SYMMETRIC INTERVAL ORDERS AND SIGNED POSETS

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1. Introduction

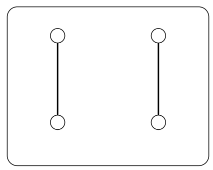
In this paper, we want to find signed poset generalizations of some results about interval orders and unit interval orders. Ordered sets, and specifically interval orders, are used extensively in other sciences such as in dating the findings in archaeology, in decision problems in economics and political science, and in queueing time-dependent processes in computer science.[2], [14]

Signed posets were first defined by Reiner [13] The motivation for such a definition comes from viewing a poset as a subset of a root system for the symmetric group S_n. He applies the same idea to the hyperoctahedral group B_n and defines a signed poset as a subset of a root system for B_n satisfying certain conditions.

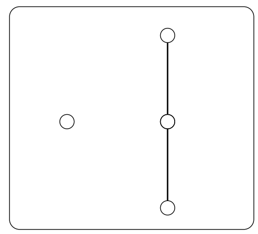
2. Background and Definitions

2.1. **Interval Orders.** Altough the idea of an interval order was first disccussed by N. Wiener(1914), it was first used explicitly by P. Fisburn(1970).

For a given set of intervals $\{[a_1, b_1], \ldots, [a_n, b_n]\}$ in R, we can order them by defining $[a_i, b_i] < [a_j, b_j]$ if $b_i < a_j$. A poset P is called an *interval order* if it can be represented as a set of closed intervals in this way. Moreover, if each interval can be made of unit length, P is called a *unit interval order*.



(2+2)



(1+3)

The poset consisting of disjoint union of two chains with cardinalities a and b is denoted by **(a+b)**.

The following two theorems characterize the posets that can represented by intervals and unit intervals.

Theorem 2.1. *(Fishburn-Mirkin Theorem, 1970) A poset is an interval order if and only if it avoids* ***(2+2)*** *as an induced subposet.*

Theorem 2.2. *(Scott-Suppes Theorem, 1958) A poset is a unit interval order if and only if it avoids* ***(2+2) and (3+1)*** *as induced subposets.*

For both theorems, there are proofs that construct the (unit) interval order explicitly [5],[12] or inductively [1].

2.2. **Signed Posets.** Before defining a signed poset, we need some preliminaries. Let B_n be the hyperoctahedral group or the group of signed permutations, which consists of all permutations and sign changes of the coordinates in $\mathbf{R^n}$.

The root system of B_n is the set of vectors

$$\Phi = \{\pm e_i : 1 \leq i \leq n\} \cup \{\pm e_i \pm e_j : 1 \leq i < j \leq n\}$$

We may choose as positive roots

$$\Phi^+ = \{+e_i : 1 \leq i \leq n\} \cup \{+e_i + e_j, +e_i - e_j : 1 \leq i < j \leq n\}$$

and the simple roots

$$\Pi = \{e_i - e_{i+1} : 1 \leq i < n\} \cup \{+e_n\}$$

A subset S of a root system Φ is isotropic if $\alpha \in S \Rightarrow -\alpha \notin S$. We define the *positive linear closure* of S in Φ by
$\overline{S}^{PLC} := \{\alpha \in \Phi : \alpha = \sum_{\beta \in S} c_\beta \beta \text{ for some } c_\beta \geq 0\}$

Now we define a **signed poset** as a positively linearly closed isotropic subset of the root system B_n.

Let $\pm[n]$ be the set $\{-n, -n+1, ..., -2, -1, 1, 2, ..., n-1, n\}$ and P be a poset on $\pm[n]$ which is self-dual in the sense that $i < j$ in P if and only if $-j < -i$ in P. An element $i \in P$ is called a bottom half element of P if $i < -i$ in P, called a top half element of P if $-i < i$ in P, and called a non-comparable element of P if i and $-i$ are incomparable. P is called a *complete* self-dual poset if it is self-dual and every top half element is above every bottom half element of P.

We know from Fischer [9] that the set of signed posets which are contained in the root system B_n is in one-to-one correspondence with the set of complete self-dual posets on $\pm[n]$. The poset $P(S)$ corresponding to S is given by

$$\pm e_i \pm e_j \in S \Leftrightarrow \pm i < \mp j \textit{ in } P(S)$$

and

$$\pm e_i \in S \Leftrightarrow \pm i < \mp i \textit{ in } P(S)$$

One can embed ordinary posets as signed posets via the following construction : given Q an ordinary poset on $[n]$, thought as a subset of $A_{n-1} = \{e_i - e_j : 0 \leq i \neq j \leq n\}$, we define the embedding as

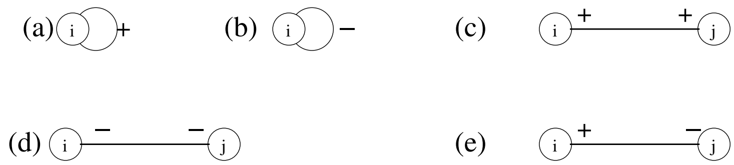
$$S_+(Q) := Q \cup \{+e_i\}_{i=1,2,\ldots,n} \cup \{+e_i + e_j\}_{1 \leq i < j \leq n}$$

A collection of intervals $I = \{[a_1, b_1], \ldots, [a_{2k}, b_{2k}]\}$, in $\mathbf{R}$ is *centrally symmetric* if there is a fixed point free involution $f : I \to I$ such that $f([a_i, b_i]) = [-b_i, -a_i]$ for all $i = 1, 2, \ldots, 2k$

We see that any such collection obviously give rise to a complete self-dual poset P, and hence the signed poset S with $P(S) = S$.

It is natural to ask whether similar statements to Theorems 2.1 and 2.2 can be made for signed posets and (unit) symmetric interval orders as was done for (unit) interval orders. That is, for a given signed poset P, we want to know what are the necessary and sufficient conditions for P to be a (unit) symmetric interval order.

Visual Represention of Signed Posets: We can visualize a signed poset inside B_n by defining its signed digraph $D(P) = (V, E)$ with the set of vertices $V = \{1, 2, \ldots, n\}$, and the set of edges E which is constructed as follows: if $+e_i \in P$ attach the loop shown in (a) below; if $-e_i \in P$ attach the loop shown in (b); if $+e_i + e_j \in P$ attach the edge shown in (c); if $-e_i - e_j \in P$ attach the edge shown in (d); if $+e_i - e_j \in P$ attach the edge shown in (e).



Two signed posets P_1 and P_2 are *isomorphic* if there exists a signed permutation $w \in B_n$ such that $wP_1 = P_2$. For a signed poset P on n elements and $T \subseteq [n]$, the *induced signed subposet of* P *on* T is the signed poset P_T on $|T|$ elements consisting of only those roots in P whose nonzero coordinates lie in T.

We have the following theorem:

Theorem 2.3. *A signed poset P is a symmetric interval order if and only if P does not have an induced signed subposet which is isomorphic to a signed poset which fits one of the 3 descriptions in Figure 1.*

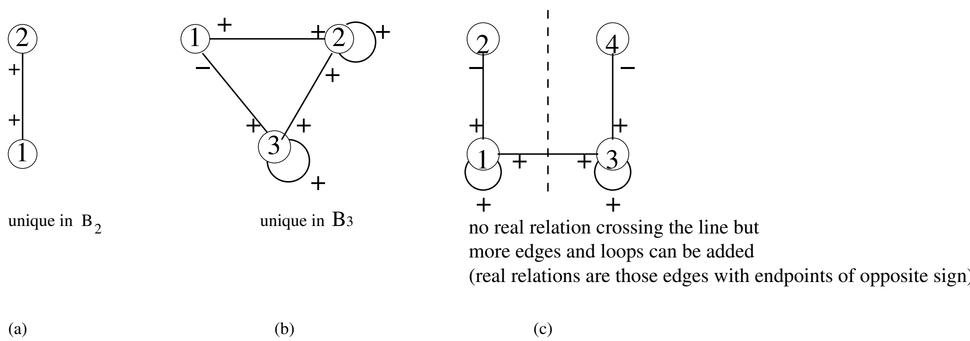


FIGURE 1.

Let's define X to be the set of all posets which fit one of the 3 descriptions in Figure 1. The theorem follows from the following lemmas.

Lemma 2.4. *Every complete self-dual poset which is an interval order is also a symmetric interval order.*

Proof. Let P be a finite complete self-dual poset on $\pm n$ which is an interval order.So in P:

- $i < j \Leftrightarrow -j < -i$
- $i < -i$ and $j < -j \Rightarrow j < -i$ and $i < -j$

From the given interval order representation of P, we want to get a symmetric interval representation without changing any relation between elements.

To construct the symmetric interval order, first we write the interval representing i as $I_i = [a_i, b_i]$. WLOG, we can assume $a_1 = 1$ is the left most point of all top-half elements, and -1 is the right most point of all bottom-half elements. Let's call the interval representing i that we will get after our construction $\hat{I}_i$.

First for all top elements i, define $\hat{I}_i = I_i$. For bottom half elements i, define $\hat{I}_i = -I_i = [-b_i, -a_i]$. If i and j are both non-comparable, we see i and j must be incomparable to each other, otherwise, if $i < j$ then we have $-j < -i$ but $i \not< -i$ and $j \not< -j$. This is impossible in an interval order, as it would give rise to a **(2+2)**. There are a finite number of top half elements and all are represented by intervals completely right of those representing the bottom half elements. We see that none of the non-comparable elements can be completely right to the 1, because if i is completely right to the 1, then i is bigger than all bottom elements, so $-i$ is smaller than all top elements, that implies $-i$ is completely left to the 1, but then we get $-i < i$, a contradiction. Similarly none of non-comparable elements can be completely left to -1.

Now for a non-comparable element $I_i = [a_i, b_i]$, if $a_i > -1$ pull a_i to the origin by stretching or contracting the interval, and if $b_i < 1$ do the same thing. With this operation we did not change any relation of non-comparable elements with top and bottom half elements, and all non-comparable elements now include the origin so they are still incomparable with each other. For a non-comparable element i, define

$$\hat{I}_i = ([-1, \infty] \cap I_i) \cup (-([1, \infty] \cap I_{-i})).$$

Here we did not change the part of I_i which is right to the -1 so did not change any relation of i with top half elements. We did not change any relation of i with bottom half elements either because we did not change any relation of $-i$ with top elements.

Finally define $\hat{I}_{-i} = -\hat{I}_i$. We completed the construction of a symmetric interval order. □

To complete the proof of Theorem 2.3, note that all of the signed posets in X create **(2+2)**'s, so it is easy to see that the signed poset of P cannot have any induced signed subposet isomorphic to a poset in X. On the other hand if P is not an interval order we can check by brute force that, up to isomorphism, this leads to the 3 cases (a), (b), (c) in Figure 1.

Conjecture 2.5. *A signed poset P is a unit symmetric interval order if and only if P does not have an induced signed subposet which is isomorphic to a signed poset which fits into one of the 6 descriptions in Figures 1 or 2.*

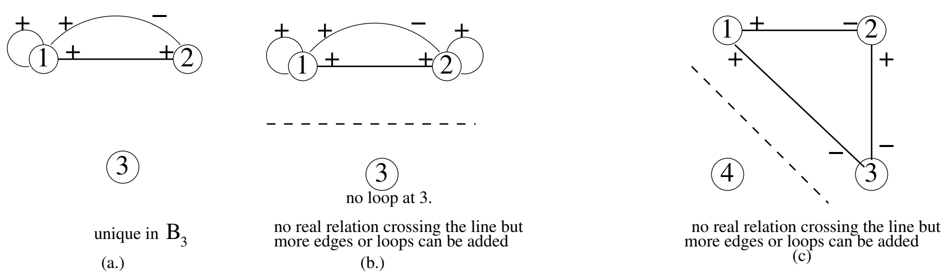


FIGURE 2.

Let's define Y to be the set of all posets which fit one of the 6 descriptions in Figures 1 or 2. Conjecture 2.5 would follow from the following conjecture and lemma.

Conjecture 2.6. *Every complete self-dual poset which is a unit interval order is also a symmetric unit interval order.*

Proof. It is on my to-do list. □

One can do a similar brute force check that, up to isomorphism, the only complete self-dual posets containing a **(3+1)** or **(2+2)** are those in Figures 1 and 2. For example, Figure 3 shows some of the different ways in which **(2+2)** can occur.

Lemma 2.7. *A complete self-dual poset P is a unit interval order if and only if its signed poset does not have an induced signed subposet which fits into one of the 6 descriptions in Figures 1 and 2.*

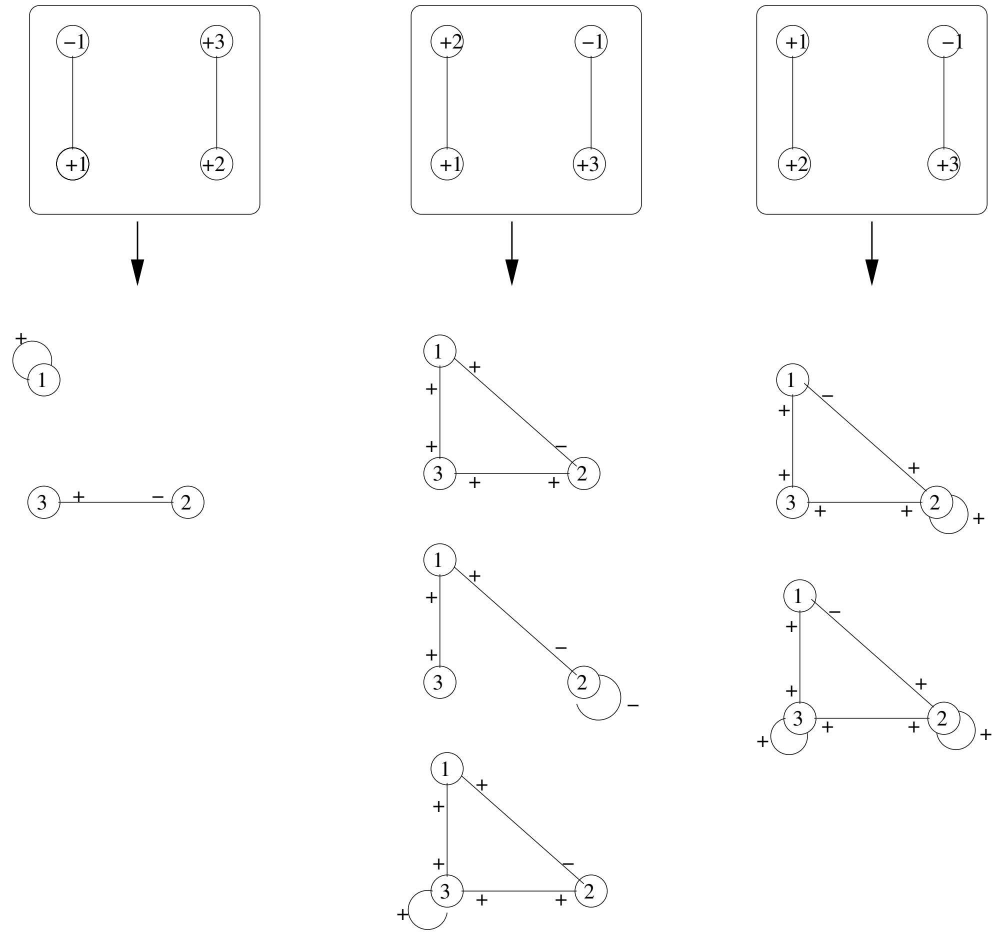


FIGURE 3.

3. Enumeration

It is known that the number of unit interval orders on $[n]$ up to isomorphism is the Catalan number C_n. Conjecture 2.6 would imply a similar result for symmetric unit interval orders as we explain here.

For a poset P labeled on n elements, define its antiadjacency matrix $A = (a_{ij})$ as $a_{ij} = 1$ if $i < j$ in P, and $a_{ij} = 0$ otherwise. We see different labelings of P can give different antiadjacency matrices. Following [15] to define a canonical labeling, first, we define *the altitude* of an element x in P to be $a(x) = \#\{u \in P | \, u < x\} - \#\{u \in P | \, u > x\}$ A labeling of P respects *altitude* if the elements are labelled in order of weakly increasing altitude.

The first part of following lemma is given in [15].

Lemma 3.1. *If P is an n element poset, and A its the antiadjacency matrix corresponding to an altitude-respecting labeling of P. Then P is a unit interval order if and only if the zero entries of A form a Ferrers shape λ in the upper right corner of A.*

Furthermore, in this situation P is a complete self-dual poset (i.e. P is a symmetric interval order) if and only if λ is anti-diagonal symmetric.

Let x_i be the element of P with label i.

Proof. For the first equivalence, first we assume P is a unit interval order. We show the zero entries of A form a Ferrers shape in the upper right corner of A. For a contradiction assume the zero entries of A do not form a Ferrers shape in the upper right corner of A. Then we must have i, j with $i <_P j$ and at least one of the following
(i) there is k such that $j < k$ and $i \nless_P k$
(ii) there is l such that $l < i$ and $l \nless_P j$

If (i) holds then $j \nless_P k$, because otherwise $i <_P j <_P k$ implies $i <_P k$, and (i) also implies $a(x_i) < a(x_j) \leq a(x_k)$, so $i \ngtr_P j$ and $i \ngtr_P k$. That shows k is incomparable to i and j. Altitude of x_k is bigger than the altitude of x_i and bigger than or equal to the altitude of x_j, so there exists at least one element l such that either $l <_P k$ and $l \nless_P k$ or $l >_P j$ and $l \ngtr_P k$. These last two requirements imply P has a **(2+2)** or **(3+1)**, respectively. Case (ii) also implies the existence of a **(2+2)** or **(3+1)** with the same kind of argument. That shows if P is a unit interval order then the zero entries of A form a Ferrers shape in the upper right corner of A.

For the converse statement, assume for a contradiction that P is not a unit interval order, so P has an induced **(2+2)** or **(3+1)**. A $(2+2)$ implies that there exist k, l, m, n such that in the antiadjacency matrix a_{ij}, we have $a_{kl} = 0$, $a_{mn} = 0$, but $a_{kl} = 0$, $a_{mn} = 0$. This says the zero entries of A do not form a Ferrers shape in the upper right corner of A. A $(3+1)$ implies that there exist k, l, m, n such that in the antiadjacency matrix a_{ij}, we have $a_{kl} = 0$, $a_{lm} = 0$, $a_{km} = 0$, but $a_{kn} = 1$, $a_{ln} = 1$, $a_{mn} = 1$. We may have $l < i$ or $i < l < j$ or $j < l < k$ or $k < l$. In each case a simple argument shows that the inequality implies the zero entries of A do not form a Ferrers shape in the upper right corner of A. So if the zero entries of A form a Ferrers shape in the upper right corner of A then P is a unit interval order.

For the second equivalence, first assume P is a complete self-dual poset. WLOG, we can choose an altitude-respecting label so that $|x_i| = |x_{2n+1-i}|$. Now, we see that self-duality of P implies the anti-diagonal symmetry of λ. Conversely, anti-diagonal symmetry of λ, together with the fact that λ is a Ferrers shape forces P to be self-dual and complete. $\square$

We do not have an exact formula for the number of interval orders on n elements, but for the unit interval orders, we have the following known result.

Corollary 3.2. *The number of unit interval orders on $[n]$ up to isomorphism is the Catalan number $\frac{1}{n+1}\binom{2n}{n}$.*

Conjecture 2.6 together with the second assertion of Lemma 3.1 would imply

Corollary 3.3. *The number of unit symmetric interval orders on $\pm[n]$ up to signed poset isomorphisms is $\binom{2n}{n}$.*

Proof. Anti-diagonal symmetric Ferrers shapes correspond to self-conjugate Ferrers shapes inside the staircase $\delta_{2n-1} = (2n-1, 2n-2, ..., 1)$, and these are in one-to-one correspondence with the lattice paths $(0,0) \to (2n,k)$ for any k staying weakly above x-axis with the allowed steps northeast and southeast, e.g., each step consists of moving one unit to the right and then moving one unit up or down. The boundary of a Ferrers shape, after rotating 45 degrees clockwise, gives the steps of the corresponding lattice. All paths $(0,1) \to (2n, \geq 1)$ are counted by $\sum_{k=0}^{n} \binom{2n}{k}$. The paths crossing the x-axis, by reflection principle, are in one-to-one correspondence with the lattice paths $(0,-1) \to (2n, \geq 1)$ and these are counted by $\sum_{k=0}^{n-1} \binom{2n}{k}$. So the paths $(0,0) \to (2n,k)$ for any k, staying weakly above x-axis are counted by the difference which is$\binom{2n}{n}$. $\square$

4. Questions for the thesis

4.1. Proving conjecture 2.6 is high on the list of things to-do.

4.2. Chain Polynomial and Real Roots. Let

$$f_p(t) = 1 + c_1 t + c_2 t^2 + ... + c_d t^d$$

be the chain polynomial or f-polynomial of a poset P, where c_i is the number of i-element chains in P, and let A be the antiadjacency matrix defined earlier for a poset P. Stanley's path counting theorem [15] says

$$f_P(t) = \det(At + I)$$

In [15], Skandera proves that when P is a unit interval order, the antiadjacency matrix A has only real eigenvalues. So $f_p(t)$ has only real zeros.

In a recent paper [6], Chudnovsky and Seymour showed that the stable set polynomial of a graph G has all real roots when G is clawfree. If we translate this into poset language via the incomparability graph of a poset, stable sets correspond to chains and clawfreeness to **(3+1)**-freeness. Then Chudnovsky and Seymour's result, improving the result mentioned above, says for a **(3+1)**-free poset P, $f_p(t)$ has only real zeros.

We first want to define two versions of a signed chain polynomial

$$f_S^{Signed}(t) = 1 + c_0^{Signed} t + ... + c_k^{Signed} t^d$$

both of which generalize $f_Q(t)$ above via the $Q \to S_+(Q)$ construction defined earlier.

The first version of c_k^{Signed} counts the number of symmetric k-chains which are $2k$-element chains in P(S) invariant under $i \to -i$. We see that, in this version, $f_{S_+(Q)}^{Signed}(t) = f_Q(t)$.

As a second version, Fisher defines a signed chain as an isotropic chain in P(S), that is one which includes at most one of each $\pm i$. Fisher and Hanlon uses this to associate a simplicial complex to a signed poset. Then their version of c_k^{Signed} counts the number of $(k-1)$-dimensional simplices inthis complex. In this version, we have $f_{S_+(Q)}^{Signed}(t) = f_Q(2t)$.

We want to prove $f_S^{Signed}(t)$ has only real roots for symmetric interval orders S with either or both versions of c_k^{Signed}. We also want to find a Type B analogue of Chudnovsky and Seymour's result for our signed chain polynomial in either or both versions.

4.3. Order Dimension. The order dimension of a poset P on the ground set $[n]$ is the least d such that intersection of d linear orders on $[n]$ gives P. Similarly, one can define the *signed order dimension* of a signed poset S by using P(S) on $\pm[n]$ and requiring the linear orders on $\pm[n]$ to be complete self-dual. Rabinovitch showed that the dimension of a unit interval order is at most three [11]. He also showed that an interval order P has $dim(P) \leq 1 + height(P)$[10]. We want to find similar upper bounds for signed order dimension of (unit) symmetric interval orders.

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