order to mimic the argument given in the proof of Theorem 1 is the almost sure convergence of $n^{-1} \sum_{i=1}^{n} \log \bar{\varphi}(\lambda, \theta^{-1} \omega) \to E_\eta [\log \bar{\varphi}(\lambda, \omega)]$, which is ensured by the ergodicity of $\eta$.

Remarks:
1. In the recurrent case, $\{ \tau_n \}$ has the same law as $\{ \tau_{-n} \}$.
2. Lemma 4 implies, by differentiating (30) at zero, that the cumulants of $\tau_1$ and $\tau_{-1}$ have the same expectation under $\eta$. In particular, $E_\eta[\tau_{-1}] = E_\eta[\tau_1]$. We note that Lemma 4 resembles results of [5], although we do not see a direct relation between the two.

For future reference, we note some easy properties of the rate function $I_\eta^{r,q}(\tau)$. We introduce the measurable function

$$ f(\lambda, \omega) := \log E_\omega [e^{\lambda \tau_1} \mathbf{1}_{\tau_1 < \infty}] = \log \varphi(\lambda, \omega), \quad (35) $$

and set

$$ G(\lambda, \eta, u) = \lambda u - \int f(\lambda, \omega) \eta(d\omega), $$

With $\tau_\omega = E_\omega[\tau_1 | \tau_1 < \infty]$, let $M_u := \{ \eta \in M^\tau(\Sigma)^+K : E_\eta[\tau_\omega] \geq u \}$. Then, for $\eta \in M_u$, one has by Lemma 1 (for $\eta \in M^\tau(\Sigma)^+K \cap M_u$) and Proposition 1 (for $\eta \in M_u \setminus M^\tau(\Sigma)^+K$) that

$$ I_\eta^{r,q}(u) = \sup_{\lambda \in \mathbb{R}} \left[ \lambda u - \int f(\lambda, \omega) \eta(d\omega) \right] = \sup_{\lambda \geq 0} \left[ \lambda u - \int f(\lambda, \omega) \eta(d\omega) \right]. \quad (36) $$

Similarly, let $M_u^- := \{ \eta \in M^\tau(\Sigma)^-K : E_\eta[\tau_\omega] \leq u \}$. Then, for $\eta \in M_u^-$, one has by Lemma 2 (for $\eta \in M^-\tau(\Sigma)^+K \cap M_u^-$) and Proposition 1 (for $\eta \in M_u^- \setminus M^\tau(\Sigma)^+K$) that

$$ I_\eta^{r,q}(u) = \sup_{\lambda \in \mathbb{R}} \left[ \lambda u - \int f(\lambda, \omega) \eta(d\omega) \right] = \sup_{\lambda \geq 0} \left[ \lambda u - \int f(\lambda, \omega) \eta(d\omega) \right]. \quad (37) $$

Next, if $\eta \in M_u^-$ then, by Jensen’s inequality,

$$ \sup_{\lambda \leq 0} G(\lambda, \eta, u) \leq \sup_{\lambda \leq 0} \left[ \lambda u - \lambda \int E_\omega[\tau_1 | \tau_1 < \infty] \eta(d\omega) \right] = - \int \log P_\omega[\tau_1 < \infty] \eta(d\omega) $$

where the last equality is due to the fact that the last supremum is achieved at $\lambda = 0$. On the other hand, the substitution $\lambda = 0$ in the above reveals that

$$ \sup_{\lambda \leq 0} G(\lambda, \eta, u) \geq - \int \log P_\omega[\tau_1 < \infty] \eta(d\omega). $$

Hence, due to (29),

$$ \sup_{\lambda \leq 0} G(\lambda, \eta, u) = \int \log \rho_0(\omega) \eta_0(\omega) \lor 0, \quad (38) $$

Similarly, if $\eta \in M_u$ then

$$ \sup_{\lambda \geq 0} G(\lambda, \eta, u) = \int \log \rho_0(\omega) \eta_0(\omega) \lor 0. \quad (39) $$
We also note that the rate function $I^\eta_q(\cdot)$ is convex, with minimum value $\int \log \rho_0(\omega) \eta(d\omega) \vee 0$ achieved at $E_\eta[\eta]$. Hence, for all $\eta \in M^\times_1(\Sigma)^K$, 

$$\sup_{\lambda \leq 0} G(\lambda, \eta, u) = \inf_{w \leq u} I^\eta_q(w), \quad (40)$$

and 

$$\sup_{\lambda \geq 0} G(\lambda, \eta, u) = \inf_{w \geq u} I^\eta_q(w). \quad (41)$$

We conclude this section with some properties of $\varphi$ in the particular case that $\eta$ is locally equivalent to the product of its marginals. These properties are useful in the study of the annealed case.

**Lemma 5** Let $\eta \in M^\times_1(\Sigma)^+K$ be locally equivalent to the product of its marginals. Then,

(i) If $\rho_{\text{max}} < 1$, then $\lambda_{\text{crit}} = \bar{\lambda} := -\frac{1}{2} \log(4\omega_{\min}(1 - \omega_{\min})) > 0$ and $\varphi(\lambda, \omega) = E_\omega[e^{\lambda\eta}] < \infty$ iff $\lambda \leq \lambda_{\text{crit}}$.

Further, $u_{\text{crit}} := \int \frac{d}{d\lambda} \log \varphi(\lambda, \omega) \bigg|_{\lambda = \lambda_{\text{crit}}} \eta(d\omega) < \infty$ unless $\eta$ is degenerate, i.e., unless $\omega = \text{const} \ \eta$-a.s.

(ii) If $\rho_{\text{max}} \geq 1$, we have $\lambda_{\text{crit}} = 0$.

Note that without the condition of local equivalence to the product of marginals, one can have $\lambda_{\text{crit}} > \bar{\lambda}$, c.f. the example in the remark following Lemma 2.

The next lemma is needed in the proof of Lemma 5. It can also be used to show that, if $\rho_{\text{max}} \leq 1$ and $\eta(\omega_0 \neq 1/2) > 0$, the random walk has a positive speed $v_\eta > 0$.

**Lemma 6** Let $(b_1(\omega), b_2(\omega), b_3(\omega), \ldots)$ be a stationary, ergodic sequence with $0 < b_1(\omega) \leq 1$, $\eta$ - a.s., and $E_\eta[b_1(\omega)] < 1$. Then we have $E_\eta[\sum_{n=1}^{\infty} b_1 \cdots b_n] < \infty$.

**Proof:** Fix $\gamma$ such that $0 < \gamma < 1$ and $\eta(b_1 \leq \gamma) > 0$. Let $t_0 := 0$, $t_1 := \inf\{n \geq 1 : b_n \leq \gamma\}$ and $t_{k+1} := \inf\{n > t_k : b_n \leq \gamma\}$. Due to our assumption on $\gamma$, the ergodic theorem implies that $E_\eta[t_1] < \infty$. Clearly, $b_1 \cdots b_k \leq \gamma^k$ and therefore

$$\sum_{n=1}^{\infty} b_1 \cdots b_n \leq \sum_{k=1}^{\infty} \gamma^k(t_k - t_{k-1}). \quad (42)$$

But $E_\eta[t_k - t_{k-1}] = E_\eta[t_1]$ due to stationarity, and taking expectations in (42) yields

$$E_\eta \left[ \sum_{n=1}^{\infty} b_1 \cdots b_n \right] \leq E_\eta[t_1] \sum_{k=1}^{\infty} \gamma^k < \infty.$$

\hfill \Box

**Proof of Lemma 5** Throughout, we take $\lambda \geq 0$.

(i) Note that for $\omega_{\min} = (\cdots, \omega_{\min}, \omega_{\min}, \omega_{\min}, \cdots)$, we have (by standard coupling) that

$$E_\omega[e^{\lambda\eta}] \leq E_{\omega_{\min}}[e^{\lambda\eta}] \quad (43)$$
Let $\varphi(\lambda) := E_{\min} \mathbb{E}^{e^{\lambda \tau_1}}$. Note that, by the same recursion used to derive (16), based on (15), one knows that if $\varphi(\lambda) < \infty$ then
\[
\varphi(\lambda) = \omega_{\min} e^\lambda + e^\lambda (1 - \omega_{\min}) (\varphi(\lambda))^2,
\]
leading to
\[
\varphi(\lambda) = \frac{1 - \sqrt{1 - e^{2(\lambda - \bar{\lambda})}}}{2 e^\lambda (1 - \omega_{\min})},
\]
as long as $\lambda \leq \bar{\lambda} = -\frac{1}{2} \log(4 \omega_{\min} (1 - \omega_{\min}))$. We have to show that for $\lambda > \bar{\lambda}$, $E_{\omega} \mathbb{E}^{e^{\lambda \tau_1}} = \infty$ for $\eta$ - a.a. $\omega$. Assume $\eta(\omega_{\min}) > 0$. In a first step, we show that for each $K > 0$, there is $\bar{A} \subset \Sigma$ with $\eta(\bar{A}) > 0$ and $E_{\omega} \mathbb{E}^{e^{\lambda \tau_1}} \geq K$ for $\omega \in \bar{A}$. Let $B_M := \{ \omega : \omega_0 = \omega_1 = \omega_2 = \ldots = \omega_{-M} = \omega_{\min} \}$. If $\eta$ is a product measure, $\eta(B_M) = (\eta(\omega_{\min}))^M > 0$. If $\eta$ is not a product measure, our assumption on $\eta$ implies that $\eta(B_M) > 0$ also, since $B_M$ depends only on $\omega_0, \omega_1, \ldots, \omega_{-M}$. For $\omega \in B_M$, we have, using a coupling argument, that
\[
E_{\omega} \mathbb{E}^{e^{\lambda \tau_1}} \geq E_{\min} \mathbb{E}^{e^{\lambda \tau_1} 1_{\{\min_X X_n \geq -M\}}} \geq K
\]
for $M = M(K)$ big enough, since
\[
\lim_{M \to \infty} E_{\min} \mathbb{E}^{e^{\lambda \tau_1} 1_{\{\min_X X_n \geq -M\}}} = E_{\min} \mathbb{E}^{e^{\lambda \tau_1}} = \infty.
\]
This proves the first step by taking $A_K = B_M(K)$. Let now $\omega \in A_K + 1$. If $E_{\omega} \mathbb{E}^{e^{\lambda \tau_1}} < \infty$ then
\[
E_{\omega} \mathbb{E}^{e^{\lambda \tau_1}} \geq \omega_{\min} e^\lambda + (1 - \omega_{\min}) e^\lambda K E_{\omega} \mathbb{E}^{e^{\lambda \tau_1}}
\]
and this is a contradiction if $1 - \omega_{\min} e^\lambda K > 1$. If $\eta(\omega_{\min}) = 0$, one has to approximate.
In order to show $u_{\mathrm{crit}} < \infty$, it is enough to prove that $\int E_{\omega} \mathbb{E}^{(\lambda_{\mathrm{crit}} \tau_1)} \eta(d\omega) < \infty$. Let $\psi(\lambda, \omega, C) := E_{\omega} \mathbb{E}^{(\lambda \wedge C) \tau_1}$. The same recursion as in (15) yields, for $\lambda \leq \lambda_{\mathrm{crit}}$,
\[
\psi(\lambda, \omega, C) \leq \omega_0 e^\lambda + (1 - \omega_0) e^\lambda \varphi(\lambda, \theta^{-1} \omega) \psi(\lambda, \omega)
\]
\[
+ (1 - \omega_0) e^\lambda \psi(\lambda, \theta^{-1} \omega, C) \varphi(\lambda, \omega) + (1 - \omega_0) e^\lambda \psi(\lambda, \omega, C) \varphi(\lambda, \theta^{-1} \omega)
\]
Note that for $\lambda \leq \lambda_{\mathrm{crit}}$, $\varphi(\lambda, \omega) \leq \varphi(\lambda)$ for all $\omega$. This implies
\[
\psi(\lambda, \omega, C) \leq a(\lambda, \omega_0) + b(\lambda, \omega_0) \psi(\lambda, \theta^{-1} \omega, C)
\]
where
\[
a(\lambda, \omega) = \frac{\omega_0 e^\lambda + (1 - \omega_0) e^\lambda (\varphi(\lambda))^2}{1 - (1 - \omega_0) e^\lambda \varphi(\lambda)}
\]
and
\[
b(\lambda, \omega_0) = \frac{(1 - \omega_0) e^\lambda \varphi(\lambda)}{1 - (1 - \omega_0) e^\lambda \varphi(\lambda)}.
\]
Iteration of (45) yields, taking $\lambda = \lambda_{\mathrm{crit}}$,
\[
\psi(\lambda_{\mathrm{crit}}, \omega_0, C) \leq a(\lambda_{\mathrm{crit}}, \omega_0) + \sum_{j=0}^{\infty} b(\lambda_{\mathrm{crit}}, \omega_0) \cdots b(\lambda_{\mathrm{crit}}, \omega_{-j}) a(\lambda_{\mathrm{crit}}, \omega_{j-1})
\]
But we know that $a(\lambda_{\mathrm{crit}}, \cdot)$ is bounded, because, using the value $\varphi(\lambda_{\mathrm{crit}}) = (\frac{\omega_0}{1 - \omega_{\min}})^{1/2}$, we have from (46) that
\[
a(\lambda_{\mathrm{crit}}, \omega_0) = \frac{\omega_0 e^{\lambda_{\mathrm{crit}}} + (1 - \omega_0) e^{\lambda_{\mathrm{crit}}} (\varphi(\lambda_{\mathrm{crit}}))^2}{1 - (1 - \omega_0) e^{\lambda_{\mathrm{crit}}} (\varphi(\lambda_{\mathrm{crit}}))^2} \leq \frac{e^{\lambda_{\mathrm{crit}}} + e^{\lambda_{\mathrm{crit}}} (\varphi(\lambda_{\mathrm{crit}}))^2}{1/2} < \infty.
\]
Further, using the same substitution,

\[ b(\lambda_{\text{crit}}, \omega_0) = \frac{1 - \omega_0}{1 - \omega_0 + 2(\omega_0 - \omega_{\text{min}})}. \]  

(49)

In particular, \(0 < b(\lambda_{\text{crit}}, \omega_0) \leq 1\) and, if \(\eta\) is not degenerate, \(E_0[b(\lambda_{\text{crit}}, \omega_0)] < 1\). Lemma 6 now enables us to integrate (48) with \(\eta\) and we see that \(\int \psi(\lambda_{\text{crit}}, \omega, C) \eta(d\omega)\) is bounded uniformly in \(C\).

Note that \(\omega_{\text{crit}} = \infty\) in the degenerate case since

\[ \lim_{\lambda \to \lambda_{\text{crit}}} \frac{d}{d\lambda} \log \varphi(\lambda, \omega_{\text{min}}) = \infty. \]

(ii) We use the same argument as in the proof of (i). Assume \(\eta_0(\omega_0 \leq 1/2) > 0\). Let \(\lambda > 0\). In a first step, we show that for each \(K > 0\), there is \(A_K \subset \Sigma\) with \(\eta(A_K) > 0\) and \(E_\omega[e^{\lambda \tau_1}] \geq K\) for \(\omega \in A_K\). Let \(B_M := \{\omega : \omega_0 \leq 1/2, \omega_{-1} \leq 1/2, \omega_{-2} \leq 1/2, \ldots, \omega_{-M} \leq 1/2\}\). If \(\eta\) is a product measure, \(\eta(B_M) = (\eta_0(\omega_{\text{min}}))^M > 0\). If \(\eta\) is not a product measure, our assumption on \(\eta\) implies that \(\eta(B_M) > 0\) also, since \(B_M\) depends only on \(\omega_0, \omega_{-1}, \ldots, \omega_{-M}\). For \(\omega \in B_M\), we have

\[ E_\omega[e^{\lambda \tau_1}] \geq E_{\omega_0, \omega_1, \omega_2, \ldots} [e^{\lambda \tau_1}]_{\omega_0 \in B_M} \geq K \]

for \(M = M(K)\) big enough, since

\[ \lim_{M \to \infty} E_{\omega_0, \omega_1, \omega_2, \ldots} [e^{\lambda \tau_1}]_{\omega_0 \in B_M} = E_{\omega_0, \omega_1, \omega_2, \ldots} [e^{\lambda \tau_1}] = \infty. \]

This proves the first step by taking \(A_K = B_M(K)\). Let now \(\omega \in A_{M+1}\). If \(E_\omega[e^{\lambda \tau_1}] < \infty\) then

\[ E_\omega[e^{\lambda \tau_1}] \geq \frac{1}{2} e^\lambda + (1 - \frac{1}{2}) e^\lambda K E_\omega[e^{\lambda \tau_1}] \]

and this is a contradiction if \(\frac{1}{2} e^\lambda K > 1\). If \(\eta_0(\omega_0 \leq 1/2) = 0\), one has to approximate. \(\square\)

**Remark** An inspection of the proof reveals that part i) of Lemma 5 still holds for any \(\eta \in M^*\Sigma^{+;K}\) satisfying \(\eta(\omega \in [\omega_{\text{min}}, \omega_{\text{min}} + \epsilon])_{\epsilon > 0} > 0\) for all \(\epsilon > 0\), where \(K_0 = K_0(\omega_{\text{min}}) := [2(\omega_{\text{min}}/(1 - \omega_{\text{min}}))^{1/2}]\).

### 3 Proofs - Annealed LDLP’s for Hitting Times.

Recall the notation \(f(\lambda, \omega) := \log E_\omega[e^{\lambda \tau_1} \eta_{\tau_1 < \infty}] = \log \varphi(\lambda, \omega)\). In what follows, \(\omega_{\text{min}}, \rho_{\text{max}}, \eta_{\text{max}}, \) etc. are always defined in terms of \(\alpha\), whereas if \(\alpha \in M^*\Sigma^{+;K}\) then \(\lambda_{\text{crit}}\) is defined as in Lemma 2, while if \(\alpha \in M^*\Sigma^{+;K} \setminus M^*\Sigma^{+;K}\) then \(\lambda_{\text{crit}} := \lambda_{\text{crit}}(\alpha_{\text{max}})\). Also, unless denoted otherwise, expectations are taken with respect to \(\alpha\) or \(P_\alpha\). We recall that \(M^*\Sigma\) is equipped with the topology of weak convergence, and define the compact set

\[ D_\alpha := \{\mu \in M^*\Sigma^K : \sup \mu_0 \subseteq \sup \alpha_0\}. \]

**Lemma 7** Assume \(\alpha \in M^*\Sigma^K\) satisfies Assumption (A) and is non-degenerate. Then, the function \((\mu, \lambda) \to \int f(\lambda, \omega) \mu(d\omega)\) is continuous on \(D_\alpha \times (-\infty, \lambda_{\text{crit}}]\).

**Proof of Lemma 7**
For $\kappa > 1$, decompose $\varphi(\lambda, \omega)$ as follows:
\[
E_\omega[e^{\lambda \tau_1} \mathbf{1}_{\tau_1 < \infty}] = E_\omega[e^{\lambda \tau_1}; \tau_1 < \kappa] + E_\omega[e^{\lambda \tau_1}; \infty > \tau_1 \geq \kappa] := \varphi_1^\kappa(\lambda, \omega) + \varphi_2^\kappa(\lambda, \omega),
\]
where $(\omega, \lambda) \rightarrow \varphi_1^\kappa(\lambda, \omega)$ is bounded and continuous. We also have
\[
0 \leq \log \left(1 + \frac{\varphi_2^\kappa(\lambda, \omega)}{\varphi_1^\kappa(\lambda, \omega)}\right) \leq \log \left(1 + \frac{\varphi_2^\kappa(\lambda_{\text{crit}}, \omega)}{\omega_{\text{min}} e^\lambda}\right).
\]
Hence, the required continuity of the function $(\lambda, \mu) \rightarrow \int f(\lambda, \omega) \mu(d\omega)$ will follow from (50) as soon as we show that for any fixed constant $C_1 < 1$,
\[
\lim_{\kappa \rightarrow \infty} \sup_{\mu \in \mathcal{D}_\alpha} \int \mu(d\omega) \log \left(1 + \frac{\varphi_2^\kappa(\lambda_{\text{crit}}, \omega)}{\omega_{\text{min}} C_1}\right) = 0. \tag{51}
\]
If $\alpha \in M^{1/2}$ (recall (1)), then $\lambda_{\text{crit}} = 0$ and then one finds for each $\epsilon > 0$ a $\kappa_{\mu} = \kappa(\epsilon, \mu)$ large enough such that,
\[
E_\mu \left[\log \left(1 + \frac{\rho_0(\omega) [\infty > \tau_1 > \kappa_{\mu}]}{\omega_{\text{min}} [\tau_1 < \infty]}\right)\right] < \epsilon.
\]
Further, in this situation, for ergodic $\mu$, c.f. (29),
\[
\int f(0, \omega) \mu(d\omega) = \left(-\int \log \rho_0(\omega) \mu(d\omega)\right) \wedge 0. \tag{52}
\]
In particular, $\mu \rightarrow \int f(0, \omega) \mu(d\omega)$, being linear, is uniformly continuous on the compact set $\mathcal{D}_\alpha$. Therefore, using (50), one sees that for each $\mu \in \mathcal{D}_\alpha$ one can construct a neighborhood $B_\mu$ of $\mu$ such that, for each $\nu \in B_\mu \cap \mathcal{D}_\alpha$,
\[
E_\nu \left[\log \left(1 + \frac{\rho_0(\omega) [\infty > \tau_1 > \kappa_{\mu} + 1]}{\omega_{\text{min}} [\tau_1 < \infty]}\right)\right] < \epsilon.
\]
By compactness, it follows that there exists an $\kappa = \kappa(\epsilon)$ large enough such that, for all $\mu \in \mathcal{D}_\alpha$,
\[
E_\mu \left[\log \left(1 + \frac{\rho_0(\omega) [\infty > \tau_1 > \kappa]}{\omega_{\text{min}} [\tau_1 < \infty]}\right)\right] < \epsilon.
\]
Using the inequality $\log(1 + cx) \leq c \log(1 + x)$, valid for $x \geq 0$, $c \geq 1$, one finds that for $\kappa$ large enough,
\[
\sup_{\mu \in \mathcal{D}_\alpha} \int \mu(d\omega) \log \left(1 + \frac{\varphi_2(0, \omega)}{C_1}\right) \leq \epsilon/C_1,
\]
proving (51) for $\alpha \in M^{1/2}$.

The case $\alpha \not\in M^{1/2}$ is simpler: suppose $\omega_{\text{min}} > 1/2$. Then, with $\lambda \geq 0$, because $\text{supp} \mu_0 \subseteq \text{supp} \alpha_0$,
\[
E_\omega[e^{\lambda \tau_1} \mathbf{1}_{\tau_1 < \infty}] \leq E_{\underline{\omega}_{\text{min}}}[e^{\lambda \tau_1} \mathbf{1}_{\tau_1 < \infty}] < \infty
\]
for $\mu$-a.e. $\omega$, where $\underline{\omega}_{\text{min}} = (\cdots, \omega_{\text{min}}, \omega_{\text{min}}, \omega_{\text{min}}, \cdots)$, and the last inequality is due to Lemma 5. On the other hand, we have $f(\lambda, \omega) \geq \lambda + \log \omega_0$. We show that $(\lambda, \omega) \rightarrow \varphi(\lambda, \omega)$ is continuous, which is enough to complete the proof of the lemma. Write as before
\[
E_\omega[e^{\lambda \tau_1} \mathbf{1}_{\tau_1 < \infty}] = E_\omega[e^{\lambda \tau_1}; \tau_1 < \kappa] + E_\omega[e^{\lambda \tau_1}; \infty > \tau_1 \geq \kappa] \tag{53}
\]
and observe that the first term in (53) is continuous as a function of $\omega$ and the second term goes to 0 for $\kappa \to \infty$, uniformly in $\omega$. More precisely,

$$E_\omega[e^{\lambda \tau_1; \infty \geq \tau_1 \geq \kappa}] \leq E_{\overline{\omega}_{\max}}[e^{\lambda \tau_1; \tau_1 \geq \kappa}]$$

where $E_{\overline{\omega}_{\max}}[e^{\lambda \omega \tau_1}] < \infty$ and therefore $P_{\overline{\omega}_{\max}}[\tau_1 \geq \kappa] \to \alpha_{\text{inv}}$ due to the transience of the random walk under the measure $\eta = \delta_{\overline{\omega}_{\max}}$. If $\omega_{\max} < 1/2$, apply the same arguments for $\alpha_{\text{inv}}$. □

**Proof of Theorem 2** We begin by proving an upper bound for $\frac{1}{n} \log P \left[ \frac{1}{n} \sum_{j=1}^{n} \tau_j \leq u \right]$, where $1 < u < \infty$. We have, for $\lambda \leq 0$,

$$P \left[ \frac{1}{n} \sum_{j=1}^{n} \tau_j \leq u \right] \leq E \left[ \exp \left( \lambda \sum_{j=1}^{n} \tau_j \right) \mathbf{1}_{\tau_j < \infty, j=1,\ldots,n} \right] e^{-\lambda nu}$$

(54)

But,

$$E \left[ \exp \left( \lambda \sum_{j=1}^{n} \tau_j \right) \mathbf{1}_{\tau_j < \infty, j=1,\ldots,n} \right] = E \prod_{j=1}^{n} E_{\omega} \left[ e^{\lambda \tau_j} \mathbf{1}_{\tau_j < \infty} \right] = E \left[ \exp \left( \sum_{j=1}^{n} \log E_{\omega} \left[ e^{\lambda \tau_j} \mathbf{1}_{\tau_j < \infty} \right] \right) \right] = E \left[ \exp \left( \sum_{j=0}^{n-1} f(\lambda, \theta^j \omega) \right) \right] = E \left[ \exp \left( n \int f(\lambda, \omega) R_n(d\omega) \right) \right]$$

where $R_n = \frac{1}{n} \sum_{j=0}^{n-1} \delta_{\theta^j \omega} \in M_1(\Sigma)$ denotes the empirical field.

By assumption, the distributions of $R_n$ satisfy a LDP with rate function $h(\cdot|\alpha)$. Lemma 7 ensures that we can apply Varadhan’s lemma (see [4, Lemma 4.3.6]) to get

$$\limsup_{n \to \infty} \frac{1}{n} \log E \left[ \exp \left( n \int f(\lambda, \omega) R_n(d\omega) \right) \right] \leq \sup_{\eta \in M_1^+(\Sigma)} \left[ \int f(\lambda, \omega) \eta(d\omega) - h(\eta|\alpha) \right].$$

(55)

Going back to (54), this yields the upper bound

$$\limsup_{n \to \infty} \frac{1}{n} \log P \left[ \frac{1}{n} \sum_{j=1}^{n} \tau_j \leq u \right] \leq \inf_{\lambda \leq 0} \sup_{\eta \in M_1^+(\Sigma)} \left[ \int f(\lambda, \omega) \eta(d\omega) - h(\eta|\alpha) \right]$$

$$= -\sup_{\lambda \leq 0} \inf_{\eta \in M_1^+(\Sigma)} \left[ \lambda u - \int f(\lambda, \omega) \eta(d\omega) + h(\eta|\alpha) \right].$$

(56)

Since $\mu \to -\int f(\lambda, \omega) \mu(d\omega) + h(\mu|\alpha)$ is lower semi-continuous and $M_1(\Sigma)$ is compact, the infimum in (56) is achieved for each $\lambda$, on measures with support of their marginal included in $K$, for otherwise $h(\eta|\alpha) = \infty$. Further, by (14), the supremum over $\lambda$ can be taken over a compact set (recall that $\infty > u > 1$). Hence, by the Minimax Theorem (see [4, Pg. 151] for Sion’s version), the min-max is equal to the max-min in (56).
Further, since taking first the supremum in in the right hand side of (56) yields a lower semicontinuous function, an achieving \( \tilde{\eta} \) exists, and then, due to compactness, there exists actually an achieving pair \( \lambda, \tilde{\eta} \). We will show that

\[
\inf_{\eta \in M^+_1(\Sigma)^{\kappa}} \sup_{\lambda \leq 0} \left( \left[ \lambda u - \int f(\lambda, \omega)\eta(d\omega) \right] + h(\eta|\alpha) \right) = \inf_{\eta \in M^+_1(\Sigma)^{\kappa}} \sup_{\lambda \leq 0} \left( \left[ \lambda u - \int f(\lambda, \omega)\eta(d\omega) \right] + h(\eta|\alpha) \right)
\]

(57)

Then,

\[
(56) = - \inf_{\eta \in M^+_1(\Sigma)^{\kappa}} \sup_{\lambda \leq 0} \left( \left[ \lambda u - \int f(\lambda, \omega)\eta(d\omega) \right] + h(\eta|\alpha) \right) = - \inf_{\eta \in M^+_1(\Sigma)^{\kappa}} \inf_{w \leq u} \left[ I^*_\eta q(w) + h(\eta|\alpha) \right],
\]

(58)

where the second equality is due to (40). Hence,

\[
\limsup_{n \to \infty} \frac{1}{n} \log P \left[ \frac{1}{n} \sum_{j=1}^{n} \tau_j \leq u \right] \leq - \inf_{w \leq u} \inf_{\eta \in M^+_1(\Sigma)^{\kappa}} \left[ I^*_\eta q(w) + h(\eta|\alpha) \right] = - \inf_{w \leq u} I^*_\alpha (w).
\]

(59)

Turning to the proof of (57), we have, due to Assumption (A), a sequence of ergodic measures with \( \eta^n \to \tilde{\eta} \) and \( h(\eta^n|\alpha) \to h(\tilde{\eta}|\alpha) \). Let \( \lambda_n \) be the maximizers in (57) corresponding to \( \eta^n \). We have

\[
\inf_{\eta \in M^+_1(\Sigma)^{\kappa}} \sup_{\lambda \leq 0} \left( \left[ \lambda u - \int f(\lambda, \omega)\eta(d\omega) \right] + h(\eta|\alpha) \right) \leq \left[ \lambda_n u - \int f(\lambda_n, \omega)\eta^n(d\omega) \right] + h(\eta^n|\alpha)
\]

(60)

W.l.o.g. we can assume, by taking a subsequence, that \( \lambda_n \to \lambda^* \leq 0 \). Using the joint continuity in Lemma 7, we have, for \( \epsilon > 0 \) and \( n \geq N_0(\epsilon) \),

\[
\lambda_n u - \int f(\lambda_n, \omega)\eta^n(d\omega) + h(\eta^n|\alpha) \leq \left[ \lambda^* u - \int f(\lambda^*, \omega)\tilde{\eta}(d\omega) \right] + h(\tilde{\eta}|\alpha) + \epsilon
\]

\[
\leq \inf_{\eta \in M^+_1(\Sigma)^{\kappa}} \sup_{\lambda \leq 0} \left( \left[ \lambda u - \int f(\lambda, \omega)\eta(d\omega) \right] + h(\eta|\alpha) \right) + \epsilon
\]

But this shows the equality in (57), since the reverse inequality there is trivial. This completes the proof of the upper bound for the lower tail (the case \( u = 1 \) being handled directly by noting that \( n^{-1} \sum_{j=1}^{n} \tau_j \leq 1 \) implies that \( \tau_j = 1, j = 1, \ldots, n \).

We next turn our attention to the upper bound for the upper tail, that is to \( \frac{1}{n} \log P \left[ \infty > \frac{1}{n} \sum_{j=1}^{n} \tau_j \geq u \right] \), where \( 1 < u < \infty \). We have, for \( \lambda \geq 0 \),

\[
P \left[ \infty > \frac{1}{n} \sum_{j=1}^{n} \tau_j \geq u \right] \leq E \left[ \exp \left( \lambda \sum_{j=1}^{n} \tau_j \right) 1_{\tau_j < \infty, j=1, \ldots, n} \right] e^{-\lambda nu}
\]

(61)

But

\[
E \left[ \exp \left( \lambda \sum_{j=1}^{n} \tau_j \right) 1_{\tau_j < \infty, j=1, \ldots, n} \right] = E \left[ \prod_{j=1}^{n} E_\omega \left[ e^{\lambda \tau_j} 1_{\tau_j < \infty} \right] \right]
\]

\[
= E \left[ \exp \left( \sum_{j=1}^{n} \log E_\omega \left[ e^{\lambda \tau_j} 1_{\tau_j < \infty} \right] \right) \right]
\]

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\[ \begin{align*}
&= E \left[ \exp \left( \sum_{j=0}^{n-1} f(\lambda, \theta^j \omega) \right) \right] \\
&= E \left[ \exp \left( n \int f(\lambda, \omega) R_n(d\omega) \right) \right].
\end{align*} \]

Lemma 7 now ensures that we can apply Varadhan’s lemma (see [4, Lemma 4.3.6]) to get

\[ \limsup_{n \to \infty} \frac{1}{n} \log E \left[ \exp \left( n \int f(\lambda, \omega) R_n(d\omega) \right) \right] \leq \sup_{\eta \in M_1^2(\Sigma)} \left[ \int f(\lambda, \omega) \eta(d\omega) - \lambda \mu \right]. \quad (62) \]

(The RHS in (62) is +∞ if \( \lambda > \lambda_{\text{crit}}(\alpha) \), simply by choosing \( \eta = \alpha \)). Going back to (61), this yields the upper bound

\[ \limsup_{n \to \infty} \frac{1}{n} \log P \left\{ \infty > \frac{1}{n} \sum_{j=1}^{n-1} \tau_j \geq u \right\} \leq \inf_{\lambda \geq 0} \sup_{\eta \in M_1^2(\Sigma)} \left[ \int f(\lambda, \omega) \eta(d\omega) - \lambda u \right], \]

\[ = -\sup_{\lambda \geq 0} \inf_{\eta \in M_1^2(\Sigma)} \left[ \lambda u - \int f(\lambda, \omega) \eta(d\omega) + \lambda \mu \right]. \quad (63) \]

Since \( \mu = -\int f(\lambda, \omega) \mu(d\omega) + \mu(\mu|\alpha) \) is lower semi-continuous and \( M_1^2(\Sigma) \) is compact, the infimum in (63) is achieved for each \( \lambda \), on measures with support of their marginal included in \( K \), for otherwise \( h(\eta|\alpha) = \infty \). Since (as can be checked using \( \eta = \alpha \)), the supremum over \( \lambda \) can be taken over the compact set \( [0, \lambda_{\text{crit}}] \) which depends only on \( \alpha \), there exists a pair \( \bar{\lambda}, \bar{\eta} \) which achieves the infimum and the supremum in (63). The Minimax Theorem (see [4, Pg. 151]) implies that the infimum and the supremum in (63) can be exchanged. Exactly as we showed (57), we prove that

\[ \inf_{\eta \in M_1^2(\Sigma)} \sup_{\lambda \geq 0} \left( \left[ \lambda u - \int f(\lambda, \omega) \eta(d\omega) \right] + h(\eta|\alpha) \right) = \inf_{\eta \in M_1^2(\Sigma)} \sup_{\lambda \geq 0} \left( \left[ \lambda u - \int f(\lambda, \omega) \eta(d\omega) \right] + h(\eta|\alpha) \right) \quad (64) \]

Then,

\[ (63) = -\inf_{\eta \in M_1^2(\Sigma)} \sup_{\lambda \geq 0} \left( \left[ \lambda u - \int f(\lambda, \omega) \eta(d\omega) \right] + h(\eta|\alpha) \right) = -\inf_{\eta \in M_1^2(\Sigma)} \inf_{\mu \geq u} \left[ I^\alpha_{\eta}(w) + h(\eta|\alpha) \right], \quad (65) \]

where the last equality is due to (41). Hence,

\[ \limsup_{n \to \infty} \frac{1}{n} \log P \left\{ \infty > \frac{1}{n} \sum_{j=1}^{n-1} \tau_j \geq u \right\} \leq -\inf_{\mu \geq u} \inf_{\eta \in M_1^2(\Sigma)} \left[ I^\alpha_{\eta}(w) + h(\eta|\alpha) \right] = -\inf_{\mu \geq u} I^\alpha(\mu). \quad (66) \]

This completes the proof of the upper bound for the upper tail. Since we show below that \( I^\alpha_{\eta}(\cdot) \) is convex, the upper bound in Theorem 2 is established.

**Proof of the lower bounds.**

We will use the following standard argument.

**Lemma 8** Let \( P \) be a probability distribution, \((F_n)\) be an increasing sequence of \( \sigma\)-fields and \( A_n \) be \( F_n\)-measurable sets, \( n = 1, 2, 3, \ldots \). Let \( (Q_n) \) be a sequence of probability distributions such that \( Q_n[A_n] \to 1 \) and

\[ \limsup_{n \to \infty} \frac{1}{n} H(Q_n|P) \big|_{F_n} \leq h \]

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where \(H(\cdot|P)\big|_{\mathcal{F}_n}\) denotes the relative entropy w.r.t. \(P\) on the \(\sigma\)-field \(\mathcal{F}_n\) and \(h\) is a positive number. Then we have

\[
\liminf_{n \to \infty} \frac{1}{n} \log P[A_n] \geq -h.
\]

**Proof of Lemma 8.** From the basic entropy inequality ([6], p. 423)

\[
Q_n[A_n] \leq \frac{\log 2 + H(Q_n|P)\big|_{\mathcal{F}_n}}{\log(1 + 1/P[A_n])}, \quad A_n \in \mathcal{F}_n
\]

we have \(-Q_n[A_n] \log P[A_n] \leq \log 2 + H(Q_n|P)\big|_{\mathcal{F}_n}\). Dividing by \(n\) and taking limits we obtain the desired result.

For \(\eta \in M^f(\Sigma)^+\), fix \(u + 1 < M < \infty\), define \(\tilde{Q}_\omega\) as in the proof of the lower bound of Theorem 1, and let \( \tilde{Q}_\eta = \tilde{Q}_\omega \otimes \eta(d\omega) \). Let \(A_n = \{|n^{-1} \sum_{j=1}^n \tau_j - u| < \delta\} \). We know already that

\[
\tilde{Q}_\omega[A_n^c] \xrightarrow{n \to \infty} 0, \ \eta - \text{a.s.},
\]

and this implies

\[
\tilde{Q}_\eta[A_n^c] \xrightarrow{n \to \infty} 0.
\]

Let \(\mathcal{F}_n := \sigma(\{\tau_i\}_{i=1}^n, \{\omega_j\}_{j=-M}^n), \mathcal{F}_n^\prime = \sigma(\{\omega_j\}_{j=-M}^n)\). Note that

\[
\tilde{Q}_\eta|_{\mathcal{F}_n} = \tilde{Q}_\omega|_{\mathcal{F}_n} \otimes \eta|_{\mathcal{F}_n^\prime}.
\]

Hence,

\[
H(\tilde{Q}_\eta|P)\big|_{\mathcal{F}_n} = H(\eta|\alpha)\big|_{\mathcal{F}_n^\prime} + \int H(\tilde{Q}_\omega|P_\omega)\big|_{\mathcal{F}_n} \eta(d\omega). \tag{67}
\]

Considering the second term in (67), we have

\[
\frac{1}{n} \int H(\tilde{Q}_\omega|P_\omega)\big|_{\mathcal{F}_n} \eta(d\omega) = -\frac{1}{n} \int \log Z_{n, \omega} \eta(d\omega) + \lambda_M(u) \int \frac{1}{n} \sum_{j=1}^n \tau_j d\tilde{Q}_\omega \eta(d\omega)
\]

\[
= -\frac{1}{n} \int \sum_{j=1}^n \log \tilde{\phi}_M(\lambda_M(u), \theta^j(\omega)) \eta(d\omega) + \lambda_M(u) \int \frac{1}{n} \sum_{j=1}^n \tau_j d\tilde{Q}_\omega \eta(d\omega)
\]

and we see, as in the proof of the lower bound of Theorem 1, that

\[
\frac{1}{n} \int H(\tilde{Q}_\omega|P_\omega)\big|_{\mathcal{F}_n} \eta(d\omega) \xrightarrow{n \to \infty} \lambda_M(u) - \tilde{\lambda}_M(\lambda_M(u)) \leq I_M(u) - C_M
\]

We already know that

\[
\limsup_{M \to \infty} (I_M(u) - C_M) \leq I^*_{\eta, q}(u),
\]

while, considering the first term in (67), we know that

\[
\limsup_{n \to \infty} \frac{1}{n} H(\eta|\alpha)\big|_{\mathcal{F}_n^\prime} = h(\eta|\alpha).
\]
Hence,
\[
\limsup_{M \to \infty} \limsup_{n \to \infty} \frac{1}{n} H(\tilde{Q}_n | P) \left\{ \mathcal{F}_n \right\} \leq I^*_{\eta}(u) + h(\eta | \alpha).
\]
and we can now apply the standard argument. As in the quenched case, one derives the LDP lower bound for \( \eta \in M_f^f(\Sigma)^K \setminus M_f^f(\Sigma)^{+,K} \) by repeating the above argument with the required (obvious) modifications.

Finally, we prove the convexity of \( I^*_{\alpha, \eta}(\cdot) \). Note that the function
\[
\sup_{\lambda \in \mathbb{R}} \inf_{\eta \in M_f^f(\Sigma)^K} \left\{ \lambda u - \int f(\lambda, \omega) \eta(\omega) + h(\eta | \alpha) \right\} = \sup_{\lambda \in \mathbb{R}} \inf_{\eta \in M_f^f(\Sigma)^K} \left\{ \lambda u - \int \left( -f(\lambda, \omega) \eta(\omega) + h(\eta | \alpha) \right) \right\},
\]
being a supremum over affine functions in \( u \), is clearly convex in \( u \), while one shows, exactly as in (57), that
\[
\inf_{\eta \in M_f^f(\Sigma)^K} \sup_{\lambda \in \mathbb{R}} \left\{ \lambda u - \int f(\lambda, \omega) \eta(\omega) + h(\eta | \alpha) \right\} = \inf_{\eta \in M_f^f(\Sigma)^K} \left\{ I^*_{\eta}(u) + h(\eta | \alpha) \right\} = I^*_{\alpha}(u).
\]
Recalling that supremum and infimum in (68) can be exchanged, this completes the proof of Theorem 2. \( \square \)

4 Proofs - LDP's for \( X_n \) and Functional LDP’s.

The results in this section are relatively straightforward applications of the work done previously. We thus emphasize in the proofs only the new elements which need to be introduced.

Proof of Theorem 3 By symmetry, it is enough to consider \( \eta \in M_f^f(\Sigma)^{+,K} \).

1. Note that \( I^*_\eta(0^+) = \lambda_{\text{crit}} \) by Lemma 2, (5) and (9). Further,
\[
\lim_{v \to 0^-} I^*_\eta(v) = \lim_{v \to 0^+} |v| I^*_{\eta}(\frac{1}{|v|}) = \lim_{v \to 0^+} \left[ v I^*_{\eta}(\frac{1}{v}) + v \int \log \rho(\omega) \eta_0(\omega) \right],
\]
and hence \( I^*_\eta(\cdot) \) is continuous at 0. Using the convexity of \( I^*_{\eta} \) and the fact that \( x \to xf(1/x) \) is convex if \( f \) is convex, one sees that \( I^*_\eta \) is convex on \((0,1]\) and on \([-1,0)\), separately. Finally, \( (I^*_\eta)'(0^+) = (I^*_\eta)'(0^-) = \int \log \rho_0(\omega) \eta_0(\omega) \geq (I^*_\eta)'(0^-) \), establishing the convexity of \( I^*_{\eta} \) on \([-1,1]\).

2. Let \( v > v_{\eta} \). We have
\[
P_\omega \left[ \frac{X_n}{n} \geq v \right] \leq P_\omega \left[ T_{[nv]} \leq n \right] = P_\omega \left[ \frac{1}{[nv]} \sum_{j=1}^{[nv]} \tau_j \leq \frac{n}{[nv]} \right].
\]

Theorem 1 now implies
\[
\limsup_{n \to \infty} \frac{1}{n} \log P_\omega \left[ \frac{X_n}{n} \geq v \right] \leq -v I^*_{\eta}(\frac{1}{v}).
\]
In the same way, we have for any $|v - v_\eta|/2 > \delta > 0$ and $0 < \epsilon < \delta/2$,
\[
P_\omega \left[ (v + \delta) \geq \frac{X_n}{n} \geq (v - \delta) \right] \geq P_\omega \left[ (1 - \epsilon)n \leq T_{[n \epsilon]} \leq n \right],
\]

hence, Theorem 1 implies
\[
\liminf_{n \to \infty} \frac{1}{n} \log P_\omega \left[ \frac{X_n}{n} \in (v - \delta, v + \delta) \right] \geq -v I_\eta^{r, q} \left( \frac{1 - \epsilon}{v} \right), \eta - a.e. \omega
\]

and the lower bound follows by letting $\epsilon \to 0$.

3. Assume $v_\eta > 0$. Let $0 < v < v_\eta$. We have
\[
P_\omega \left[ \frac{X_n}{n} \leq v \right] \leq P_\omega \left[ T_{[n \epsilon]} \geq n \right]
\]
and, for any $\delta/2 > \epsilon > 0$, and $\delta < |v - v_\eta|/2$,
\[
P_\omega \left[ (v - \delta) \leq \frac{X_n}{n} \leq (v + \delta) \right] \geq P_\omega \left[ n(1 + \epsilon) \geq T_{[n \epsilon]} \geq n(1 - \epsilon) \right]
\]
The conclusion follows from Theorem 1.

4. The upper bound for general subsets of $[0, 1]$ follows by noting that the rate function $I_\eta^q(\cdot)$ is convex.

5. The proof concerning deviations to the left follows the same path, replacing $T_n$ by $T_{-n}$.

Proof of Theorem 4. All the statements follow from Theorem 2 by a rerun of the derivation of Theorem 3 from Theorem 1, except for the convexity of $I_\alpha^q$. From the convexity of $I_\alpha^{r, q}$ it is clear that $P_\alpha$ is convex separately on $[-1, 0]$ and on $[0, 1]$. If $\lambda_{\text{crit}} = 0$ we have $0 \leq I_\alpha^q(0) \leq I_\alpha^q(0) = 0$, and then $I_\alpha^q$ is convex on $[-1, 1]$ in this case. It remains to consider the case $\lambda_{\text{crit}} > 0$. We will assume that $\rho_{\text{max}} < 1$, the case $\rho_{\text{min}} > 1$ being proved with the same arguments for $\alpha^{\text{inv}}$ instead of $\alpha$. Then
\[
I_\eta^{r, q}(u) \geq \lambda_{\text{crit}} u - \int \log \varphi(\lambda_{\text{crit}}, \omega) \eta(d\omega) \geq \lambda_{\text{crit}} u - \log \bar{\varphi}(\lambda_{\text{crit}}),
\]
where $\bar{\varphi}(\lambda_{\text{crit}}) = E_{\eta(\alpha)}[e^{\lambda_{\text{crit}} \cdot}] < \infty$ for any $\eta$ with $h(\eta|\alpha) < \infty$ (and in particular, $\rho_{\text{max}}(\eta) < 1$, as in the proof of Lemma 5. Hence,
\[
P_\alpha^q(0) = \lim_{u \to \infty} u^{-1} I_\alpha^{r, q}(u) \geq \lambda_{\text{crit}}.
\]
Since we already know that $I_\alpha^q(0) \leq I_\alpha^q(0) = \lambda_{\text{crit}}$, we conclude that $I_\alpha^q(0) = \lambda_{\text{crit}}$. Due to separate convexity it is enough, in order to prove convexity of $I_\alpha^q$ on $[-1, 1]$, to show that for $v > 0$
\[
I_\alpha^q(v) + I_\alpha^q(-v) \geq 2I_\alpha^q(0)
\]
(70)
since it would imply that $I_\alpha^q(0^-) \leq I_\alpha^q(0^+)$. But
\[
I_\alpha^q(v) = v I_\alpha^{r, q}\left(\frac{1}{v}\right)
\]
\[
= v \inf_{\eta \in M_{[1]}(\Sigma)^{\kappa}} \left[ I_\eta^{r, q}\left(\frac{1}{v}\right) + h(\eta|\alpha) \right]
\]
\[
\geq v \inf_{\eta \in M_{[1]}(\Sigma)^{\kappa}} \left[ \lambda_{\text{crit}} \frac{1}{v} - \int \log \varphi(\lambda_{\text{crit}}, \omega) \eta(d\omega) + h(\eta|\alpha) \right]
\]

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by the substitution $\lambda = \lambda_{\text{crit}}$ in (5). With a similar computation for $I_{\alpha}(v)$ we then get

$$I_{\alpha}(v) + I_{\alpha}(-v) \geq 2\lambda_{\text{crit}} + v \inf_{\eta, \eta' \in M^\infty_\tau(\Sigma)^K} \left[ -\int \log \varphi(\lambda_{\text{crit}}, \omega) \eta(d\omega) - \int \log \varphi(\lambda_{\text{crit}}, \omega) \eta'(d\omega) + h(\eta|\alpha) + h(\eta'|\alpha) \right]$$

(71)

with $\varphi$ defined in (26), and we finally derive (70) by showing that

$$\int \log \varphi(\lambda_{\text{crit}}, \omega) \eta(d\omega) + \int \log \varphi(\lambda_{\text{crit}}, \omega) \eta'(d\omega) \leq 0$$

(72)

for all $\eta, \eta' \in M^\infty_\tau(\Sigma)^K$ such that $h(\eta|\alpha) + h(\eta'|\alpha) < \infty$. Recall (33) and note that

$$\varphi(\lambda_{\text{crit}}, \omega) \varphi(\lambda_{\text{crit}}, \theta^{-1}\omega) \leq 1$$

holds for all $\omega$ with $\omega_i \geq \omega_{\text{min}}, i \in \mathbb{Z}$. The point here is, that $\varphi(\lambda_{\text{crit}}, \theta^{-1}\omega)$ [resp., $\varphi(\lambda_{\text{crit}}, \omega)$] is measurable with respect to the $\sigma$-algebra $\mathcal{F}^-$ generated by $\omega_i, i < 0$ [resp., $\mathcal{F}^+$ generated by $\omega_i, i \geq 0$]. Taking logarithms in the last inequality and integrating for the measure $\eta_{\omega^-} \otimes \eta_{\omega^+}$, we get

$$\int \log \varphi(\lambda_{\text{crit}}, \theta^{-1}\omega) \eta(d\omega) + \int \log \varphi(\lambda_{\text{crit}}, \omega) \eta'(d\omega) \leq 0$$

proving (72) since $\eta$ is translation invariant.

\[ \square \]

**Proof of Theorem 5.** Fix $\Delta > 0$ (eventually, $\Delta \to 0$). For $\phi \in \mathcal{L}$, let

$$\theta_0 = 0, \theta_j = \min\{t \geq \theta_{j-1} : |\phi(t) - \phi(\theta_{j-1})| = \Delta\} \land 1, j = 1, \ldots, J,$$

define $Y_j = \phi(\theta_j)$, and say that $1 \leq j \in I^+$ if $Y_j > Y_{j-1}$ and $j \in I^-$ otherwise. Define next the random times

$$\xi_0 = 0, \xi_j = \min\{k > \xi_{j-1} : X_k = n\Delta \lfloor \frac{Y_k}{\Delta} \rfloor \} \land n, j = 1, \ldots, J.$$

Consider the event

$$A^\phi_{\Delta, \delta} = \bigcap_{j=1}^J \left\{ \left| \frac{1}{n}(\xi_j - \xi_{j-1}) - (\theta_j - \theta_{j-1}) \right| \leq \delta \right\}.$$

We begin by proving the

**Lemma 9**

$$\lim_{\delta \to 0} \lim_{n \to \infty} \sup \frac{1}{n} \log P_\omega( A^\phi_{\Delta, \delta} ) \leq - \sum_{j \in I^+} \Delta I_{\varphi\omega}(\theta_j - \theta_{j-1}) - \sum_{j \in I^-} \Delta I_{\varphi\omega}(\theta_j - \theta_{j-1}), \eta \text{ a.e.},$$

(73)

and

$$\lim_{\delta \to 0} \lim_{n \to \infty} \inf \frac{1}{n} \log P_\omega( A^\phi_{\Delta, \delta} ) \geq - \sum_{j \in I^+} \Delta I_{\varphi\omega}(\theta_j - \theta_{j-1}) - \sum_{j \in I^-} \Delta I_{\varphi\omega}(\theta_j - \theta_{j-1}), \eta \text{ a.e.}.$$ (74)

**Proof of Lemma 9.** The proof is no more than an exercise in book-keeping. Indeed, let $M^+ = \max Y_j$,

$M^- = -\min Y_j$,

and note that $[-M^-, M^+] = \bigcup_{k=1}^{M^+ + M^- - \Delta} L_k$, where $L_k = [-M^- + (k-1)\Delta, -M^- + k\Delta]$.

With $R_j = [Y_j \land Y_{j+1}, Y_j \lor Y_{j+1}]$, one obtains a partition of $j \in I^+$ ($j \in I^-$) into sets $K^+_k$, ($K^-_k$), such that $R_j = L_k$ for $j \in K^+_k$ ($j \in K^-_k$). Note that $|K^+_k| - |K^-_k| = 0$ or 1, and $|K^+_k| \leq \Delta^{-1}$. 24
Next, let \( \{ \tau_{k\ell} \}_{k=1}^{\infty} \) be independent (given \( \omega \)) copies of the random variable \( \tau_{\ell} \), and, with \( \bar{\tau}_{-i} = \inf \{ t \geq T_i : X_i = i - 1 \} - T_i \), let \( \{ \bar{\tau}_{-i} \}_{i=1}^{\infty} \) denote independent (given \( \omega \)) copies of \( \bar{\tau}_{-i} \). Then, with respect to \( P_\omega \),

\[
A_{\Delta, \delta}^{\phi} \stackrel{d}{=} \left(\bigcap_{k=1}^{(M^+ + M^-)/\Delta} \left\{ \left\{ \frac{1}{n} \sum_{t \in K^+_n} \tau_{k\ell} \in (\theta_{\ell} - \theta_{\ell-1} - \delta, \theta_{\ell} - \theta_{\ell-1} + \delta) \right\} \right\} \right) \cap \left\{ \left\{ \frac{1}{n} \sum_{t \in K^-_n} \bar{\tau}_{-i} \in (\theta_{\ell} - \theta_{\ell-1} - \delta, \theta_{\ell} - \theta_{\ell-1} + \delta) \right\} \right\}.
\]

An application of Theorem 1 now yields the lemma.

Lemma 9 possesses an analogue stated in terms of the process \( X_i \) itself. Its proof repeats the same argument and is therefore omitted. For simplicity in notations, we assume that \( \Delta^{-1} \) is integer valued. Define (note that \( \Delta \) now denotes discretization in time, not space!)

\[
B_{\Delta, \delta}^{\phi} \equiv \bigcap_{j=1}^{1/\Delta} \left\{ \left| \frac{1}{n} X_{nj\Delta} - \phi_j \right| \leq \delta \right\}.
\]

**Lemma 10**

\[
\lim_{\delta \to 0} \limsup_{n \to \infty} \frac{1}{n} \log P_\omega (B_{\Delta, \delta}^{\phi}) \leq - \sum_{j=1}^{1/\Delta} \Delta I_\eta^\phi \left( \frac{\phi_j - \phi_{j-1}}{\Delta} \right), \eta \text{ a.e.,}
\]

and

\[
\lim_{\delta \to 0} \liminf_{n \to \infty} \frac{1}{n} \log P_\omega (B_{\Delta, \delta}^{\phi}) \geq - \sum_{j=1}^{1/\Delta} \Delta I_\eta^\phi \left( \frac{\phi_j - \phi_{j-1}}{\Delta} \right), \eta \text{ a.e.}
\]

We may now return to the proof of Theorem 5.

1. In view of the compactness of \( \mathcal{L} \), the only issue is the lower-semicontinuity of \( I_\eta^\phi \). This however is obvious due to the convexity of \( I_\eta^\phi (\cdot) \) on \( \mathbb{R} \).

2. In view of the compactness of \( \mathcal{L} \) and the projective limits method, c.f. [4, Ch. 5.1], having established Lemma 10, all that is needed is to prove that for any \( \phi \in \mathcal{L} \),

\[
\lim_{\Delta \to 0} \sum_{j=1}^{1/\Delta} \Delta I_\eta^\phi \left( \frac{\phi_j - \phi_{j-1}}{\Delta} \right) = \int_0^1 I_\eta^\phi (\phi (t)) dt.
\]

But this is obvious from dominated convergence since \( \phi \) is differentiable a.e. (Lebesgue) with derivative bounded in absolute value by 1.

\[
\square
\]

5 Properties of the rate functions

We gather in this section some detailed properties of the various annealed and quenched rate functions \( I_\eta^{\tau, q} \), \( I_\alpha^{\tau, \alpha} \), \( I_\eta^\tau \), \( I_\alpha^\tau \) encountered in this paper. Throughout, we assume that \( \eta \) is ergodic and locally equivalent to
the product of its marginals, whereas \( \alpha \in M^\uparrow(\Sigma)^K \) will be taken to be a \textbf{product measure}. Note that under these assumptions, all the above rate functions are convex and (c.f. (8) and (11)),

\[
I^{\tau\cdot q}_\alpha(u) \leq I^{\tau\cdot q}_\alpha(v), \quad u \geq 1, \quad \text{and} \quad I^{\tau\cdot q}_\alpha(v) \leq I^0_\alpha(v), v \in [-1,1].
\]

With some abuse, we say that a measure \( \eta \in M^\uparrow(\Sigma)^K \) is transient to the right (transient to the left, recurrent) if \( X_\eta \) is transient to \( +\infty \) (transient to \( -\infty \), recurrent), \( \eta \)-a.s. We also introduce the notation \( \langle \tau \rangle_\eta = E_\eta[\tau_\omega] \), where we recall that \( \tau_\omega = E_\omega[\tau_0|\tau_1 < \infty] \). The following summarizes our main results concerning the quenched rate functions. Additional details, e.g. the precise slopes of certain linear pieces of the rate functions, are mentioned inside the proofs. We remind the reader that the term “increasing” includes the case of not strictly increasing, etc.

**Proposition 2** Assume that \( \eta \) is ergodic, locally equivalent to the product of its marginals and non-degenerate, i.e. not concentrated on one point. Then,

**Case A.** \( \int \log \rho_0(\omega)\eta(d\omega) = 0 \), i.e. \( \eta \) is recurrent. Then, \( I^{\tau\cdot q}_\eta \) and \( I^0_\eta \) are strictly convex, \( I^{\tau\cdot q}_\eta \) is decreasing on \([1,\infty)\) with \( \lim_{u \to \infty} I^{\tau\cdot q}_\eta(u) = 0 \), while \( I^0_\eta(0) = 0 \) and \( I^0_\eta \) increasing on \([0,1]\), decreasing on \([-1,0] \) and \( I^0_\eta \) is symmetric (see Figures 1 and 6).

**Case B.** \( \int \log \rho_0(\omega)\eta(d\omega) < 0 \), \( \langle \tau \rangle_\eta = \infty \), i.e. \( \eta \) is transient to the right with zero speed. Then, \( I^{\tau\cdot q}_\eta \) and \( I^0_\eta \) have the same properties as in case A except that \( I^0_\eta \) is not symmetric (see Figure 7).

**Case C.** \( \eta \in M^{1/2} \), \( \int \log \rho_0(\omega)\eta(d\omega) < 0 \), and \( \langle \tau \rangle_\eta < \infty \), i.e. \( \eta \) is transient to the right with mixed drifts and positive speed. Then, \( I^{\tau\cdot q}_\eta \) is strictly convex and decreasing on \([1,\langle \tau \rangle_\eta]\), while \( I^{\tau\cdot q}_\eta = 0 \) on \([\langle \tau \rangle_\eta,\infty)\). \( I^0_\eta \) is monotone increasing on \([0,1]\), monotone decreasing on \([-1,0]\), strictly convex on \([-1,-v_\eta]\cup[v_\eta,1]\), \( I^0_\eta(v) = |v|\int \log \rho_0(\omega)\eta(d\omega) \) for \( v \in [-v_\eta,0] \), and \( I^0_\eta = 0 \) on \([0,v_\eta]\) (see Figures 3 and 8).

**Case D.** \( \rho_{\max} < 1 \), i.e. all drifts point to the right and the walk is transient to \( +\infty \). Define \( \lambda_{\crit} \) and \( u_{\crit} \) as in Lemma 5. Then, \( I^{\tau\cdot q}_\eta \) is strictly convex and decreasing on \([1,\langle \tau \rangle_\eta]\), is strictly convex and increasing on \([\langle \tau \rangle_\eta,u_{\crit}\]\, and is linear on \([u_{\crit},\infty)\). Further, \( I^{\tau\cdot q}_\eta(v_{\eta}^{-1}) = 0 \). The rate function \( I^0_\eta \) is decreasing and strictly convex on \([-1,-u_{\crit}^{-1}]\), decreasing linearly on \([-u_{\crit}^{-1},0]\), decreasing linearly (with a smaller slope) on \([0,u_{\crit}^{-1}]\), and strictly convex on \([u_{\crit}^{-1},1]\), with \( I^0_\eta(v_{\eta}) = 0 \) (see Figures 4 and 9).

**Case E.** \( \eta \in M^{1/2} \), \( \eta \) is transient to \( -\infty \), with \( \langle \tau \rangle_\eta \) either finite or infinite. Then \( I^{\tau\cdot q}_\eta \) is strictly convex and decreasing on \([1,\langle \tau \rangle_\eta]\), and \( I^{\tau\cdot q}_\eta(u) = E_\eta[\log \rho_0] > 0 \) for \( u \geq \langle \tau \rangle_\eta \). For \( I^0_\eta \), simply consider Cases B and C under the transformation \( v \mapsto -v \) (see Figure 2).

**Case F.** \( \omega_{\max} < 1/2 \), i.e. all drifts point to the left. With \( \langle \tau \rangle_\eta < \infty \), \( I^{\tau\cdot q}_\eta \) is strictly convex and decreasing on \([1,\langle \tau \rangle_\eta]\), strictly convex and increasing on \([\langle \tau \rangle_\eta,u_{\crit}(\eta_{\text{inv}})]\), and linearly increasing on \([u_{\crit}(\eta_{\text{inv}}),\infty)\). The rate function \( I^0_\eta \) is obtained from Case D by the transformation \( v \mapsto -v \) (see Figure 5).

We note that we do not discuss the regularity properties of \( I^0_\eta \) at 0. In the case of \( \eta \) a product measure, some information on analyticity, obtained by considering the continued fraction defining \( \varphi(\lambda,\omega) \), may be found in [10].

**Proof of Proposition 2** In Theorem 1 we have already shown that \( I^{\tau\cdot q}_\eta \) is decreasing and convex on \([1,E_\eta[\tau_\omega]]\) and increasing and convex on \([E_\eta[\tau_\omega],\infty)\). It follows from Lemma 1, Lemma 2 and Lemma 5 that
if \( \eta \in M^*_\alpha (\Sigma)^+ \) then \( I^\alpha_\eta \) is strictly convex on \([1, u_{\text{crit}}]\), and that for \( u > u_{\text{crit}} \) one has that
\[
I^\alpha_\eta(u) = \lambda_{\text{crit}} u - \int \log \varphi(\lambda_{\text{crit}}, \omega) \eta(d\omega).
\] (78)

Note also that \( \lambda_{\text{crit}} = 0 \) in cases A, B, C. Proposition 1 then allows one to make the appropriate transfer of the results to all \( \eta \in M^*_\alpha (\Sigma)^K \). Finally, the results for \( I^\alpha_\eta \) follow those for \( I^\alpha_\eta \) by using the representation (9), which allows for the transfer of strict convexity from the time variable to the space variable.

We turn next to the annealed rate functions. Introduce the product measure \( \hat{\alpha} \in M^*_\alpha (\Sigma)^K \) as follows:
\[
\frac{d\tilde{\alpha}_0}{d\alpha_0} = \frac{1}{\rho_0} \left( \int \frac{1}{\rho_0(\omega_0)} \alpha_0(d\omega_0) \right)^{-1}.
\] (79)

Let \( u^* := \langle \tau \rangle_\alpha (\omega) \). Note that the formula in the remark following Lemma 1 implies that \( u^* < \infty \) if \( \omega_{\text{max}} < 1/2 \), and define
\[
b = \begin{cases} 
  u^*, & \omega_{\text{max}} < 1/2, \\
  \infty, & \alpha \in M^{1/2}, \\
  \langle \tau \rangle_\alpha, & \omega_{\text{min}} > 1/2.
\end{cases} \\
b' = \begin{cases} 
  \langle \tau \rangle_\alpha, & \alpha \in M^*_\alpha (\Sigma)^+, \\
  u^*, & \hat{\alpha} \in M^*_\alpha (\Sigma)^-, \\
  \infty, & \text{otherwise}.
\end{cases}
\]

Always, \( b \geq \langle \tau \rangle_\alpha \) and \( b' \leq b \). Note that \( \hat{\alpha} \in M^*_\alpha (\Sigma)^- \) implies that
\[
0 \leq \int \log \rho_0(\omega_0) \tilde{\alpha}_0(d\omega_0) = \int \rho_0^{-1} \log \rho_0(\omega_0) \alpha_0(d\omega_0) \leq \int \log \rho_0(\omega_0) \alpha_0(d\omega_0),
\]
(where we used Jensen’s inequality to show the last inequality) and hence \( \alpha \in M^*_\alpha (\Sigma)^- \).

**Proposition 3** Assume \( \alpha \in M^*_\alpha (\Sigma)^K \) is a product measure, and non-degenerate. Then \( I^\alpha_\eta(\cdot) \) is increasing on \([b, \infty)\), constant on \([b', b]\), and decreasing on \([1, b']\).

**Remark:** In fact, one sees from the proof below that whenever \( b \neq b' \) then \( I^\alpha_\eta(u) = I^\alpha_\eta(b') \) for \( u > b' \).

More detailed information is also available. The classification of different cases follows the one in Proposition 2.

**Proposition 4** Assume \( \alpha \in M^*_\alpha (\Sigma)^K \) is a product measure, and not concentrated on one point.

**Case A.** \( I^\alpha_\eta \) is strictly decreasing with limit 0 at infinity. \( I^\alpha_\eta \) is strictly decreasing on \([-1, 0]\) and strictly increasing on \([0, 1]\), with \( I^\alpha_\eta(0) = 0 \) (and is not necessarily symmetric).

**Case B.** Same as Case A.

**Case C.** \( I^\alpha_\eta \) is strictly decreasing on \([1, \langle \tau \rangle_\alpha]\), and is zero on \([\langle \tau \rangle_\alpha, \infty)\). \( I^\alpha_\eta \) is zero on \([0, \nu_\alpha]\) and is strictly increasing on \([\nu_\alpha, 1]\). Further, define \( d = E_\alpha [\nu_\alpha^2]/E_\alpha [\nu_\alpha] \), with \( v^* = (1 - d)/(1 + d) \) if \( d < 1 \) and \( v^* = 0 \) otherwise. Then, \( I^\alpha_\eta \) is strictly decreasing on \([-1, 0]\) and is linear on \([-v^*, 0]\).

**Case D.** \( I^\alpha_\eta \) is strictly decreasing on \([1, \langle \tau \rangle_\alpha]\) and strictly increasing on \([\langle \tau \rangle_\alpha, \infty)\), with \( I^\alpha_\eta(\langle \tau \rangle_\alpha) = 0 \). \( I^\alpha_\eta \) is strictly decreasing on \([-1, \nu_\alpha]\) and strictly increasing on \([\nu_\alpha, 1]\), with \( I^\alpha_\eta(\nu_\alpha) = 0 \) and \( I^\alpha_\eta(0) = I^\alpha_\eta(0) = \lambda_{\text{crit}} \).

Assume in addition that there exists a non degenerate minimizer \( \eta^+ \) of \( \eta \mapsto -E_\eta[f(\lambda_{\text{crit}}, \omega)] + h(\eta|\alpha) \) for which the conclusions of Lemma 3, part 1 hold true and such that \( \lambda_{\text{crit}}(\eta^+) = \lambda_{\text{crit}} \) (with \( \lambda_{\text{crit}} := \lambda_{\text{crit}}(\alpha) \)).

In this case, \( I^\alpha_\eta \) is linear on \([u^+, \infty)\) with \( u^+ = u_{\text{crit}}(\eta^+) = E_{\eta^+}[\tau_1 e^{\lambda_{\text{crit}} \tau_1}]/E_{\eta^+}[e^{\lambda_{\text{crit}} \tau_1}] < \infty \), and \( I^\alpha_\eta \) is linear on \([0, (u^+)^{-1}]\).
Case E. Set $\rho^* = E_\alpha[\rho_0^{-1}] / E_\alpha[\rho_0^{-1}]$, and $u^* = (1 + \rho^*) / (1 - \rho^*)$ if $\rho^* < 1$, $u^* = \infty$ otherwise. Then, $I_{\alpha}^{\rho^*}$ is strictly decreasing on $[1, u^*]$ and $I_{\alpha}^{\rho^*}(u) = -\log E_\alpha[\rho_0^{-1}] > 0$ on $[u^*, \infty)$. For $I_{\alpha}^0$, simply consider Cases B and C under the transformation $v \mapsto -v$.

Case F. With $\rho^* < 1$ and $u^*$ as in Case E, $I_{\alpha}^{\rho^*}$ is strictly decreasing on $[1, u^*]$ and strictly increasing on $[u^*, \infty)$. Further, $I_{\alpha}^{\rho^*}(u^*) = -\log E_\alpha[\rho_0^{-1}] > 0$. For $I_{\alpha}^0$, simply consider Case D under the transformation $v \mapsto -v$. Assume in addition that there exists a non-degenerate minimizer $\eta^-$ of $\eta \mapsto -E_\eta[f(\lambda_{\text{crit}}, \omega)] + h(\eta|\alpha)$ for which the conclusions of Lemma 5, part i) hold true and such that $\lambda_{\text{crit}}(\eta^-) = \lambda_{\text{crit}}$. In this case, $I_{\alpha}^{\rho^*}$ is linear on $[u^-, \infty)$ with $u^- = u_{\text{crit}}(\eta^-) = E_{\eta^-}[E_\omega[\tau_1 \leq \infty]/E_\omega[e^{\lambda_{\tau_1}} \tau_1 < \infty]] < \infty$.

Remark
1. We will see examples at the end of this section where the additional assumption in Cases D and F is satisfied. Checking instead the stronger assumption that $\eta^+$ [resp., $\eta^-$] is locally equivalent to the product of its marginal with $\lambda_{\text{crit}}(\eta^+) = \lambda_{\text{crit}}$ and $\eta^+$ [resp., $\lambda_{\text{crit}}(\eta^-) = \lambda_{\text{crit}}$ and $\eta^-$] non-degenerate, turns out to be far more difficult.

2. It is worthwhile to note that in Case D, if in addition a minimizer $\eta^-$, satisfying the additional assumptions, exists for $\alpha^{\text{inv}}$ (which belongs to case F) then $I_{\alpha}^{\rho^*}$ is linear on the interval $(-u^- \omega^{-1}, 0]$ defined in Case F.

Before proving the above propositions, we state and prove some auxiliary facts.

**Lemma 11.** 1. For any product measure $\alpha$ and any bounded continuous function $\Psi$,

$$\inf_{\eta \in M^1(\Sigma)} \left[ h(\eta|\alpha) + \int \Psi(\omega_0) \eta_0(d\omega_0) \right] = H(\tilde{\alpha}|\alpha_0) + \int \Psi(\omega_0) \tilde{\alpha}_0(d\omega_0),$$

where $\tilde{\alpha}$ is a product measure and $d\tilde{\alpha}_0 / d\alpha_0 = \exp(-\Psi(\omega_0)) / \int \exp(-\Psi(\omega_0)) \alpha_0(d\omega_0)$.

2. Let $\Theta = \{ \eta \in M^1(\Sigma) : \int \Psi(\omega_0) \eta_0(d\omega_0) \geq 0 \}$. If $\alpha \in \Theta$ then

$$\inf_{\eta \in M^1(\Sigma)} \left[ h(\eta|\alpha) + \left( 0 \vee \int \Psi(\omega_0) \eta_0(d\omega_0) \right) \right] = \inf_{\eta \in \Theta} \left[ h(\eta|\alpha) + \left( 0 \vee \int \Psi(\omega_0) \eta_0(d\omega_0) \right) \right]. \tag{80}$$

In particular, if also $\tilde{\alpha} \in \Theta$ then

$$\inf_{\eta \in M^1(\Sigma)} \left[ h(\eta|\alpha) + \left( 0 \vee \int \Psi(\omega_0) \eta_0(d\omega_0) \right) \right] = H(\tilde{\alpha}|\alpha_0) + \int \Psi(\omega_0) \tilde{\alpha}_0(d\omega_0). \tag{81}$$

**Proof of Lemma 11.** 1. We have, with $\mathcal{F}_n = \sigma(\omega_0, \omega_1, \ldots, \omega_{n-1})$,

$$h(\eta|\alpha) = \sup_n \frac{1}{n} H(\eta|\alpha)|\mathcal{F}_n.$$

Therefore,

$$h(\eta|\alpha) \geq H(\eta|\alpha_0) \geq -\int \Psi(\omega_0) \eta(d\omega_0) - \log \int e^{-\Psi(\omega_0)} \alpha_0(d\omega_0)$$

where the second inequality is due to the variational characterization of relative entropy, c.f. for example [4, Lemma 6.2.13]. Hence

$$h(\eta|\alpha) + \int \Psi(\omega_0) \eta(d\omega_0) \geq -\log \int e^{-\Psi(\omega_0)} \alpha(d\omega_0)$$


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and equality is achieved for the measure $\bar{\alpha}$.

2. Assume (80) does not hold true. Then there exists a $\eta^* \not\in \Theta$ such that

$$h(\eta^*|\alpha) + \left(\inf_{\eta \in \Theta} h(\eta|\alpha) + 0 \vee \int \Psi(\omega_0)\eta_0(d\omega_0)\right) = h(\eta^|\alpha) + \left(\inf_{\eta \in \Theta} h(\eta|\alpha) + 0 \vee \int \Psi(\omega_0)\eta_0(d\omega_0)\right) ,$$

Because $\alpha \in \Theta$, $\eta^* \neq \alpha$, and further, $\int \Psi(\omega_0)\alpha_0(d\omega_0) > 0$, otherwise $\eta^* = \alpha$ is a global minimizer. Take a convex combination $\eta_{\theta} := \theta \alpha + (1 - \theta) \eta^*$ such that $\int \Psi(\omega_0)\eta_{\theta}(d\omega_0) = 0$. Since the product measure $\alpha$ satisfies Assumption (A), one can find a sequence $\eta_{n \theta} \in M_\Sigma^\tau$ such that $\eta_{n \theta} \rightarrow \eta_{\theta}$ weakly and $h(\eta_{n \theta}|\alpha) \rightarrow h(\eta_{\theta}|\alpha)$. Therefore,

$$\limsup_{n \rightarrow \infty} h(\eta_{n \theta}|\alpha) + \left(0 \vee \int \Psi(\omega_0)\eta_{n \theta}(d\omega_0)\right) = h(\eta_{\theta}|\alpha) < h(\eta^*|\alpha) ,$$

a contradiction.

It is worthwhile to note that actually, one may compute explicitly the optimal $\eta$ in (80) even when $\bar{\alpha} \not\in \Theta$: it is a product measure with marginal $Z^- (\beta) \exp(\beta \Psi(\omega_0))\alpha_0(d\omega_0)$, where $-1 \leq \beta \leq 0$ is chosen such that $\int \exp(\beta \Psi(\omega_0))\Psi(\omega_0)\alpha_0(d\omega_0) = 0$. Using this observation one may relax the assumptions in the lemma to $\Psi$ being merely bounded measurable.

**Proof of Proposition 3** Since $I_{\alpha}^\tau$ is convex, it is enough to show that whenever $b' < \infty$ then it is a minimizer of $I_{\alpha}^\tau$, that the latter is constant on $[b', b]$; and that if $b' = \infty$ then $I_{\alpha}^\tau$ is decreasing. We divide the proof into the following cases:

1. $\alpha \in M_\Sigma^\tau$. Then, $\beta = b' = \langle \tau \rangle_ \alpha \in (1, \infty]$, and $I_{\alpha}^\tau(b') = 0$ (if $b' < \infty$) while, if $b' = \infty$, $\lim_{u \rightarrow \infty} I_{\alpha}^\tau(u) = 0$.

2. Assume $\bar{\alpha} \in M_\Sigma^\tau$ (and then, as noted above, also $\alpha \in M_\Sigma^\tau$). Then $b' = \infty$. Assume first $u^* < \infty$. Then,

$$I_{\alpha}^\tau(u^*) \leq I_{\alpha}^\tau(u^*) + \log \int \rho_0(\omega_0)^{-1} \alpha_0(d\omega_0) ,$$

as can be checked by an explicit computation involving the definition of $\bar{\alpha}$. On the other hand, using in the first equality the exact value of the minimum of $I_{\alpha}^\tau(\cdot)$, see the comment before (40), and (81) in the second equality (with $\Psi(\rho_0) = \log \rho_0$),

$$\inf_{u} I_{\alpha}^\tau(u) = \inf_{\eta \in M_\Sigma^\tau} \left[0 \vee \int \log \rho_0(\omega_0)\eta_0(d\omega_0)\right] = - \log \int \rho_0(\omega_0)^{-1} \alpha_0(d\omega_0) .$$

Hence, $u^*$ is a global minimizer of $I_{\alpha}^\tau$ in this case.

If $u^* = \infty$, (83) still holds true while, for any $u < \infty$,

$$I_{\alpha}^\tau(u) \leq I_{\alpha}^\tau(u) + \log \int \rho_o(\omega_0)^{-1} \alpha_0(d\omega_0) ,$$

$$\lim_{u \rightarrow \infty} - \log \int \rho_0(\omega_0)^{-1} \alpha_0(d\omega_0) ,$$

since $\bar{\alpha} \in M_\Sigma^\tau$. It follows that $\inf_u I_{\alpha}^\tau(u) = \lim_{u \rightarrow \infty} I_{\alpha}^\tau(u)$, as required.

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3. Assume \( \hat{\alpha} \in M_f^-(\gamma) \) but \( \alpha \in M_f^+(\gamma) \). In this case \( b' = \infty \), and one repeats the previous argument, using this time that

\[
\lim_{u \to \infty} I_\alpha^{\tau,a}(u) \leq \inf_{\eta \in M_1^f(\Gamma)} \left\{ \int \log \rho_0(\omega_0) \eta(d\omega_0) \geq 0 \right\} \left[ \int \log \rho_0(\omega_0) \eta(d\omega_0) + h(\eta|\alpha) \right],
\]

while, using now (80),

\[
I_\alpha^{\tau,a}(u) \geq \inf_{\eta \in M_1^f(\Gamma)} \left[ h(\eta|\alpha) + \left( 0 \vee \int \log \rho_0(\omega_0) \eta_\theta(d\omega_0) \right) \right]
\]

\[
= \inf_{\eta \in M_1^f(\Gamma): \int \log \rho_0(\omega_0) \eta(d\omega_0) \geq 0} \left[ \int \log \rho_0(\omega_0) \eta(d\omega_0) + h(\eta|\alpha) \right],
\]

implying as before that \( \inf_u I_\alpha^{\tau,a}(u) = \lim_{u \to \infty} I_\alpha^{\tau,a}(u) \).

\[\Box\]

**Proof of Proposition 4**

1. **Properties of \( I_\alpha^{\tau,a} \)** The monotonicity of \( I_\alpha^{\tau,a} \) on the claimed intervals is a direct consequence of Proposition 3, while the convexity is stated in Theorem 2. Further, (8) implies that if \( I_\alpha^{\tau,a}(u) = 0 \) then \( I_\alpha^{\tau,a}(u) = 0 \), yielding the claimed zero values for \( I_\alpha^{\tau,a} \).

To see the claimed strict monotonicity of \( I_\alpha^{\tau,a} \) in case \( \Lambda - D \), note that by convexity, it is enough to show that \( I_\alpha^{\tau,a}(u) > 0 \) at a point \( u \) in order to show that it is strictly monotone there. But \( I_\alpha^{\tau,a}(u) = 0 \) only if \( I_\alpha^{\tau,a}(u) = 0 \) by (8) and the fact that the infimum there is attained, leading to the monotonicity claim.

Cases E-F require slightly more work. Assume first that \( u^* < \infty \), we already know, c.f. Proposition 3, that \( u^* \) is a global minimum of \( I_\alpha^{\tau,a} \).

To prove the strict monotonicity of \( I_\alpha^{\tau,a} \) on \([1, u^*]\) when \( u^* < \infty \), in both cases E and F, we check that \( I_\alpha^{\tau,a}(u) > -\log E_\alpha[\rho^{-1}_0] \) for \( u < u^* \), and then the convexity of \( I_\alpha^{\tau,a} \) proves the required strict monotonicity. To this end, note that \( \hat{\alpha} \) is transient to the left (because \( \rho^* < 1 \)). But, for any \( \eta \),

\[
I_\eta^{\tau,a}(u) + h(\eta|\alpha) \geq E_{\eta_0} [\log \rho_0] + H(\eta_0|\alpha_0) \geq -\log \int \rho^{-1}_0(\omega)\alpha_0(d\omega),
\]

where the first inequality is achieved only on product measures transient to the left, and the second, due to Lemma 11, only when \( \eta_0 = \hat{\alpha}_0 \). But in the latter case, the first inequality is strict because \( u < u^* \) and \( E_{\eta}[\tau_{\omega}^*] = u^* \). Since the infimum over \( \eta \) is always achieved in the definition of \( I_\alpha^{\tau,a} \), we conclude that necessarily

\[
\inf_{\eta \in M_1^f(\Gamma)} \left[ I_\eta^{\tau,a}(u) + h(\eta|\alpha) \right] > I_\alpha^{\tau,a}(u^*),
\]

as claimed.

The strict monotonicity on \([u^*, \infty)\) in Case F is proved similarly, using that in Case F the quenched rate function is strictly monotone, and repeating the above argument.

Finally, it remains to check the strict monotonicity on \([1, \infty)\) in case E when \( u^* = \infty \). The argument given above actually shows that \( I_\alpha^{\tau,a}(\infty) := \lim_{u \to \infty} I_\alpha^{\tau,a}(u) = -\log E_\alpha[\rho^{-1}_0] \), and hence it suffices to check that \( I_\alpha^{\tau,a}(u) > I_\alpha^{\tau,a}(\infty) \). The argument is the same as above and therefore omitted.
We turn now to the linear part of $I^{\tau,a}_\eta$ in Case D. We checked in the proof of Proposition 2 that if the conclusions of Lemma 5, part i) are satisfied, then $I^{\tau,q}_\eta$ is linear on $[u^+, \infty)$. More precisely, like in (78) it holds for $u \geq u^+$ that

$$I^{\tau,q}_\eta(u) = \lambda_{\text{crit}} u - \int f(\lambda_{\text{crit}}, \omega)\eta^+(d\omega)$$

Hence we have

$$I^{\tau,a}_\eta(u) \leq I^{\tau,q}_\eta(u) + h(\eta^+|\alpha) = \lambda_{\text{crit}} u - \int f(\lambda_{\text{crit}}, \omega)\eta^+(d\omega) + h(\eta^+|\alpha).$$

(84)

On the other hand, it follows from the large deviation lower bound together with the substitution $\lambda = \lambda_{\text{crit}}$ in (63) that

$$-I^{\tau,a}_\eta(u) \leq \limsup_{n \to \infty} \frac{1}{n} \log P\left[\frac{T_n}{n} \geq u\right]$$

$$\leq -\inf_{\eta \in M^*_1(\Sigma)} \left[\lambda_{\text{crit}} u - \int f(\lambda_{\text{crit}}, \omega)\eta(d\omega) + h(\eta|\alpha)\right]$$

$$= -\lambda_{\text{crit}} u + \int f(\lambda_{\text{crit}}, \omega)\eta^+(d\omega) - h(\eta^+|\alpha)$$

since $\eta^+$ is a minimizer. Therefore the equality holds in (84), and $I^{\tau,a}_\eta$ is linear on $[u^+, \infty)$. The proof of existence of a linear part in Case F is similar.

2. Properties of $I^\alpha_\tau$. All the stated properties of $I^\alpha_\tau$ follow immediately, using (10), from the properties of $I^{\tau,a}_\eta$, except for checking that in Case D, $I^\alpha_\tau(0) = \lambda_{\text{crit}}$. But this was obtained in the proof of Theorem 4.□

We conclude this section by providing a class of examples where the additional assumption in Proposition 4, Cases D and F, is satisfied, resulting with the existence of linear pieces for $I^{\tau,a}_\eta$. We concentrate on Case D, as the construction for Case F is similar.

Choose $\omega_{\text{min}} > 1/2$, $\alpha_0(\omega_{\text{min}}) > 0$, $\alpha_0(\omega_{\text{max}}) > 0$, and $\omega_{\text{max}} - \omega_{\text{min}}$ small enough. (What is meant by small enough will become clear in the course of the construction). Due to the remark below the proof of Lemma 5 it is enough to ensure that any ergodic minimizer $\eta^+$ of the function $F(\eta) = -\int f(\lambda_{\text{crit}}, \omega)\eta(d\omega) + h(\eta|\alpha)$, satisfies, for a fixed $K_0$ depending on $\omega_{\text{min}}$ only, that $\eta^+\left(\omega_i = \omega_{\text{min}}\right)_{i=0}^{K_0+1} > 0$, and that $\eta_0^+(\omega_{\text{max}}) > 0$. We argue by contradiction. Assume that $\eta^+\left(\omega_i = \omega_{\text{min}}\right)_{i=0}^{K_0+1} = 0$. Then,

$$h(\eta^+|\alpha) \geq \frac{1}{K_0 + 2}H(\eta^+|\alpha)_{[a_0, a_{n+1}]} \geq \frac{1}{K_0 + 2} \log \frac{1}{1 - \alpha_0(\omega_{\text{min}})K_0+2} =: \delta.$$  

(85)

Recall that $\varphi(\lambda_{\text{crit}}, \omega ) \leq \varphi(\lambda_{\text{crit}}) = (\omega_{\text{min}}/(1 - \omega_{\text{min}}))^{1/2}$, and note that estimates similar to (44) and the substitution of the value for $\lambda_{\text{crit}}$ lead to the bound

$$\varphi(\lambda_{\text{crit}}, \omega) \geq \varphi(\lambda_{\text{crit}}, (\omega_{\text{max}}, \omega_{\text{max}}, \omega_{\text{max}}, \ldots))$$

$$= \frac{\sqrt{\omega_{\text{min}}(1 - \omega_{\text{min}}) - \sqrt{\omega_{\text{min}}(1 - \omega_{\text{min}}) - \omega_{\text{max}}(1 - \omega_{\text{max}})}}}{1 - \omega_{\text{max}}} := \varphi(\lambda_{\text{crit}})$$

(86)

(86)

From the estimates $F(\alpha) \leq -\log \varphi(\lambda_{\text{crit}})$ and $F(\eta^+) \geq \delta - \log \varphi(\lambda_{\text{crit}})$ (following from (85)), it is clear that if $\omega_{\text{max}}$ is close enough to $\omega_{\text{min}}$ then $F(\eta^+) > F(\alpha)$, a contradiction. The proof that $\eta^+_0(\omega_{\text{max}}) > 0$ being similar, only simpler, we conclude our construction.
6 Concluding Remarks and Open Problems

1. Our quenched results cover the case when \( \eta \) is ergodic, without being locally equivalent to the product of its marginals. However, the shape of the quenched rate function in this case can be different. For instance, one can construct examples where there are no linear pieces in \( I_\eta^\varphi \) in Case C above.

2. In general, we do not know how to solve the annealed variational problem in (11), and hence we do not have explicit expressions for \( I_\alpha^\varphi \). One case where this problem can be solved is when \(|v| = 1\). More precisely, for \( v = 1 \) we have

\[
P_\omega \left[ \frac{X_n}{n} = 1 \right] = \prod_{i=0}^{n-1} \omega_i,
\]

and

\[
P \left[ \frac{X_n}{n} = 1 \right] = E \left[ P_\omega \left[ \frac{X_n}{n} = 1 \right] \right] = E \left[ \prod_{i=0}^{n-1} \omega_i \right] = \prod_{i=0}^{n-1} E_{\alpha_0}[\omega_i].
\]

Hence, taking logarithms, dividing by \( n \) and taking limits, one concludes that

\[
I_\alpha^\varphi (1) = - \int \log \omega_\alpha \alpha_0 (d\omega_0),
\]

\[
I_\alpha^\varphi (1) = - \log \int \omega_\alpha \alpha_0 (d\omega_0).
\]

In particular, \( I_\alpha^\varphi (1) < I_\alpha^\varphi (1) \) as soon as \( \alpha \) is non-degenerate.

3. The minimizing measures for the variational problem (11) are of particular interest, for they hint at the environment which creates atypical behavior. In general - except for the points where the rate function is linear - we do not know how to compute the minimizing \( \eta \), or even how to find some of its properties, not even if \( \alpha \) is a product measure, although we have exhibited situations where we strongly suspect the minimizer is not a product measure (in fact, we believe it is Gibbsian with a long-range potential determined by the non-local, non-smooth function \( \varphi \)). An interesting exception is for \( v = 1 \). If \( \alpha \) is a product measure, then the minimizer in (11) is a product measure with

\[
\frac{d\eta_0}{d\alpha_0} = \frac{\omega_0}{\int \omega_\alpha \alpha_0 (d\omega_0)}.
\]

Of course, the case \( v = -1 \) is solved by duality.

4. We suspect, but have not proved, that \( I_\alpha^\varphi (u) \neq I_\alpha^\varphi (u) \) for any \( u \) such that \( I_\alpha^\varphi (u) \) is strictly convex at \( u \).

5. We speculate that the extra assumption stated in Proposition 4, cases D and F, is always satisfied, and is not limited to the class of example constructed at the end of the last section. Recall that these extra assumptions imply the existence of linear pieces for \( I_\alpha^\varphi \).

6. As in the i.i.d. environment case studied in length in [3], [9], [15], [14], one may look for refined asymptotics in the flat pieces of \( I_\alpha^\varphi (u) \) or \( I_\alpha^\varphi (u) \). When \( \eta \) is equivalent to the product of its marginals, we believe it to exhibit the same qualitative behavior as in the i.i.d. case, that is polynomial decay in the case \( \omega_\text{min} < 1/2 < \omega_\text{max} \) and sub-exponential decay when \( \omega_\text{min} = 1/2 \). Refined asymptotics for the multi-dimensional case were obtained in [17]. Some explicit computations are possible in the Markov environment case, we do not pursue this direction here.

7. When the support of \( \alpha_0 \) includes the points 0 or 1, our proofs break down (even if \( \alpha_0([0] \cup \{1\}) = 0 \))

under strong enough assumptions on the rate of decay of the \( \alpha_0([0, 1] \setminus [c, 1 - c]) \), we believe that the analysis can still be pushed through.

8. The multi-dimensional case presents many challenges. Important works in this domain are [21], [17], but many questions remain open, most notably what happens when \( 0 \not\in \text{conv supp } \alpha_0 \), what is the annealed rate function, and what is the relation of the latter to the quenched rate function.
References