# Structural Aspects of Differential Posets

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There are many people that have earned my gratitude for their contribution to my time in graduate school.

# Dedication

To those who held me up over the years

**Abstract**

This thesis investigates some structural properties of differential posets and answers some open questions. There are three main results.

First, a proof is given that the largest rank size of the $n$th rank of an $r$-differential poset is given by the $n$th rank of the Fibonacci $r$-differential poset. This solves a question of Stanley.

Second, a proof is given that the only 1-differential lattices are Young's lattice and the Fibonacci 1-differential poset. This also solves a question of Stanley.

Third, it is shown that any quantized $r$-differential poset has the Fibonacci $r$-differential poset as its underlying $r$-differential poset. This negatively answers a question of Lam.

Further, a method for computing all partial 1-differential posets up to a given rank is described. The results of this computation up to rank 10 are also included.

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# Chapter 1

# Introduction

Differential posets are combinatorial objects that were introduced independently by Stanley [1] and Fomin [2] (actually, Fomin introduced dual graded graphs which are a more general concept). A poset is *$r$-differential* if it is locally finite, has a $\hat{0}$, and its Hasse diagram exhibits the following two properties.

- There is a 2 edge path that goes down from a vertex and then up to another vertex if and only if there is a 2 edge path that goes up from the first vertex and then down to the second vertex.
- Each vertex has $r$ more edges going up than going down.

A picture of the bottom 4 ranks of a 1-differential poset is given in Figure 1.1.

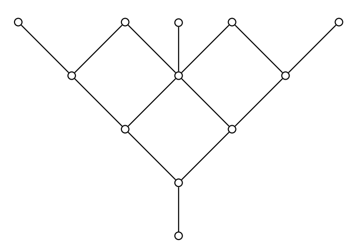


Figure 1.1: Bottom 4 ranks of a 1-differential poset

It turns out that the structure of the bottom 4 ranks of a 1-differential poset is forced by the definition. After that there are options to create different 1-differential posets. For instance, ranks 3 through 5 could look like either Figure 1.2 or Figure 1.3.

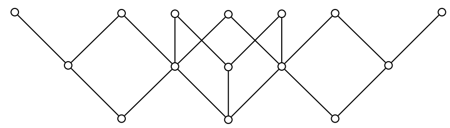


Figure 1.2: Ranks 3 through 5 of $Y$

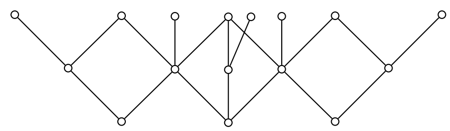


Figure 1.3: Ranks 3 through 5 of $Z(1)$

Surprisingly, this simple definition leads to relatively simple questions that appear hard to answer. An example of this is:

**Question 1.1.** *What is the fewest possible number of vertices in the nth rank of a* 1-*differential poset?*

Question 1.1 was actually asked in Stanley's original paper [1]. The conjecture is still open even though the numbers in question are conjectured to be well known numbers (specifically, the number of partitions of $n$).

Besides being inherently mathematically interesting, the study of differential posets also has other motivations. One of the initial motivations was that differential posets are a generalization of Young's lattice. It turns out that many of the combinatorial properties of Young's lattice are true of all differential posets. So, one can hope to obtain simpler proofs and/or a better understanding of some of the properties of Young's lattice by only considering Young's lattice as a differential poset. From this point of view, the hope is that the definition of differential posets encapsulates some of the "important" properties of Young's lattice. Further, there is some hope that new properties of Young's lattice may be found by only focusing on the properties of differential posets. An example of this is Proposition 3.1 in [1].

Another motivation for studying differential posets is based on certain operators $U$ and $D$ defined on certain vector spaces. The defining equation for differential posets turns out to be $DU - UD = rI$. This equation seems fairly natural to consider since it is a commutator relation. Further, it turns out that equations with the same or similar form to $DU - UD = rI$ have been studied in other contexts. So, one may hope to shed light on these situations by studying differential posets. One particular example is in representation theory where $U$ can be thought of as induction on representations and $D$ as restriction on representations. Another area where such equations have been used is in studying convergence rates of Markov chains [3].

In this thesis, some of the structural properties of differential posets are investigated. Much of the existing work on differential posets has focused on algebraic techniques. This thesis will instead use more elementary and straightforward techniques. Not only has this approach seemed to be productive, but the hope is that the results will also be more accessible.

This thesis contains three main theorems.

The first result deals with the largest possible rank sizes of $r$-differential posets. The statement of the result will use $r$-Fibonacci numbers which are recursively defined by $F_r(0) = 1$, $F_r(1) = r$, and $F_r(n) = F_r(n-2) + rF_r(n-1)$. Note that the 1-Fibonacci numbers are the usual Fibonacci numbers.

**Theorem 1.2.** *Let $p_0, p_1, \ldots$ be the rank sizes of an $r$-differential poset. Then, $p_i \leq F_r(i)$.*

Theorem 1.2 was first conjectured by Stanley in [1] (see Problem 6 in Section 6).

The second result deals with which 1-differential posets are lattices. Recall that a lattice is a poset such that greatest lower bounds and least upper bounds are defined.

**Theorem 1.3.** *The only 1-differential lattices are Young's lattice and the 1-differential Fibonacci poset.*

Theorem 1.3 answers a question of Stanley in [1] (see Problem 2 in Section 6). It should be noted that a proof of Theorem 1.3 appears in [4]. However, this proof is incorrect.

The third result deals with quantized differential posets which are a generalization of differential posets introduced by Thomas Lam [5].

**Theorem 1.4.** *Every quantized $r$-differential poset has the the $r$-differential Fibonacci poset as its underlying differential poset.*

Theorem 1.4 resolves a question of Lam (see Problem 1 in [6]). Actually, Lam asked for a quantization of Young's lattice. Theorem 1.4 shows that no such quantization exists.

## 1.1 Summary of Chapter 2

Chapter 2 provides information on background material. This includes partitions, posets, and graphs.

## 1.2 Summary of Chapter 3

In this chapter three equivalent definitions of differential posets are introduced. One of the definitions is new, but is really just a different way of looking at an existing definition. This chapter also discusses some basic terminology related to differential posets. Young's lattice and the Fibonacci 1-differential poset are also introduced.

## 1.3 Summary of Chapter 4

This chapter discusses an algorithm that has been used to find all partial 1-differential posets up to rank ten. The chapter starts with a discussion of how the algorithm works. Next is a discussion about the amount of time needed to run the algorithm (note that there are not proven asymptotic running times, only actual running times). Then, the number of differential posets found at each rank is discussed. The chapter concludes with a discussion about some of the uses that have been found for a library of partial 1-differential posets.

## 1.4 Summary of Chapter 5

Chapter 5 discusses results dealing with the rank sizes of differential posets. The primary result in this chapter is a proof of Theorem 1.2. Known results dealing with rank sizes of differential posets are also covered. Further, the chapter includes a discussion about conjectures dealing with rank sizes of differential posets and to what extent the data from Chapter 4 support these conjectures.

## 1.5 Summary of Chapter 6

Chapter 6 examines under what conditions a differential poset is a lattice. The primary result in this section is Theorem 1.3. Finally, what is known about the existence of $r$-differential lattices for $r > 1$ is discussed.

## 1.6 Summary of Chapter 7

Chapter 7 examines various generalizations and variations of differential posets. These include sequential differential posets, dual-graded graphs, signed differential posets, and quantized differential posets. The primary result in this section is Theorem 1.4.

# Chapter 2

# Background

In this chapter some of the basic combinatorial objects that will be used in this thesis are covered.

## 2.1 Partitions

A *partition* of $n$ is a vector of non-increasing integers that add up to $n$. More formally,

**Definition 2.1.** *A partition of $n$ is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathbb{N}^k_{>0}$ such that $\lambda_i \geq \lambda_{i+1}$ and $\sum_{i=1}^{k} \lambda_i = n$. One says that $\lambda$ has $k$ parts.*

Sometimes one writes $(\lambda_1, \ldots, \lambda_k)$ as $\lambda_1 + \ldots + \lambda_k$. Other times one writes $\lambda = (\lambda_1, \ldots, \lambda_k)$ as $b_1^{e_1} b_2^{e_2} \ldots b_t^{e_t}$ where $\lambda$ begins with $e_1$ entries equal to $b_1$ followed by $e_2$ entries of $b_2$ and so on.

As an example, the 7 partitions of 5 using the three notations are:

- $(5) = 5 = 5$
- $(4, 1) = 4 + 1 = 41$
- $(3, 2) = 3 + 2 = 32$
- $(3, 1, 1) = 3 + 1 + 1 = 31^2$
- $(2, 2, 1) = 2 + 2 + 1 = 2^2 1$

- $(2,1,1,1) = 2+1+1+1 = 21^3$
- $(1,1,1,1,1) = 1+1+1+1+1 = 1^5$

Often one wishes to add or subtract 1 from one of the entries in the sequence defining a partition. For this, one can use the following notation.

**Definition 2.2.** *Let $\lambda = (\lambda_1, \dots, \lambda_k)$ be a partition. Let $\lambda^{a+}$ be the vector defined as $\lambda^{a+} = (\mu_1, \dots, \mu_k)$ where $\mu_i = \lambda_i$ for $i \neq a$ and $\mu_a = \lambda_a + 1$. Similarly, let $\lambda^{b-}$ be the vector defined as $\lambda^{a-} = (\tau_1, \dots, \tau_k)$ where $\tau_i = \lambda_i$ for $i \neq a$ and $\tau_a = \lambda_a - 1$.*

In short, $\lambda^{a+}$ means 1 is added to the $a$th entry of $\lambda$ and $\lambda^{b-}$ means 1 is subtracted from the $a$th entry of $\lambda$. Similarly, one can use notation such as $\lambda^{a+,c+,b-}$ to be the vector that results from adding 1 to the $a$th and $c$th entries of $\lambda$ and subtracting 1 from the $b$th entry of $\lambda$. Some examples:

- $(5,4,4,3,2,1)^{2+} = (5,5,4,3,2,1)$
- $(5,4,4,3,2,1)^{4-} = (5,4,4,2,2,1)$
- $(5,4,4,3,2,1)^{2+,4-,6+} = (5,5,4,2,2,2)$

## 2.2 Graphs

Let $V$ be a finite set. Let $E$ be a set of subsets of $V$ all of which have cardinality 2. Then, $G = (V, E)$ will be called a *graph*. An element of $V$ is called a *vertex*. An element of $E$ is called an *edge*.

Let $v \in V$. The *degree* of $v$ will be $|\{e \in E \mid v \in e\}|$. The degree of $v$ will be denoted $d_G(v)$ or just $d(v)$ if the underlying $G$ is understood by context. If $w$ is another element of $V$, then $v$ and $w$ are *adjacent* if $\{v, w\} \in E$.

Further information of graphs can be found in [7].

## 2.3 Posets

A poset is a partially ordered set. The idea is to impose an order on some set of elements, but not require that every pair of elements is comparable. More formally,

**Definition 2.3.** *A poset is a pair $P = (A, B)$ where $A$ is a set and $B \subset A \times A$ such that:*

- *$(A, B)$ is reflexive: $(x, x) \in B$ for all $x \in A$.*
- *$(A, B)$ is antisymmetric: if $(x, y) \in B$ with $x \neq y$, then $(y, x) \notin B$.*
- *$(A, B)$ is transitive: if $(x, y) \in B$ and $(y, z) \in B$, then $(x, z) \in B$.*

Given a poset $P = (A, B)$, $(x, y) \in B$ will often be denoted as $x \leq_P y$. In this way a poset gives a $\leq$ comparison on $A$ that behaves in the way one expects. The only potential exception is that it is possible for $x, y \in A$ to be *incomparable*, which means that neither $x \leq_P y$ nor $y \leq_P x$.

One can say that $x$ *covers* $y$ in $P$ if $x \neq y$ and $x \geq z \geq y$ implies that either $z = x$ or $z = y$. Sometimes the notation $x \gtrdot y$ will be used for "$x$ covers $y''$. Further, let $d_P(x) = \{y \in P \mid x \gtrdot y\}$.

One can say that $x$ *is covered by* $y$ in $P$ if $x \neq y$ and $x \leq z \leq y$ implies that either $z = x$ or $z = y$. Sometimes the notation $x \lessdot y$ will be used for "$x$ is covered by $y$". Further, let $u_P(x) = \{y \in P \mid x \lessdot y\}$.

The *Hasse diagram* of a poset is a pictorial way of looking at a poset. A Hasse diagram of a poset $P$ is a realization of a graph where there is a vertex for each element of $P$. If $x <_P y$, then $y$ is drawn above $x$. Also, if $x$ is covered by $y$, then there is an edge $(x, y)$ in the graph. An example is Figure 2.1 where the cover relations are $x \lessdot z$, $x \lessdot w$, and $y \lessdot z$. Further note that in Figure 2.1 $y$ and $w$ are incomparable (despite $w$ being drawn above $y$).

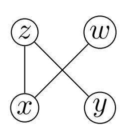



Figure 2.1: Hasse diagram example

Let $P$ be a poset. For $x, y \in P$ the *greatest lower bound* or *meet* of $x$ and $y$ is an element $z$ such that if $w \leq x$ and $w \leq y$, then $z \geq w$. Similarly, the *least upper bound* or *join* of $x$ and $y$ is an element $r$ such that if $s \geq x$ and $s \geq y$, then $r \leq s$. Note that meets and joins do not always exist (there may be maximal lower bounds that are incomparable, for instance). If $P$ is a poset such that the meet of $x$ and $y$ and the join of $x$ and $y$ exist for all $x$ and $y$, then $P$ is called a *lattice*. If $P$ is a poset such that the meet of $x$ and $y$ exists for all $x$ and $y$ (but possibly joins do not exist), then $P$ is called a *meet semi-lattice*.

Further information on posets can be found in Chapter 3 of [8].

# Chapter 3

# Definitions

In this chapter differential posets are defined in three (equivalent) ways.

## 3.1 Cover Definition of Differential Posets

The first definition of differential posets is based directly on the cover relations of the poset in question.

**Definition 3.1.** *An* $r$-differential poset *is a locally finite poset with a* $\hat{0}$ *such that:*

1. *If* $x$ *covers* $k$ *elements, then* $x$ *is covered by* $k+r$ *elements.*
2. *If* $x \neq y$ *are vertices that both cover exactly* $t$ *vertices, then there are exactly* $t$ *vertices that cover both* $x$ *and* $y$*.*

Stanley made the observation that in Definition 3.1 $t$ must be either 0 or 1 [1].

Figure 3.1 is an example of the bottom 6 ranks of a 1-differential poset.

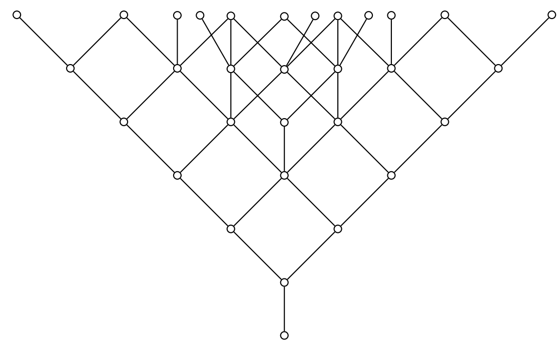


Figure 3.1: Bottom 6 ranks of a 1-differential poset

Figure 3.2 is an example of the bottom 3 ranks of a 2-differential poset.

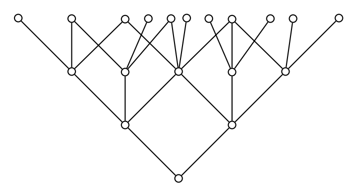


Figure 3.2: Bottom 3 ranks of a 2-differential poset

Throughout this thesis "differential posets" will mean "$r$-differential posets". Generally, this will be done in a setting where the $r$ is arbitrary. It should be noted that Definition 3.1 is slightly different than what is usually found in the literature. In the literature, differential posets are generally defined to be graded. But, it turns out that Definition 3.1 implies that the poset in question is graded.

**Proposition 3.2.** *All $r$-differential posets are graded.*

*Proof.* Let $P$ be an $r$-differential poset. Choose $a \in P$ so that $\hat{0}, x_1, x_2, \ldots, x_n, a$ and $\hat{0}, y_1, y_2, \ldots, y_k, a$ are maximal chains in $P$ such that $n > k$ and $k$ is minimal amongst all such possible choices. Since $a$ covers both $x_n$ and $y_k$, there exists $b \in P$ with $x_n$ and $y_k$ covering $b$.

Let $\hat{0}, z_1, z_2, \ldots, z_t, b$ be a maximal chain from $\hat{0}$ to $b$. A picture of the situation is given in Figure 3.3.

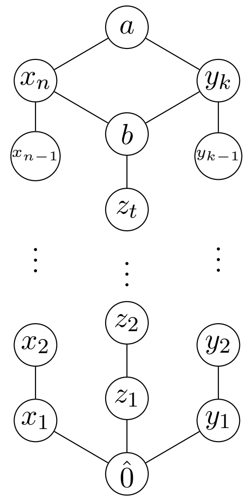



Figure 3.3: Picture for proof of Proposition 3.2

If $t \neq k-2$, then $\hat{0}, z_1, z_2, \ldots, z_t, b, y_k$ is a maximal chain from $\hat{0}$ to $y_k$ that is of a different length than $\hat{0}, y_1, y_2, \ldots, y_k$. This cannot happen since $k$ was chosen to be minimal. So, $t = k-2$.

But that means that $\hat{0}, z_1, z_2, \ldots, z_t, b, x_n$ and $\hat{0}, x_1, x_2, \ldots, x_n$ are maximal chains from $\hat{0}$ to $x_n$ of different lengths with the shorter chain of length less than $k$. This is a contradiction.

So, $a \in P$ cannot be chosen with maximal chains from $\hat{0}$ to $a$ of different length. That is, $P$ must be graded. $\square$

There are two examples of 1-differential posets that have received attention in the literature. The most well known is Young's lattice.

### 3.1.1 Young's Lattice

Young's lattice is a frequently studied combinatorial object that is sometimes denoted as $Y$. $Y$ is a poset defined on the set of all partitions. In $Y$ one has that $\vec{\lambda} = (\lambda_1, \ldots, \lambda_{t_a}) \leq \vec{\mu} = (\mu_1, \ldots, \mu_{t_b})$ if and only if $t_a \geq t_b$ and $\lambda_i \geq \mu_i$ for all $1 \leq i \leq t_b$. Figure 3.4 is a picture of the bottom 9 ranks of $Y$:

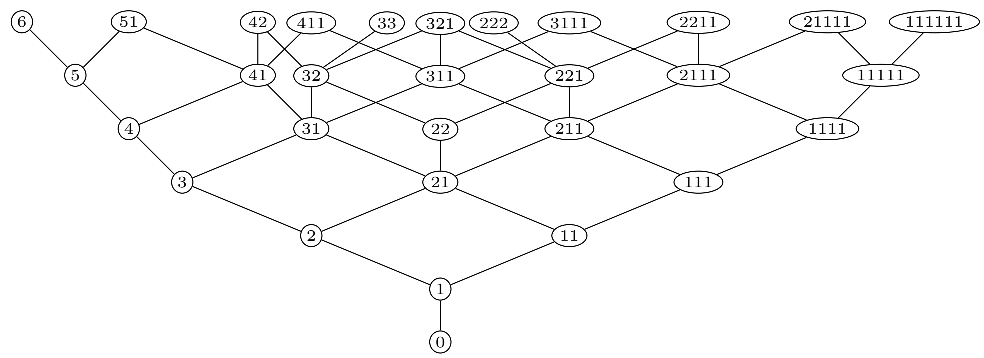



Figure 3.4: Bottom 9 ranks of Young's lattice

$Y$ has many nice properties which is a reason that $Y$ has been well studied. Some of the properties of Young's lattice are the following.

- Young's lattice is ranked. The rank of a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ is $|\lambda| = \sum_{i=1}^{n} \lambda_i$.
- Young's lattice is locally finite.
- Young's lattice is a 1-differential poset.
- Young's lattice is, in fact, a lattice. Given two partitions $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mu = (\mu_1, \ldots, \mu_k)$ with $n \geq k$, the least upper bound (or join) of $\lambda$ and $\mu$ is $\tau = (\tau_1, \ldots, \tau_n)$ where $\tau_i = \max\{\lambda_i, \mu_i\}$ for $1 \leq i \leq k$ and $\tau_i = \lambda_i$ for $i > k$. Likewise the greatest lower bound (or meet) of $\lambda$ and $\mu$ is $\gamma = (\gamma_1, \ldots, \gamma_k)$ where $\gamma_i = \min\{\lambda_i, \mu_i\}$ for $1 \leq i \leq k$.
- Young's lattice is distributive. In fact, every $r$-differential distributive lattice is isomorphic to $Y^r$. See Proposition 5.5 in [1].
- Young's lattice plays a role in the representation theory of the symmetric group. In particular, the irreducible characters of $\Sigma_n$ can be indexed by elements of $Y_n$. If $x$ is an element of $Y$, the character associated to $x$ will be written $\chi_x$. Further, it turns out that:

$$\chi_\lambda \uparrow_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} = \sum_{\mu \gtrdot \lambda} \chi_\mu$$

and

$$\chi_\lambda \downarrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} = \sum_{\mu \lessdot \lambda} \chi_\mu$$

- There is a well known bijection, known as the Robinson-Schensted correspondence, between pairs of saturated chains of length $n$ staring at $\hat{0}$ in Young's lattice and permutations on $n$ letters.

### 3.1.2 Fibonacci Differential Poset

The Fibonacci $r$-differential poset is a less often studied combinatorial object.

**Definition 3.3.** *The* Fibonacci $r$-differential poset*, denoted $Z(r)$, is the poset whose ground set is $\{1_1, 1_2, \ldots, 1_r, 2\}^*$ and where $x = x_1x_2 \ldots x_t$ is covered by $y = y_1y_2 \ldots y_k$ if and only if one of the following is true:*

- *$k = t$ and there exists $a$ with $x_i = y_i$ for $i \neq a$, $x_i = 2$ for $i < a$, $x_a \in \{1_1, \ldots, 1_r\}$, and $y_a = 2$.*
- *$k = t + 1$ and there exists $a$ with $x_i = y_i = 2$ for $i < a$, $y_a \in \{1_1, \ldots, 1_r\}$, and $x_j = y_{j+1}$ for $j \geq a$.*

Less formally, $x$ is covered by $y$ if either:

- $y$ is formed from $x$ by changing the first 1 to a 2.
- $y$ is formed from $x$ by inserting a 1 somewhere before the first 1.

Thinking about the cover relations the opposite way is also sometimes useful. It turns out that $x$ is covered by $y$ if either:

- $x$ is formed from $y$ by changing any 2 before the first 1 into a 1.
- $x$ is formed from $y$ by removing the first 1.

Figure 3.5 is a picture of the bottom 6 ranks of the Fibonacci 1-differential poset:

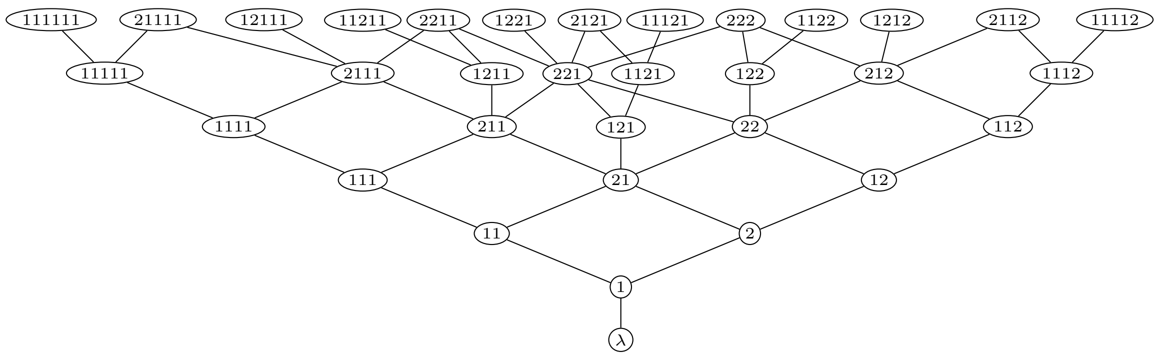



Figure 3.5: Bottom 6 ranks of the Fibonacci 1-differential poset

There is another way to build the Fibonacci $r$-differential poset first discovered by Wagner. Let $P_{[0,n]}$ be a graded poset up to rank $n$. *Reflection-extension* creates a new poset $P'_{[0,n+1]}$ where $P_{[0,n]} \cong P'_{[0,n]}$, $P'_{n+1} = \{y' \mid y \in P_{n-1}\} \cup \{x_1, \ldots, x_r \mid x \in P_n\}$, and the cover relations of $P'_{[n,n+1]}$ are given by:

- If $x \in P_n$ and $1 \leq i \leq r$, then $x \lessdot_{P'} x_i$.
- If $y \in P_{n-1}$ and $x \in P_n$ with $x \lessdot_P y$, then $y \lessdot_{P'} y'$.

The Fibonacci $r$-differential poset can be created by starting with $P_0$ being a single element and repeatedly applying reflection-extensions. Note that given $y$ in the Fibonacci $r$-differential poset that $y'$ is created by adding a 2 to the front of $y$. Also, $x_i$ is created by adding $1_i$ to the front of $x$.

Many of the properties of Young's lattice are also true of the Fibonacci 1-differential poset. Some of these are the following.

- The Fibonacci 1-differential poset is ranked. The rank of an element $x_1 x_2 \ldots x_n$ is $\sum_{i=1}^{n} x_i$.
- The Fibonacci 1-differential poset is locally finite.
- The Fibonacci $r$-differential poset is, in fact, an $r$-differential poset.
- The Fibonacci 1-differential poset is a lattice. A proof is given in [1]. However, it does not seem that there is a known way to construct the least upper bound or

greatest lower bound of two elements of $Z(1)$. The proof in [1] only shows that least upper bounds and greatest lower bounds exist.

- The Fibonacci 1-differential poset is modular. Actually, every $r$-differential lattice must be modular. Further, any $r$-differential lattice such that every complemented interval has length at most two is isomorphic to $Z(r)$. See Proposition 1.3 and Proposition 5.4 in [1].

- In [9] Okada showed that for each $n$ there exists a semisimple algebra, $\mathfrak{F}_n$, whose irreducible representations are indexed by the elements of $Z(1)_n$. Further, if $V_x$ is the representation that corresponds to element $x \in Z(1)$

$$V_x \uparrow_{\mathfrak{F}_{n-1}}^{\mathfrak{F}_n} = \sum_{y \gtrdot x} V_y$$

and

$$V_x \downarrow_{\mathfrak{F}_{n-1}}^{\mathfrak{F}_n} = \sum_{z \lessdot x} V_z$$

- Roby showed that as for Young's lattice there is a bijection between pairs of saturated chains of length $n$ staring at $\hat{0}$ in $Z(1)$ and permutations on $n$ letters [10]. Cameron and Killpatrick showed that a similar correspondence can be extended to $Z(2)$ [11]. In a later paper, Cameron and Killpatrick showed that such a correspondence can be extended to $Z(r)$ for any $r$ [12].

### 3.1.3 Properties of differential posets

One of the first facts one tends to discover about differential posets is that the $k$ in Definition 3.1 must be either 0 or 1. Another way to state this is that a certain induced subposet is forbidden in a differential poset.

**Proposition 3.4** (Stanley [1])**.** *Let $P$ be an $r$-differential poset. Let $x, y \in P_n$. Then there do not exist two distinct elements $w$ and $z$ of $P_{n+1}$ such that $w$ and $z$ each cover both of $x$ and $y$.*

*Proof.* In short one needs to show that the poset in Figure 3.6 does not occur as an induced subposet in a differential poset.

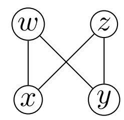


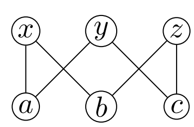

Figure 3.6: A subposet not induced in any differential poset

Suppose that the subposet in Figure 3.6 exists in an $r$-differential $P$. Choose $n$ so that $n$ is minimal amongst all such possibilities. Once $n$ is chosen, choose $x$, $y$, $z$, and $w$ as in the statement of the proposition. By Definition 3.1 there must be at least two elements, $a$ and $b$, that are covered by both $x$ and $y$. But, then $a$, $b$, $x$, and $y$ would form an induced subposet of $P$ isomorphic to Figure 3.6 that occurs at a lower rank than $n$. This contradicts the minimality of $n$. Thus, no induced subposet isomorphic to Figure 3.6 exists in $P$.

□

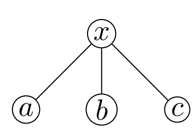
There are two induced subposets that do occur in differential posets that will be important when considering the possible rank sizes of differential posets.

**Definition 3.5.** *A* crown *is an induced subposet of the following form:*




*One can also say that* $\{x, y, z\}$ *is a crown covering* $\{a, b, c\}$*.*

**Definition 3.6.** *A* cap *is an induced subposet of the following form:*




*One can also say that* $x$ *is a cap covering* $\{a, b, c\}$*.*

The term "crown" is due to Joel Lewis in [13]. The author believes the term "cap" is new in this thesis.

Caps and crowns appear to play central roles in both of the well known examples of $r$-differential posets.

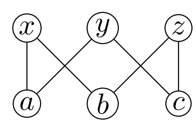
**Proposition 3.7.** *The Fibonacci $r$-differential poset does not have a crown as an induced subposet.*

*Proof.* Clearly $Z(r)_{[0,1]}$ does not have an induced crown. Since each successive $Z(r)_{[n,n+1]}$ is created by reflection-extension, the result follows. □

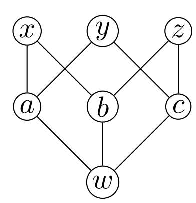
In fact, one could define $Z(r)$ as the unique $r$-differential poset that does not have an induced crown. Young's lattice turns out to have a property that is somewhat opposite.

**Proposition 3.8.** *In Young's lattice every crown is covered by a cap and every cap covers a crown.*

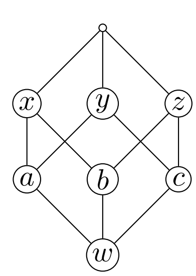
*Proof.* Suppose the following is a crown in Young's lattice with $a$, $b$, and $c$ at a minimal rank amongst all such choices.




Since the rank of $a$, $b$, and $c$ is minimal the picture must actually look like the following.




Thus, the Young diagrams corresponding to $a$, $b$, and $c$ arise from the Young diagram for $w$ by adding a single box. Since any of the boxes added to get to $a$, $b$, or $c$ could independently be added to $w$ it turns out that one must get the following induced boolean subposet.



Thus, every crown must be covered by a cap.

The same induced boolean subposet argument also shows that every cap in Young's lattice must cover a crown. □

When discussing differential posets it is often useful to talk about truncating a differential poset after $n$ ranks. Such a situation will be called an *$r$-differential poset up to rank $n$.*

**Definition 3.9.** *An* $r$-differential poset up to rank $n$ *is a graded, locally finite poset with a minimum element $\hat{0}$ and with $n$ ranks such that:*

1. *If $x \in P_{[0,n-1]}$ covers $k$ elements, then $x$ is covered by $k + r$ elements.*
2. *If $x \in P_{[0,n-1]}$ and $y \in P_{[0,n-1]}$ both cover $w$, then there exists $z$ that covers both $x$ and $y$.*

One of the primary uses of differential posets up to rank $n$ is to build differential posets rank-by-rank. Generally, this can be done by determining in what ways a differential poset up to rank $n$ can be extended to a differential poset up to rank $n+1$. Here is an example using Definition 3.9.

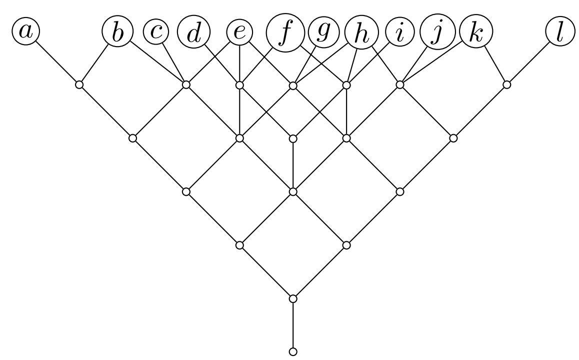



Figure 3.7: A partial 1-differential poset up to rank 6

Suppose one wishes to extend Figure 3.7 from a partial differential poset up to rank 6 to a partial differential poset up to rank 7. The first thing to consider is which nodes at rank 6 need to be covered by a common element at rank 7. The following is a list of the common covers. A set of size greater than 2 means that each pair of elements in the set must be covered by a common element at rank 7.

- $\{a, b\}$
- $\{b, c, e\}$
- $\{d, e, f\}$
- $\{e, g, h\}$
- $\{f, h, i\}$
- $\{h, j, k\}$
- $\{k, l\}$

Each set of size 2 must be covered by a node at rank 7 that covers no other elements. For a set of size greater than 2, choices can be made. The only requirements are that each pair in a set must be covered by exactly one common element and that each node at rank 6 is covered by exactly 1 (or $r$) more nodes than it covers. Two possible choices for how to deal with the sets $\{b, c, e\}$ and $\{h, j, k\}$ are given in Figure 3.8. Note that $\{b, c, e\}$ is covered by a cap and $\{h, j, k\}$ is covered by a crown.

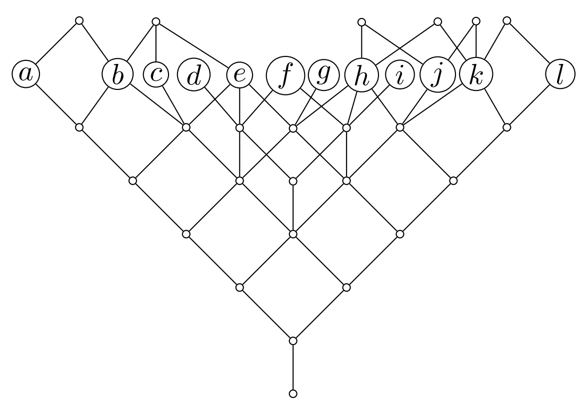



Figure 3.8: Building toward a partial 1-differential poset up to rank 7

One can then continue for the other sets of size 3 with the only requirement to be sure that the number of potential covers of any element of rank 6 is not exceeded. One possibility for the new extension is as in Figure 3.9.

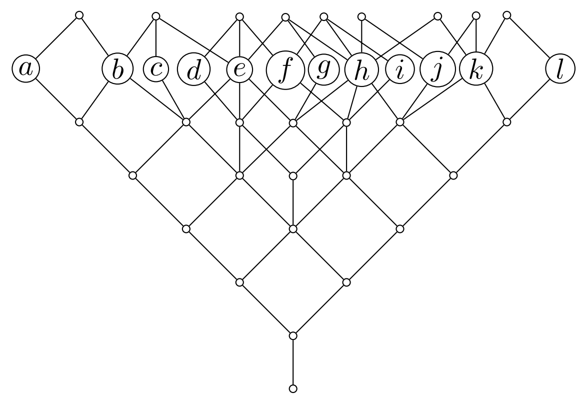



Figure 3.9: Building toward a partial 1-differential poset up to rank 7

The last step to extend from a partial 1-differential poset up to rank 6 to a partial 1-differential poset up to rank 7 is to possibly add one or more vertices covering each of the vertices at rank 6. Such a vertex is only added if the vertex at rank 6 does not already have 1 more vertex covering it than it covers. In Figure 3.9 this means a vertex needs to be added to cover each of $a$, $b$, $c$, $d$, $e$, $f$, $g$, $i$, and $l$. The finished extension is shown in Figure 3.10.

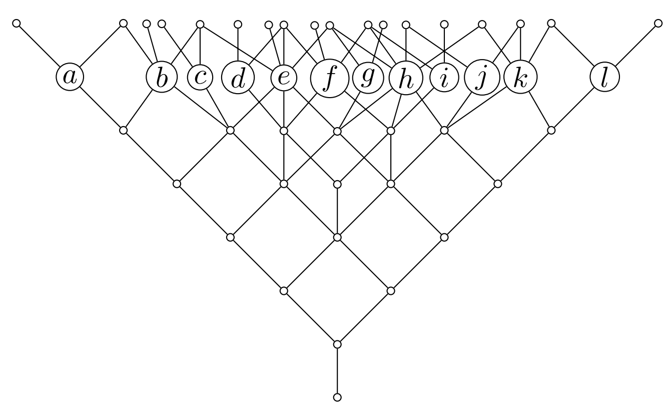



Figure 3.10: Completed partial 1-differential poset up to rank 7

It turns out that there are at least two other ways that differential posets can be defined. The next two sections will discuss the other definitions.

## 3.2 $U$, $D$ Definition of Differential Posets

One such definition involves what are referred to as "up maps" and "down maps" of posets.

Let $(P, <_P)$ be a poset such that $\mathcal{U}(x) = \{y \in P \mid y \text{ covers } x\}$ and $\mathcal{D}(x) = \{y \in P \mid y \text{ covered by } x\}$ are both finite. Let $\mathbb{C}P$ be the complex vector space whose basis is $P$. Define linear maps $U$, $D$, and $I$ on $\mathbb{C}P$ by letting $U$, $D$, and $I$ act on $x \in P$ by:

$$Ux = \sum_{y \gtrdot x} y,$$

$$Dx = \sum_{y \lessdot x} y,$$

$$Ix = x.$$

For an example, consider Figure 3.11. For Figure 3.11 some example up and down map results are as follows.

- $Ua = d + e + f$
- $Ub = e + g$
- $Dd = a$
- $Df = a + c$

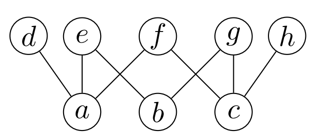



Figure 3.11: Example for up and down maps

$U$ is often called an "up map" and $D$ is often called a "down map". $I$ is the identity map. Up maps and down maps lead to an alternative way to define differential posets.

**Definition 3.10.** *Let $P$ be a locally finite poset with a $\hat{0}$. Let $\mathbb{C}P$ be the complex vector space with basis $P$. $P$ is called an $r$-*differential poset *if:*

$$DU - UD = rI.$$

One can now check that the two definitions of differential posets agree.

**Proposition 3.11** (Stanley [1])**.** *Let $P$ be a locally finite poset with a $\hat{0}$. Then the following are equivalent:*

- *$P$ is an $r$-differential poset according to Definition 3.1.*
- *$P$ is an $r$-differential poset according to Definition 3.10.*

*Proof.* A proof can be found as Theorem 2.2 in [1]. Essentially the idea is to look at the $y$ coordinate of $(DU - UD)x$ and $rIx$ for $x$ and $y$ of the same rank.

If $x \neq y$, then the $y$ coordinate of $rIx$ is 0. So for $DU - UD = rI$ to be true, the $y$ coordinates of $DUx$ and $UDx$ must be the same. That is if $x$ and $y$ both cover $k$ elements, then $x$ and $y$ are both covered by $k$ elements.

If $x = y$, then the $y$ coordinate of $rIx$ is $r$. The $y$ coordinate of $(DU-UD)x$ is then the difference between the number of elements covering $x$ and the number of elements covered by $x$.

□

Differential posets as defined in Definition 3.10 are a somewhat natural object to study because the defining equation, $DU - UD = rI$ is a commutator relation. For a more in depth study of similar relations see [14].

It should be noted that Definition 3.10 is the reason behind differential posets being called "differential". This is because the $D$ operator acts as a derivative from a certain point of view. See [1] for details.

## 3.3 Hypergraph Definition of Differential Posets

A third way to define differential posets involves hypergraphs. A review of hypergraphs is in order before the differential poset definition.

### 3.3.1 Hypergraphs

Roughly speaking, a hypergraph is a graph where the edges are allowed to be any size.

Let $V$ be a finite set. Let $E$ be a set of subsets of $V$ (of any size). Then, $G = (V, E)$ will be called a *hypergraph*. An element of $V$ is called a *vertex*. An element of $E$ is called an *edge*. Edges with exactly one element are called *loops*. Note that graphs are hypergraphs where each edge has cardinality two. Figure 3.12 gives a picture of a hypergraph. Shaded regions denote edges. Rectangular vertices denote that the vertex is a loop.

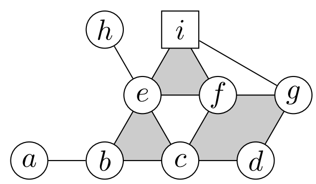



Figure 3.12: $G$, an example of a hypergraph

Most of the terminology for graphs carries through to hypergraphs. Let $G = (V, E)$ be a hypergraph. Let $v \in V$. The *degree* of $v$ will be $|\{e \in E \mid v \in e\}|$. The degree of $v$ will be denoted $d_G(v)$ or just $d(v)$ if the underlying $G$ is understood by context. In Figure 3.12, $d_G(c) = 2$ and $d_G(i) = 3$. If $w$ is another element of $V$, then $v$ and $w$ are *adjacent* if there exists $e \in E$ with $\{v, w\} \subset e$. Note that in Figure 3.12 $d$ and $f$ are adjacent since they are both part of edge $\{c, d, f, g\}$.

The *skeleton* of a hypergraph $H$ will be the graph $G$ with the same vertex set and where $\{u, v\}$ is an edge of $G$ if and only if $u$ and $v$ are adjacent in $H$. Figure 3.13 shows the skeleton of $G$ from Figure 3.12.

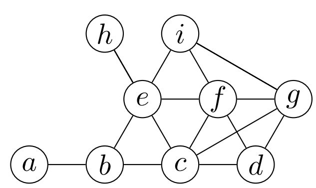



Figure 3.13: The skeleton of $G$

The *dual* of a hypergraph $G = (V, E)$ is another hypergraph that will be denoted as $G^* = (V^*, E^*)$. Here, $V^* = E$. Also, $e^* \in E^*$ if and only if there exists $v \in V$ such that $e^* = \{e \in E \mid v \in e\}$. Figure 3.14 shows the dual of the hypergraph $G$ from Figure 3.12. Table 3.1 shows how the vertices of $G^*$ correspond to the edges of $G$.

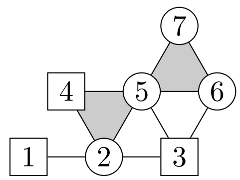



Figure 3.14: $G^*$, the dual of $G$

Table 3.1: Correspondence of vertices of $G^*$ and edges of $G$

| vertex of $G^*$ | edge of $G$ |
|---|---|
| 1 | $\{a, b\}$ |
| 2 | $\{b, c, e\}$ |
| 3 | $\{c, d, f, g\}$ |
| 4 | $\{e, h\}$ |
| 5 | $\{e, f, i\}$ |
| 6 | $\{g, i\}$ |
| 7 | $\{i\}$ |

For further details on hypergraphs see [15].

### 3.3.2 Differential posets as hypergraphs

Now a definition of differential posets using hypergraphs can be given.

**Definition 3.12.** *An* $r$-differential poset *is a pair of sequences of hypergraphs* $G_0 = (V_0, E_0), G_1 = (V_1, E_1), \ldots$ *and* $G'_0 = (V'_0, E'_0), G'_1 = (V'_1, E'_1), \ldots$ *such that:*

- $G^*_n \cong G'_{n+1}$

- $V_n = V'_n$
- *$G_n$ and $G'_n$ have the same skeleton.*
- $d_{G_n}(v) = d_{G'_n}(v) + r$
- $|V_0| = 1$
- $E'_0 = \emptyset$

Given a differential poset, $P$, by Definition 3.1, the idea behind Definition 3.12 is that $G_i$ is a hypergraph whose vertices are the elements of $P_i$ and where an element $y \in P_{i+1}$ covering elements $\{x_1, \dots, x_d\} \in P_i$ gives rise to the edge $\{x_1, \dots, x_d\}$ of $G_i$. Thus, $G_i$ is keeping track of $U_i$. Similarly, an element $z \in P_{i-1}$ covered by elements $\{w_1, \dots, w_d\} \in P_i$ gives rise to the edge $\{w_1, \dots, w_d\}$ of $G'_i$. Thus, $G'_i$ is keeping track of $D_i$. Note that one could omit either $G$ or $G'$ from the definition (since $D_i = U^t_{i-1}$). However, both are included for later notational simplicity.

An example of $G_4$ and $G'_4$ for a 1-differential poset according to Definition 3.12 is as follows. Figure 3.15 gives the Hasse diagram of $P_{[3,5]}$. $P_{[3,4]}$ determines $G'_4$ and $P_{[4,5]}$ determines $G_4$. Figure 3.16 shows $G_4$ and $G'_4$. A shaded region indicates that all of the edges in that region form an edge of the hypergraph. A rectangular node indicates that there is a loop (edge of cardinality 1) consisting of that node.

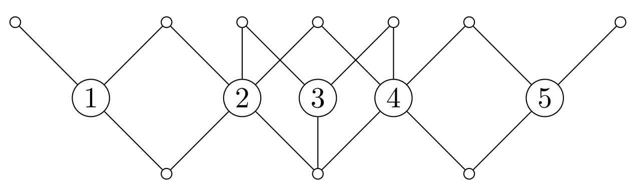
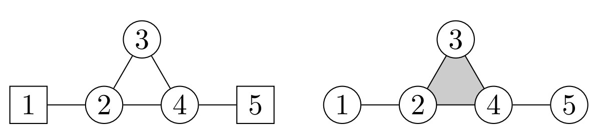



Figure 3.15: Hasse diagram of $P_{[3,5]}$




(a) $G_4$ (b) $G'_4$

Figure 3.16: $G_4$ and $G'_4$

Similarly, an example of $G_5$ and $G'_5$ for a 1-differential poset according to Definition 3.12 is as follows. Figure 3.17 gives the Hasse diagram of $P_{[4,6]}$. $P_{[4,5]}$ determines $G'_5$ and $P_{[5,6]}$ determines $G_5$. Figure 3.18 shows $G_5$ and $G'_5$. Again, a shaded region indicates that all of the edges in that region form an edge of the hypergraph. A rectangular node indicates that there is a loop (edge of cardinality 1) consisting of that node.

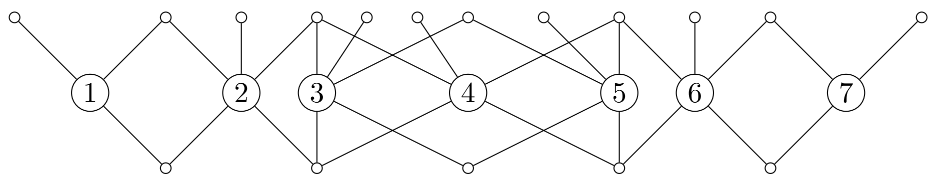



Figure 3.17: Hasse diagram of $P_{[4,6]}$

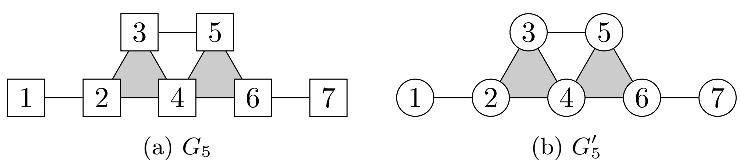



Figure 3.18: $G_5$ and $G'_5$

An example of $G_2$ and $G'_2$ for a 2-differential poset according to Definition 3.12 is as follows. Figure 3.19 gives the Hasse diagram of $P_{[1,3]}$. $P_{[1,2]}$ determines $G'_2$ and $P_{[2,3]}$ determines $G_2$. Figure 3.20 shows $G_2$ and $G'_2$. A shaded region indicates that all of the edges in that region form an edge of the hypergraph. A rectangular node indicates that there is at least one loop (edge of cardinality 1) consisting of that node.

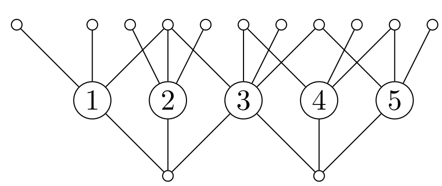
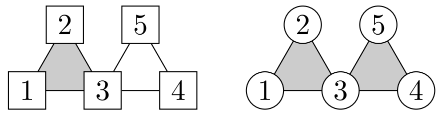



Figure 3.19: Hasse diagram of $P_{[1,3]}$




(a) $G_2$ (b) $G_2'$

Figure 3.20: $G_2$ and $G_2'$

An example of $G_3$ and $G_3'$ for a 2-differential poset according to Definition 3.12 is as follows. Figure 3.21 gives the Hasse diagram of $P_{[2,4]}$. $P_{[2,3]}$ determines $G_3'$ and $P_{[3,4]}$ determines $G_3$. Figure 3.22 shows $G_3$ and $G_3'$. A shaded region indicates that all of the edges in that region form an edge of the hypergraph. A rectangular node indicates that there is at least one loop (edge of cardinality 1) consisting of that node.

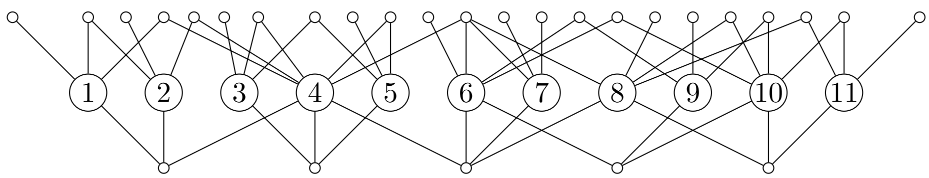
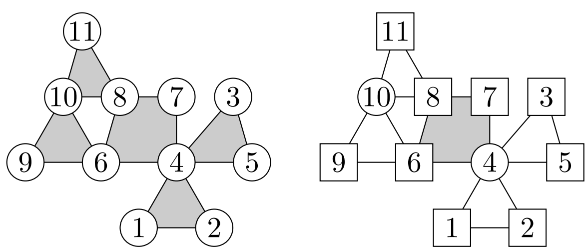



Figure 3.21: Hasse diagram of $P_{[2,4]}$




(a) $G_3$ (b) $G_3'$

Figure 3.22: $G_3'$

One can show that the three different definitions of differential posets presented in this chapter are equivalent.

**Proposition 3.13.** *Let $P$ be a locally finite poset with a $\hat{0}$. Then the following are* *equivalent:*

1. *$P$ is an $r$-differential poset according to Definition 3.1.*

2. *$P$ is an $r$-differential poset according to Definition 3.10.*

3. *$P$ is an $r$-differential poset according to Definition 3.12.*

*Proof.* 1 and 2 are equivalent by Proposition 3.11.

Let $G_0 = (V_0, E_0), G_1 = (V_1, E_1), \ldots$ and $G'_0 = (V'_0, E'_0), G'_1 = (V'_1, E'_1), \ldots$ be two sequences of linear hypergraphs that satisfy Definition 3.12.

For each $n \geq 0$, let $f_n : V(G'_{n+1}) \to V(G^*_n) = E(G_n)$ be a bijection that determines the isomorphism $G'_{n+1} \cong G^*_n$.

Build a poset to satisfy Definition 3.1 as follows. Let the ground set of the poset be $A = \bigcup_{i=0}^{\infty} V_i$. Let $x \lessdot y$ if and only if the following are true.

- $x \in V_i$ and $y \in V_{i+1}$ for some $i \geq 0$.

- $x \in f_i(y)$

Let $x$ be in $V_n$. The number of elements that cover $x$ is $|\{y \in V_{n+1} \mid x \in f_n(y)\}|$. Since $f_n$ is a bijection this means that the number of elements that cover $x$ is $d_{G_n}(x)$.

The number of elements covered by $x$ is $|\{z \in V_{n-1} \mid z \in f_{n-1}(x)\}| = |f_{n-1}(x)| = d_{G'_n}(x)$. Since $d_{G_n}(x) = d_{G'_n}(x) + r$ one gets that the number of elements that cover $x$ is $r$ more than the number of elements that $x$ covers. Thus, part 1 of Definition 3.1 is satisfied.

Suppose $x$ and $y$ are both in $V_n$. Then for $z \in V_{n-1}$:

$$\begin{aligned}
x \text{ and } y \text{ both cover } z &\Leftrightarrow z \in f_n(x) \cap f_n(y) \\
&\Leftrightarrow x \text{ and } y \text{ are adjacent in } G^* \\
&\Leftrightarrow x \text{ and } y \text{ are adjacent in } G' \\
&\Leftrightarrow x \text{ and } y \text{ are adjacent in } G \\
&\Leftrightarrow \exists e \in E_n \text{ such that } x, y \in e \\
&\Leftrightarrow \exists w \in V(G'_{n+1}) \text{ such that } x, y \in f_n(w) \\
&\Leftrightarrow \exists w \in V_{n+1} \text{ such that } w \text{ covers both } x \text{ and } y
\end{aligned}$$

Thus, part 2 of Definition 3.1 is satisfied.

Let $P$ be an $r$-differential poset by Definition 3.1. Create a sequence of linear hypergraphs $G_0 = (V_0, E_0), G_1 = (V_1, E_1), \ldots$ as follows. Let $V_i$ be the set of all elements of $P$ at rank $i$. Let $B$ be an edge in $E_i$ if and only if $B$ is exactly the set of elements that cover some element, $x$, of rank $i$. Also, create a sequence of linear hypergraphs $G_0' = (V_0', E_0'), G_1' = (V_1', E_1'), \ldots$ as follows. Let $V_i'$ be the set of all elements of $P$ at rank $i$. For $i \geq 1$ Let $B$ be an edge in $E_i'$ if and only if $B$ is exactly the elements that are covered by some element, $x$, of rank $i$. Let $E_0' = \emptyset$.

Now the conditions of Definition 3.12 need to be checked. It is clear that $V_i = V_i'$, $|V_0| = 1$, and $E_0' = \emptyset$. Two vertices of $G_n$ are adjacent if they are covered by a common vertex in $P$. Also, two vertices of $G_n'$ are adjacent if they both cover a common vertex in $P$. Thus, $G_n$ and $G_n'$ have the same skeletons.

Note that $d_{G_n}(v)$ is the number of elements that cover $v$. Also, $d_{G_n'}(v)$ is the number of elements that are covered by $v$. Thus, $d_{G_n}(v) = d_{G_n'}(v) + r$.

Finally, it needs to be shown that $G_n^* \cong G_{n+1}'$. Let $f : V(G_n^*) \to V(G_{n+1}')$ be defined as follows. Suppose $A \in V(G_n^*)$. That means, $A$ is an edge of $G_n$. So, $A$ is a subset of $V_n$ such that all elements of $A$ are covered by a common element, $x$ of $P$. Let $f_n(A) = x$. Now it needs to be shown that $f_n$ is a hypergraph isomorphism of $G_n^*$ and $G_{n+1}'$. Clearly $f_n$ is a bijection between $V(G_n^*)$ to $V(G_{n+1}')$. Suppose that $A$ and $B$ are adjacent vertices of $G_n^*$. That means that as edges of $G_n$, $A$ and $B$ both contain a common $v$ in $V(G_n)$. Thus, $f_n(A)$ and $f_n(B)$ both cover $v$. So, $f_n(A)$ and $f_n(B)$ are adjacent in $G_{n+1}'$. This process is reversible. So the proof is complete.

$\square$

The advantage of having three equivalent ways to define differential posets is that sometimes one definition is easier to work with than another. Generally speaking, Definition 3.1 will be viewed as the standard definition and will be used unless there is a reason to use another definition. Definition 3.10 will tend to be used in algebraic situations. Definition 3.12 is useful when one thinks of building differential posets and wants to visualize the structure of the bipartite graphs $P_{[n,n+1]}$ of Definition 3.1. The primary example of the usefulness of Definition 3.12 in this thesis will be Chapter 6 where the constructions involved are more difficult to understand as bipartite graphs.

As with the cover definition of differential posets, one can consider only differential posets up to a certain rank using the hypergraph definition.

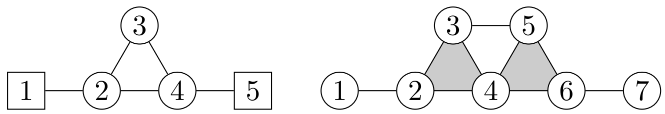
**Definition 3.14.** *A* (partial) $r$-differential poset up to rank $k$ *is a pair of sequences of hypergraphs* $G_0 = (V_0, E_0), G_1 = (V_1, E_1), \ldots, G_{n-1} = (V_{k-1}, E_{k-1})$ *and* $G'_0 = (V'_0, E'_0), G'_1 = (V'_1, E'_1), \ldots, G'_n = (V'_k, E'_k)$ *such that:*

- $G^*_n \cong G'_{n+1}$ *for* $0 \leq n \leq k-1$*.*
- $V_n = V'_n$ *for* $0 \leq n \leq k$*.*
- $G_n$ *and* $G'_n$ *have the same skeleton for* $0 \leq n \leq k-1$*.*
- $d_{G_n}(v) = d_{G'_n}(v) + r$ *for all* $v \in V_n = V'_n$*.*
- $|V_0| = 1$
- $E'_0 = \emptyset$

Building differential posets up to rank $n$ is relatively straightforward using Definition 3.9, but building a differential poset up to rank $n$ using Definition 3.14 is somewhat more complicated. Suppose $G_0 = (V_0, E_0), \ldots, G_{k-1} = (V_{k-1}, E_{k-1})$ and $G'_0 = (V'_0, E'_0), \ldots G'_k = (V'_k, E'_k)$ form a 1-differential poset up to rank $k$. One method of finding a possible extension to a 1-differential poset up to rank $k+1$ is as follows.

1. Draw the skeleton of $G_k$. This is the same as the skeleton of $G'_k$.
2. Note the degree of each vertex of $G_k$. This is known by the equation $d_{G_n}(v) = d_{G'_n}(v) + r$.
3. Find any covering of the skeleton of $G_k$ that gives the correct degrees and skeleton. Note that edges of size 1 (loops) may be used.
4. Let $G'_{k+1}$ be the dual of $G_k$.

As an example consider the potential $G_4$ and $G'_5$ shown in Figure 3.23.




(a) $G_4$ (b) $G'_5$

Figure 3.23: $G_4$ and $G'_5$

The skeleton of $G_5$ is shown in Figure 3.24.

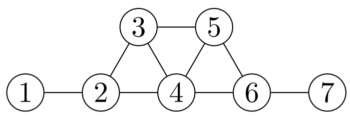
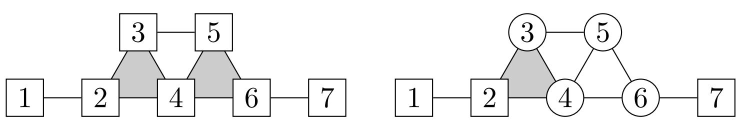



Figure 3.24: Skeleton of $G_5$

Since $d_{G'_5}(v) = 1$ for $v \in \{1, 7\}$ the covering of $G_5$'s skeleton must give vertices 1 and 7 degree 2. Since $d_{G'_5}(v) = 2$ for $v \in \{2, 3, 4, 5, 6\}$ the covering of $G_5$'s skeleton must give vertices $2, 3, 4, 5, 6$ degree 3. Two possible such coverings are given in Figure 3.25.




(a) Possible $G_5$ (b) Another possible $G_5$

Figure 3.25: Two possible $G_5$'s

All that is left to do is let $G'_6$ be the dual of the chosen $G_5$.

# Chapter 4

# Computing 1-Differential Posets

In this chapter a computation to find all partial 1-differential posets up to rank ten is described. The chapter starts with a discussion of the algorithm used. Then, the experimental running times of the algorithm are discussed. The chapter concludes with a discussion of the results of the computation and some of the ways the results have been used.

## 4.1 The algorithm

This section describes an algorithm used to compute all partial 1-differential posets up to rank ten.

The rough idea of the algorithm is to start with a partial 1-differential poset up to rank $n$, $P_{[0,n]}$. Next, find all possible extensions to $P_{[0,n+1]}$. Recursively repeating this procedure until $n = 9$ will lead to a list of all partial 1-differential posets up to rank ten.

Suppose $P_{[0,n]}$ is a partial 1-differential poset up to rank $n$. To find all possible partial 1-differential $P_{[0,n+1]}$ one can do the following:

- Build a graph $G$. The vertices of $G$ are the elements of $P_n$. Include an edge $\{v, w\}$ in $G$ if $v$ and $w$ are adjacent in $P_{[0,n]}$. To check this, just check to see if there exists a vertex $x \in P_{n-1}$ such that both $v$ and $w$ cover $x$.

- Find all complete subgraphs of $G$.

- Find all sets of complete subgraphs of $G$ that cover $G$ and have no common edges. Call such a covering set of complete graphs an *independent cover* of $G$.
- Check if each independent cover leads to a 1-differential extension of $P_{[0,n]}$ to a $P_{[0,n+1]}$.

  Given an independent cover $\{H_1, H_2, \ldots, H_t\}$ where each $H_i$ is a complete subgraph of $G$ check if $\{H_1, \ldots, H_t\}$ extends to a 1-differential $P_{[0,n+1]}$ as follows:

  – Place one element $h_i$ in $P_{n+1}$ for each $H_i$.

  – If $H_i = \{v_1, \ldots, v_k\}$, then let $h_i$ cover $v_1, \ldots, v_k$ in $P_n$.

  – For each $v \in P_n$ do the following. Let $c$ be the number of elements that $v$ covers. If $v$ is covered by more than $c+1$ elements, then $\{H_1, \ldots, H_t\}$ will not extend to a 1-differential $P_{[0,n+1]}$. Otherwise, if necessary add one or more vertices that cover $v$ so that $v$ is covered by $c+1$ elements.

Note that the algorithm was not designed to be as fast as possible. One research direction would be to examine the asymptotic running time of the algorithm. Further, the author suspects that the running time of the algorithm could be improved by simple techniques such as being more judicious in memory allocation for arrays. Hopefully, such an improvement in running time would lead to more data being collected.

## 4.2 Running times of the algorithm

Note that all partial 1-differential posets are identical up to rank 4. The running time of the algorithm for $5 \leq n \leq 10$ can be found in Table 4.1. A running time of 0 seconds means the computation took less than 1 second.

Table 4.1: Running times to find all partial 1-differential posets up to rank $n$

| $n$ | time (sec) |
|---|---|
| 5 | 0 |
| 6 | 0 |
| 7 | 0 |
| 8 | 0 |
| 9 | 27 |
| 10 | 46029 |

## 4.3 Results of the algorithm

The result of the algorithm is a list of all partial 1-differential posets up to rank $n$ for a given $n$. The posets are listed out using notation compatible with Stembridge's posets package for Maple [16].

One of the basic sets of data from the algorithm is the number of non-isomorphic partial 1-differential posets up to rank $n$. These data can be found in Table 4.2.

Table 4.2: Number of partial 1-differential posets up to rank $n$

| $n$ | Number of non-isomorphic 1-differential posets |
|---|---|
| 1 | 1 |
| 2 | 1 |
| 3 | 1 |
| 4 | 1 |
| 5 | 2 |
| 6 | 5 |
| 7 | 35 |
| 8 | 643 |
| 9 | 44,606 |
| 10 | 29,199,636 |

## 4.4 Uses of results

One of the useful aspects of having a list of all partial 1-differential posets up to a rank is the ability to check conjectures. Examples where the data from this chapter have been used as such include the following.

- In [17] Miller and Reiner study Smith normal forms for the map $DU + kI$ for a differential poset where $k$ is any integer. One question that arose was about the first differences of rank sizes of differential posets. It was asked whether $p_{n+1} - p_n \geq p_n - p_{n-1}$ for all 1-differential posets. The data from this chapter were able to show that there are 1-differential posets with $p_{n+1} - p_n < p_n - p_{n-1}$. Based on the counterexamples, it seems unclear how one would find such 1-differential posets without an extensive list of examples.

- In the proof of Theorem 1.3 one must start with a poset up to rank $n$. The proof looks at details of ranks all the way up to rank $n + 6$. One might wonder if this proof could be simplified to not consider as many ranks. However, amongst the data for 1-differential posets is a poset that is isomorphic to $Z(1)$ up to rank 5 and is a meet semi-lattice up to rank 10. This example indicates that a proof in the same spirit as that in Chapter 6 must go up to at least rank $n + 6$.

# Chapter 5

# Rank Sizes of Differential Posets

One area of inquiry into differential posets deals with rank sizes occur for differential posets. In this chapter what is known about the rank sizes is discussed. The main result will be a proof of Theorem 1.2.

Perhaps the first result dealing with the rank sizes of differential posets was Stanley's Corollary 4.3 in [1]. The theorem is restated here:

**Theorem 5.1** (Stanley [1]). *Let $p_0, p_1, \ldots$ be the rank sizes of an $r$-differential poset. Then, $p_i \geq p_{i-1}$ for all $i \geq 1$.*

In order to prove Theorem 5.1, Stanley used the following proposition (in slightly different language).

**Proposition 5.2** (Stanley [1]). *Let $P$ be an $r$-differential poset. Then, $ri$ is an eigenvalue of $UD_n$ with multiplicity $\Delta p_{n-i} = p_{n-i} - p_{n-i-1}$.*

Recently, Miller was able to extend Theorem 5.1 to the following theorem [18].

**Theorem 5.3** (Miller [18]). *Let $p_0, p_1, \ldots$ be the rank sizes of an $r$-differential poset. Then, $p_i > p_{i-1}$ for all $i \geq 1$ except when $r = i = 1$.*

One of the remarkable things about Theorems 5.1 and 5.3 is that the data on 1-differential posets seem to indicate that the rank sizes are growing much more quickly. From the data for 1-differential posets it seems likely that for any constant $A$ there exists $N$ such that $p_n - p_{n-1} > A$ for all $n \geq N$. It would be interesting to find a stronger bound on the increase in rank sizes.

A recent result of Stanley and Zanello [19] does make some progress in this direction.

**Theorem 5.4** (Stanley, Zanello [19])**.** *Let $P$ be an $r$-differential poset with rank sizes $p_0, p_1, \ldots$. Then, there exists a constant $a$ such that*

$$p_n >> n^a e^{2\sqrt{rn}}.$$

While Theorem 5.4 does not explicitly show any growth for consecutive ranks, it does indicate that the long term pattern is for rank sizes of differential posets to increase. Clearly, there are basic questions about the rank sizes of differential posets that remain to be solved.

The main objective of this section will be to prove an upper bound on the rank sizes of $r$-differential. In fact, the bound is tight. Here is a restatement of Theorem 1.2.

**Theorem 1.2.** *Let $p_0, p_1, \ldots$ be the rank sizes of an $r$-differential poset. Then, $p_i \leq F_r(i)$.*

Theorem 1.2 was conjectured by Stanley in [1] (see Problem 6 in Section 6).

The proof is a relatively straightforward induction proof and will use three lemmas.

The first lemma shows that if $P$ is a differential poset, then $P_{[n,n+1]}$ is "connected" in a particular sense. To state the lemma one needs to define a graph for each rank of a differential poset.

Let $G_n(P)$ be the graph whose vertices are $P_n$ and where there is an edge between two vertices $u$,$v$ if and only if $u$ and $v$ both cover a common element of $P_{n-1}$. Equivalently, there is an edge between two vertices $u$, $v$ if and only if $u$ and $v$ are both covered by a common element of $P_{n+1}$. Say that a vertex of $G_n(P)$ covers (is covered by) a vertex of $G_{n-1}(P)$ ($G_{n+1}(P)$) if the corresponding element of $P_n$ covers (is covered by) the corresponding element of $P_{n-1}$ ($P_{n+1}$).

The first lemma can now be stated.

**Lemma 5.5.** *For $n \geq 0$ and $P$ an $r$-differential poset, $G_n(P)$ is connected.*

*Proof.* Let $k$ be the smallest rank for which $G_k(P)$ is not connected. Let $C$ be a connected component of $G_k(P)$. Let $A$ be the set of vertices of $G_{k-1}(P)$ covered by the vertices in $C$.

Suppose $A$ does not contain all of the vertices of $P_{k-1}$. Since $G_{k-1}(P)$ is connected there exists an edge $\{v, w\}$ in $G_{k-1}(P)$ with $v \in A$ and $w \notin A$. Let $x \in C$ cover $v$ and $y \notin C$ cover $w$. Since $v$ and $w$ are adjacent in $G_{k-1}(P)$ there exists $z \in P_k$ that covers both $x$ and $y$. So, there are edges $\{x, z\}$ and $\{z, y\}$ in $G_k(P)$. Since $x \in C$ and $y \notin C$, this contradicts $C$ being a component of $G_k(P)$.

So, $A = P_{k-1}$. Let $s \in P_k - C$. Let $a \in P_{k-1}$ be covered by $s$. So $a \in A$ and there exists $t \in C$ that covers $a$. But, that means $\{s, t\}$ is an edge in $G_k(P)$. Since $t \in C$ and $s \notin C$ this contradicts $C$ being a component of $G_k(P)$.

So, no minimum $k$ exists with $G_k(P)$ disconnected. Thus, $G_n(P)$ is connected for all $n$.

$\square$

The second lemma is an identity involving $r$-Fibonacci numbers.

**Lemma 5.6.** *For $n \geq 1$,*

$$F_{n-1}(r) = 1 + r\left(\sum_{k=0}^{n-2} F_k(r)\right) - F_{n-2}(r)$$

*Proof.* The proof will be by induction. The case when $n = 1$ is clear (take $F_k(r) = 0$ for $k < 0$).

Suppose $F_{n-1}(r) = 1 + r\left(\sum_{k=0}^{n-2} F_k(r)\right) - F_{n-2}(r)$. Then,

$$\begin{aligned}
1 + r\left(\sum_{k=0}^{n-1} F_k(r)\right) - F_{n-1}(r) &= 1 + r\left(\sum_{k=0}^{n-1} F_k(r)\right) - \left(1 + r\left(\sum_{k=0}^{n-2} F_k(r)\right) - F_{n-2}(r)\right) \\
&= rF_{n-1}(r) + F_{n-2}(r) \\
&= F_n(r)
\end{aligned}$$

$\square$

The third lemma is an identity involving two different ways of looking at rank sizes of a differential poset. Here $d_{up}(v)$ is used to denote the number of elements that cover $v$ and $d_{down}(v)$ is used to denote the number of elements that are covered by $v$.

**Lemma 5.7.** *For $P$ an $r$-differential poset and $n \geq 0$,*

$$\sum_{v \in P_n} d_{up}(v) = r \sum_{k=0}^{n} p_k$$

*Proof.* The proof is by induction. For the $n = 0$ case one gets $r = r$ since every $r$-differential poset has $p_0 = 1$ and $p_1 = r$. Suppose $\sum_{v \in P_n} d_{up}(v) = r \sum_{k=0}^{n} p_k$. Then,

$$\begin{aligned}
\sum_{v \in P_{n+1}} d_{up}(v) &= \sum_{v \in P_{n+1}} (d_{down}(v) + r) \\
&= \left( \sum_{v \in P_{n+1}} d_{down}(v) \right) + rp_{n+1} \\
&= \left( \sum_{v \in P_n} d_{up}(v) \right) + rp_{n+1} \\
&= r \left( \sum_{k=0}^{n} p_k \right) + rp_{n+1} \\
&= r \sum_{k=0}^{n+1} p_k
\end{aligned}$$

□

Now the proof of Theorem 1.2 can be completed.

*Proof.* The proof is by induction on $n$. The base case holds since $p_0 = F_0 = 1$. Since $G_n(P)$ is connected by Lemma 5.5 one can order the vertices of $G_{n-1}(P)$ such that each vertex is adjacent to a previous vertex in the ordering. Thus, the number of vertices at rank $n$ in $P$ is at most the sum of $d_{up}(v)$ for $v \in P_{n-1}$ minus $(p_{n-1} - 1)$ since each vertex added after the first vertex has one of its covers already accounted for. So,

$$\begin{aligned}
p_n &\leq \left(\sum_{v \in P_{n-1}} d_{up}(v)\right) - (p_{n-1} - 1) \\
&= r\left(\sum_{k=0}^{n-1} p_k\right) - p_{n-1} + 1 && (5.1) \\
&= 1 + r\sum_{k=0}^{n-2} p_k + (r-1)p_{n-1} \\
&\leq 1 + r\left(\sum_{k=0}^{n-2} F_k(r)\right) + (r-1)F_{n-1}(r) && (5.2) \\
&= \left(1 + r\left(\sum_{k=0}^{n-2} F_k(r)\right) - F_{n-1}(r)\right) + rF_{n-1}(r) && (5.3) \\
&= F_{n-2}(r) + rF_{n-1}(r) \\
&= F_n(r)
\end{aligned}$$

Note that (5.1) is true due to Lemma 5.7, (5.3) is true due to Lemma 5.6, and (5.2) holds by induction.

□

## 5.1 Further directions and conjectures

There are many remaining questions about rank sizes of differential posets.

One of the more prominent conjectures is the following which was originally asked by Stanley in [1] (see Problem 6).

**Conjecture 5.8.** *$Y^r$ has the smallest $n$th rank of any $r$-differential poset for all $n$.*

The data from Chapter 4 show that Conjecture 5.8 is true for $r = 1$ up to $n = 10$. Interestingly, there exist 1-differential posets with the same rank size as $Y$ for some $n$. However, all of the known examples of such 1-differential posets eventually are forced to have larger rank sizes than $Y$.

Even more appears to be true from the upper bound perspective than Theorem 1.2. The following conjecture was introduced by Stanley in [1] (see Problem 6).

**Conjecture 5.9.** *Let $P$ be an $r$-differential poset with rank sizes $p_0, p_1, \ldots$. Then, $p_n \leq rp_{n-1} + p_{n-2}$.*

Note that Conjecture 5.9 asks if the $r$-Fibonacci recurrence relation holds as an upper bound for arbitrary rank sizes.

The data from Chapter 4 show that Conjecture 5.9 is true for $r = 1$ up to $n = 10$. Also, it appears that there is a unique extension of a partial 1-differential poset $P_{[0,n-1]}$ that satisfies the inequality of Conjecture 5.9 at equality. This unique extension can be achieved via reflection-extension.

Another conjecture is that the possible rank sizes for partial 1-differential posets up to rank $n$ actually form the lattice points of a polytope.

**Conjecture 5.10.** *Let $n \geq 0$. Let $A$ be defined as*

$$A = \{(p_0, p_1, \ldots, p_n) \mid P \text{ is a } 1\text{-differential poset up to rank } n\}.$$

*Then, there exists a polytope $Q_n$ such that $A = Q_n \cap \mathbb{Z}^n$.*

The author believes Conjecture 5.10 is new. However, similar ideas have been studied in [19].

The data from Chapter 4 show that Conjecture 5.10 is true for $0 \leq n \leq 10$. On the other hand, the analog of Conjecture 5.10 is not true for 4-differential posets (see Theorem 2.7 in [19]).

If Conjecture 5.10 is true, then it would be interesting to determine the facets of $Q_n$. Based on computer experiments it appears that one facet is given by the inequality $p_n \leq p_{n-1} + p_{n-2}$. Note that this is the inequality from Conjecture 5.9.

# Chapter 6

# Differential Lattices

The two known "natural" examples of 1-differential posets are both lattices. One might wonder if there are any more 1-differential lattices. A proof that there are no additional 1-differential lattices is the content of this chapter. This proposition has a bit of history. The question of if there were any additional 1-differential lattices was first asked by Stanley [1]. In 2008, Yulan Qing showed in a master's thesis that there were no additional 1-differential lattices [4]. However, the proof is not correct. This chapter patches the holes in the proof. The same general outline of Qing's proof is followed, but the proof in this chapter provides the needed details.

The outline of the proof is as follows:

1. Show that if $P$ is a 1-differential lattice that is isomorphic to Young's lattice up to rank $n$, then $P$ is isomorphic to Young's lattice up to rank $n+1$.

2. Show that if $P$ is a 1-differential lattice that is isomorphic to $Z(1)$ up to rank $n$, then $P$ is isomorphic to $Z(1)$ up to rank $n+1$.

The difficulty in the proof is the latter part. In that part, the ramifications of the lattice property end up being traced all the way up to rank $n+6$. With an example, it can also be shown that one does need to consider the effects of being a 1-differential lattice up to at least that rank. So, there is not much hope that a similar proof could be simplified by looking at fewer ranks, as in the proof of Qing which only considers implications up to rank $n+4$.

The following lemma from Qing's thesis [4] will be helpful throughout the proof and is the only fact used in the proof that distinguishes 1-differential lattices from 1-differential posets.

**Lemma 6.1** (Qing [4])**.** *If $P$ is a 1-differential lattice with a crown of $\{x, y, z\}$ covering $\{a, b, c\}$, then there cannot be a crown covering $\{x, y, z\}$.*

*Proof.* Suppose that $P$ is a 1-differential poset such that $\{x, y, z\}$ is a crown covering $\{a, b, c\}$ and such that $\{p, q, r\}$ is a crown covering $\{x, y, z\}$. The induced subposet of $P$ is shown in Figure 6.1.

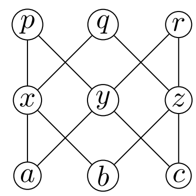



Figure 6.1: A crown covering a crown

Suppose $k$ was the greatest lower bound of $p$ and $z$. Since $b < p$ and $b < z$ one gets that $b \leq k$. Since $c < p$ and $c < z$ one gets that $c \leq k$. Since $k \leq z$ this forces $k = z$. But, $z \not< p$. This is a contradiction which completes the proof. □

## 6.1 Young's lattice case

In this section it will be shown that a 1-differential lattice that is isomorphic to Young's lattice up to rank 5 will be isomorphic to Young's lattice for all ranks. The proof is pretty straightforward, but some technicalities are needed before beginning the proof.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k, 0, 0, \ldots)$ be a partition of $n$. For $1 \leq t \leq k+1$ define $\lambda^{t+} = (\mu_1, \mu_2, \ldots, \mu_{k+1}, 0, 0, \ldots)$ where $\mu_t = \lambda_t + 1$ (taking $\lambda_{k+1} = 0$) and $\mu_i = \lambda_i$ for $i \neq t$. For $1 \leq t \leq k$ define $\lambda^{t-} = (\mu_1, \mu_2, \ldots, \mu_k, 0, 0, \ldots)$ where $\mu_t = \lambda_t - 1$ and $\mu_i = \lambda_i$ for $i \neq t$. Note that $\lambda^{t+}$ and $\lambda^{t-}$ need not be partitions.

The following is a lemma that will be needed for the proof.

**Lemma 6.2.** *Let $n \geq 4$ and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k, 0, 0, \ldots)$ be a partition of $n$ such that $\lambda^{a+}$, $\lambda^{b+}$, and $\lambda^{c+}$ are all partitions with $a$, $b$, $c$ distinct. Then, there exists $\{x, y\} \subset$*

*$\{a, b, c\}$ with $x \neq y$ and $1 \leq z \leq k$ with $z \notin \{x, y\}$ such that the following are all partitions:*

- $\lambda^{x+,z-}$
- $\lambda^{y+,z-}$
- $\lambda^{x+,y+,z-}$

*Proof.* Note that $\lambda$ must have at least 2 distinct part sizes in order for $\lambda^{a+}$, $\lambda^{b+}$, and $\lambda^{c+}$ to be partitions. Without loss of generality assume that $a < b < c$. Consider the following cases:

1. $\lambda$ has at least 3 distinct part sizes.

   Let $x = a$ and $y = b$. Choose $z$ so that $\lambda_z > \lambda_{z+1}$ and $\lambda_z = \lambda_c$.

2. $\lambda$ has 2 distinct part sizes and $\lambda_i - \lambda_{i+1} > 1$ for some $i$.

   Let $x = i + 1$, $y = k + 1$, and $z = i$.

3. $\lambda$ has 2 distinct part sizes and $\lambda_k \geq 2$.

   Let $x = 1$, $y = k + 1$, and $z = k$.

4. $\lambda$ has 2 distinct part sizes, $\lambda_1 = 2$, and $\lambda_k = 1$.

   If $\lambda_2 = 2$, then let $x = 1$, $y = k + 1$, and $z$ be such that $\lambda_z = 2$, $\lambda_{z+1} = 1$.

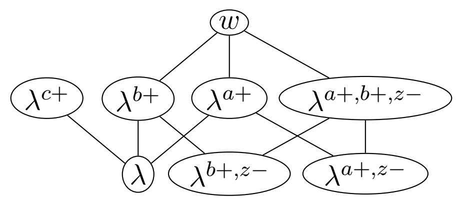
   If $\lambda_2 = 1$, then let $x = 1$, $y = 2$, and $z = k$. Note that $k \geq 3$ because $n \geq 4$.

Note that if $\lambda$ has 2 distinct part sizes, then $|\{t \mid \lambda^{t+} \text{ is a partition}\}| = 3$. This means that since $\lambda^{x+}$ is a partition and $\lambda^{y+}$ is a partition one gets that $\{x, y\} \subset \{a, b, c\}$ for all the cases that did not set $x$ and $y$ explicitly to one of $\{a, b, c\}$. $\square$

The main result of this section can now be introduced.

**Proposition 6.3.** *If $P$ is a 1-differential lattice that is isomorphic to Young's lattice up to rank $n$ for $n \geq 5$, then $P$ is isomorphic to Young's lattice (at all ranks).*

*Proof.* Let $\lambda \in P_{n-1}$ have up degree at least 3. Let $a, b, c$ be distinct and such that $\lambda^{a+}$, $\lambda^{b+}$, and $\lambda^{c+}$ are partitions. Without loss of generality let $\{a, b\} = \{x, y\}$ as in Lemma 6.2. Also, let $z$ be as in Lemma 6.2. By Lemma 6.2, $\lambda^{a+,z-}$, $\lambda^{b+,z-}$, and $\lambda^{a+,b+,z-}$ are all partitions. Now, $\{\lambda^{a+}, \lambda^{b+}, \lambda^{a+,b+,z-}\}$ is a crown covering $\{\lambda, \lambda^{a+,z-}, \lambda^{b+,z-}\}$. Thus, since $P$ is a 1-differential lattice Lemma 6.1 shows that there exists $w \in P_{n+1}$ that covers $\{\lambda^{a+}, \lambda^{b+}, \lambda^{a+,b+,z-}\}$. The following is a picture of the situation.




Note that $\lambda^{c+}$ and $\lambda^{a+,b+,z-}$ are not adjacent in Young's lattice. So, $w$ cannot cover $\lambda^{c+}$. This means there must be a crown covering $\{\lambda^{a+}, \lambda^{b+}, \lambda^{c+}\}$ by Proposition 3.4.

Note that it has been shown that any set of 3 vertices of $P_n$ that are mutually adjacent and cover a common element must be covered by a crown. Thus, $P$ is isomorphic to Young's lattice up to rank $n+1$. By induction, $P$ is isomorphic to Young's lattice (at all ranks). $\square$

## 6.2 Fibonacci lattice case

It now needs to be shown that if $P$ is a 1-differential lattice that is isomorphic to $Z(1)$ up to rank 5, then $P$ will be isomorphic to $Z(1)$. The proof for this case will be more complicated than the proof for the Young's lattice case. Before beginning with the Fibonacci lattice part of the proof, a review of notations is included.

### 6.2.1 Notations

Let $P$ be a poset. The *up degree* of a vertex $x$ is the number of elements that cover $x$. The up degree of $x$ will be denoted $d_{up}(x)$. Similarly, the *down degree* of a vertex $y$ is the number of elements covered by $y$. The down degree of $y$ will be denoted $d_{down}(y)$. Note that in an $r$-differential poset $d_{up}(x) = d_{down}(x) + r$.

The set of elements that cover a vertex $x$ is called the *up set* of $x$. The up set of $x$ will be denoted as $\mathcal{U}(x)$. Similarly, the set of elements covered by $y$ is called the *down set* of $y$ and will be denoted as $\mathcal{D}(y)$. Note that $|\mathcal{U}(x)| = d_{up}(x)$ and $|\mathcal{D}(y)| = d_{down}(y)$.

Further if $A$ is a set of vertices of a poset $P$, then denote the set of all elements of $P$ that cover at least one element of $A$ as $\mathcal{U}(A)$. Similarly, denote the set of all elements of $P$ that are covered by at least one element of $A$ as $\mathcal{D}(A)$.

### 6.2.2 Proof of Fibonacci lattice case

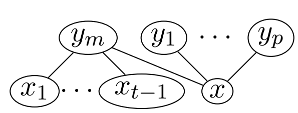
A lemma that restricts the possibilities to consider will be helpful. This lemma appears in [4], however the proof there is incorrect.

**Lemma 6.4.** *Let $P$ be a 1-differential poset such that $P_{[0,n]} \cong Z(1)_{[0,n]}$. Let $x \in P_{n-1}$ with $d_{up}(x) \geq 4$. Then, there exists $w \in P_{n+1}$ such that $\mathcal{D}(w) = \mathcal{U}(x)$.*

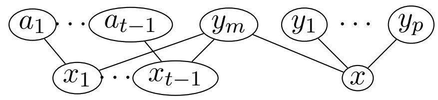
*Proof.* The proof will be somewhat complicated/technical. The basic outline is as follows.

- For each element, $y$, covering $x$ find a sufficient number of distinct elements that cover $y$ and do not cover any other element that covers $x$.
- That will leave $y$ with only 2 remaining up edges.
- Eliminate the possibility that $y$ uses both remaining up edges to cover other elements that cover $x$.
- Setting $w$ to be the unique element that covers $y$ and all the other elements that cover $x$ completes the proof.

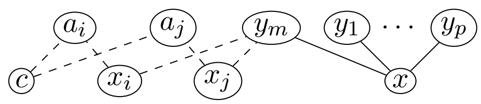
Let $\phi : P_{[0,n]} \to Z(1)_{[0,n]}$ be an isomorphism. Let $\{y_1, y_2, \ldots, y_p\} = \mathcal{U}(x)$. Choose $1 \leq m \leq p$. Let $\{x, x_1, x_2, x_3, \ldots, x_{t-1}\} = \mathcal{D}(y_m)$. So far the picture looks like the following.




For $1 \leq i \leq t-1$ choose $a_i \in \mathcal{U}(x_i) \setminus \{y_m\}$. The new picture looks like the following.



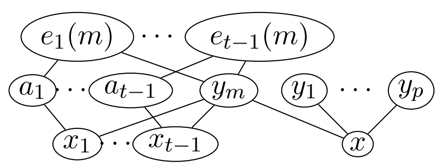

Suppose there exists $c \in \mathcal{D}(a_i) \cap \mathcal{D}(a_j)$ for $1 \leq i < j \leq t-1$. Then, $\{a_i, a_j, y_m\}$ is a crown covering $\{c, x_i, x_j\}$. The crown can be seen by the dashed lines in the following picture.



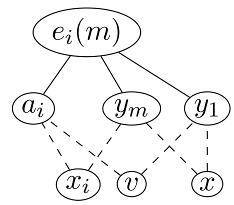

But, this means $\{\phi(a_i), \phi(a_j), \phi(y_m)\}$ is a crown covering $\{\phi(c), \phi(x_i), \phi(x_j)\}$ in $Z(1)$. This is a contradiction since $Z(1)$ has no crowns by Prop 3.7. So, such a $c$ could not have been chosen. This means that $\mathcal{D}(a_i) \cap \mathcal{D}(a_j) = \emptyset$ for $i \neq j$.

Since $\mathcal{D}(a_i) \cap \mathcal{D}(a_j) = \emptyset$ for any choices of $a_i$ and $a_j$, it must be that $\mathcal{U}(\mathcal{U}(x_i) \setminus \{y_m\}) \cap \mathcal{U}(\mathcal{U}(x_j) \setminus \{y_m\}) = \emptyset$ for $i \neq j$ by the definition of differential posets.

For $1 \leq i \leq t-1$ let $e_i(m) \in \mathcal{U}(\mathcal{U}(x_i) \setminus \{y_m\}) \cap \mathcal{U}(y_m)$. Note that such $e_i(m)$'s exist (and are actually uniquely determined) since $P$ is differential and $x_i \in \mathcal{D}(\mathcal{U}(x_i) \setminus \{y)m\}) \cap \mathcal{D}(y_m)$. Also note that $e_i(m) \neq e_j(m)$ for $i \neq j$ since $\mathcal{U}(\mathcal{U}(x_i) \setminus \{y_m\}) \cap \mathcal{U}(\mathcal{U}(x_j) \setminus \{y_m\}) = \emptyset$. The picture with the $e_i(m)$'s added looks like the following.




Suppose $e_i(m) \in \mathcal{U}(y_k)$ for some $1 \leq i \leq t-1$ and $1 \leq k \leq p$ with $k \neq m$. Then, $a_i$, $y_m$, and $y_k$ are mutually adjacent in $P$ because $e_i(m)$ covers all 3 of them. Since $x_i \in \mathcal{D}(a_i) \cap \mathcal{D}(y_m)$ and $x \in \mathcal{D}(y_m) \cap \mathcal{D}(y_k)$, one element cannot be covered by all 3 of $a_i$, $y_m$, and $y_k$. This means that there exists $v \in \mathcal{D}(a_i) \cap \mathcal{D}(y_k)$. But, this means that $\{a_i, y_m, y_k\}$ is a crown covering $\{x, x_i, v\}$ in $P$. The picture looks like the following with the crown edges dashed.

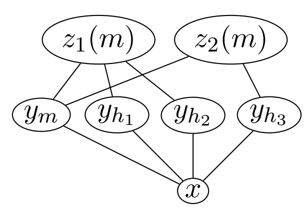



So, $\{\phi(a_i), \phi(y_m), \phi(y_k)\}$ is a crown covering $\{\phi(x), \phi(x_i), \phi(v)\}$ in $Z(1)$. This is a contradiction since $Z(1)$ has no crowns. Thus, $\mathcal{U}(\mathcal{U}(x_i) \setminus \{y_m\}) \cap \mathcal{U}(y_m) \cap \mathcal{U}(y_k) = \emptyset$ for any $m$, $k$, and $i$ with $m \neq k$, $1 \leq i \leq t-1$, and $1 \leq m, k \leq p$. Thus, $e_i(m) \notin \mathcal{U}(y_k)$ for $1 \leq i \leq t-1$ and $1 \leq k \leq p$ with $k \neq m$.

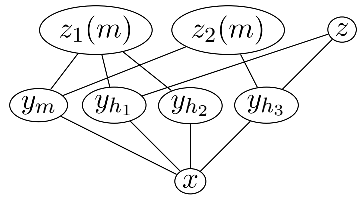
Let $C(m) = \mathcal{U}(y_m) \cap \bigcup_{i=1, i\neq m}^{p} \mathcal{U}(y_i)$. Since $e_i(m) \neq e_j(m)$ for $1 \leq i < j \leq t-1$, $e_i(m) \notin \mathcal{U}(y_h)$ for $h \neq m$, and $|\mathcal{U}(y_h)| = t+1$ for $1 \leq h \leq p$; it must be that $|C(h)| \leq 2$ for $1 \leq h \leq p$. $|C(h)| \neq 0$ since $x \in \mathcal{D}(y_h) \cap \mathcal{D}(y_i)$ for any $i \neq h$ and $P$ is a differential poset.

If $|C(h)| = 1$ for any $1 \leq h \leq p$, then letting $w \in C(h)$ satisfies the requirements of the lemma. So, it needs to be shown that $|C(h)| \neq 2$ for at least one $h$.

Suppose $|C(h)| = 2$ for all $h$ with $1 \leq h \leq p$. For $j \in \{1, 2\}$ let $C(m) = \{z_1(m), z_2(m)\}$. Let $A_j = \{i \neq m \mid y_i \in \mathcal{D}(z_j(m))\}$. Note that $A_j \neq \emptyset$ since $|C(m)| = 2$ and that $A_1 \cap A_2 = \emptyset$ by Prop 3.4. So, $\{A_1, A_2\}$ is a partition of $[p] \setminus \{m\}$. Since $p = |\mathcal{D}(x)| \geq 4$ one can assume that $|A_1| \geq 2$ without loss of generality. Let $h_1, h_2 \in A_1$ with $h_1 \neq h_2$. Let $h_3 \in A_2$. The picture now looks like the following.




Since $x \in \mathcal{D}(y_{h_1}) \cap \mathcal{D}(y_{h_3})$, $\mathcal{U}(y_{h_1}) \cap \mathcal{U}(y_{h_3}) \neq \emptyset$ (since $P$ is differential). Let $z \in \mathcal{U}(y_{h_1}) \cap \mathcal{U}(y_{h_3})$. The picture now looks like the following.




It has been shown that $C(h_3) = \{z_2(m), z\}$. Also, $x \in \mathcal{D}(y_{h_2}) \cap \mathcal{D}(y_{h_3})$ so $\mathcal{U}(y_{h_2}) \cap \mathcal{U}(y_{h_3}) \neq \emptyset$. Thus, one of $z_2(m)$ or $z$ is in $\mathcal{U}(y_{h_2})$. If $z_2(m) \in \mathcal{U}(y_{h_2})$, then $\{z_1(m), z_2(m)\} \subset \mathcal{U}(y_m) \cap \mathcal{U}(y_{h_2})$. This is a contradiction since by Prop 3.4 $|\mathcal{U}(y_m) \cap \mathcal{U}(y_{h_2})| \leq 1$. Similarly, if $z \in \mathcal{U}(y_{h_2})$ we have that $\{z_1(m), z\} \subset \mathcal{U}(y_{r_1}) \cap \mathcal{U}(y_{h_2})$ which is also a contradiction. Thus, it is not true that $|C(h)| = 2$ for all $1 \leq h \leq p$.

Now pick any $h$ with $|C(h)| = 1$ (actually what has been shown implies that $|C(h)| = 1$ for all $h$). Choosing $w \in C(h)$ completes the proof. $\square$

Note that the previous lemma does not require that $P$ be a 1-differential lattice, only that $P$ is a 1-differential poset.

The impact of the previous lemma is as follows. Suppose one is looking for a 1-differential lattice, $P$, that starts out being isomorphic to $Z(1)$ and that eventually differs from $Z(1)$. The lemma states that the first difference between $P$ and $Z(1)$ must be "caused" by a vertex of up degree *exactly* 3. It is now time to begin the (tedious) proof of this section's main proposition.

**Proposition 6.5.** *Let $P$ be a 1-differential lattice such that $P_{[0,n]} \cong Z(1)_{[0,n]}$ for $n \geq 5$. Then, $P \cong Z(1)$.*

*Proof.* Suppose the proposition is false. By Lemma 6.4 this means that there must exist $x \in P_{n-1}$ with $|\mathcal{U}(x)| = 3$ such that there is a crown covering $\mathcal{U}(x)$. Let $\phi : P_{[0,n]} \to Z(1)_{[0,n]}$ be an isomorphism. As an element in $Z(1)$ of rank at least 4 and with up degree exactly 3, $\phi(x)$ must be one of the following:

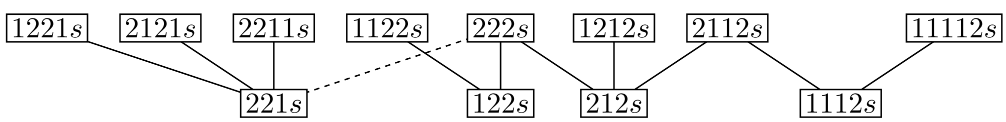
1. $212s$ with $s \in \{1,2\}^*$

2. $22$

3. $211s$ with $s \in \{1,2\}^*$

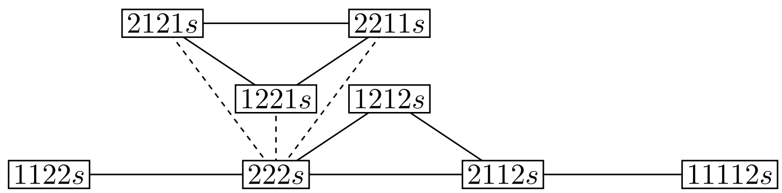
The structure of the proof is as follows. For each of the 3 possibilities for $\phi(x)$ build up part of $P$ up to some rank $k$. Then, try to extend $P_{[0,k]}$ to $P_{[0,k+1]}$. To do this let $\Gamma_i, \Lambda_i \subset P_i$ where one can think of $\Gamma_i$ as the part of $P_i$ where one knows most (or even all) of the structure of $P|_\Gamma$ and $\Delta_i$ as another part of $P_i$ for which one does not know as much about the total structure of $P|_\Delta$. But, it turns out that one will know enough of the structure of $P|_\Delta$ to be able to complete the proof. Generally, one finds $\Gamma_i$, $\Gamma_{i+1}$, $\Delta_i$, and $\Delta_{i+1}$ and then determines what one needs to make $\Gamma_{i+2}$ and $\Delta_{i+2}$. The general tool used to do this is looking at the hypergraphs $G$ and $G'$ of $P$ restricted to $\Gamma \cup \Delta$. Eventually, one can show that one cannot continue these extensions. But, one will need to go up several levels from $n-1$ to reach this contradiction.

Suppose $\phi(x) = 212s$ with $s \in \{1,2\}^*$. For simplicity identify $\phi(y) \in P$ with $y$. So, instead of writing $\phi(x)$, we will just write $x$ for the element of $P$ that maps to $x \in Z(1)$. Let $\Gamma_{n-1} = \{122s, 212s, 1112s\}$, $\Gamma_n = \{1122s, 222s, 1212s, 2112s, 11112s\}$, $\Delta_{n-1} = \{221s\}$, and $\Delta_n = \{1221s, 2121s, 2211s\}$. The bipartite graph of $P_{[n-1,n]}$ restricted to $\Gamma \cup \Delta$ is:

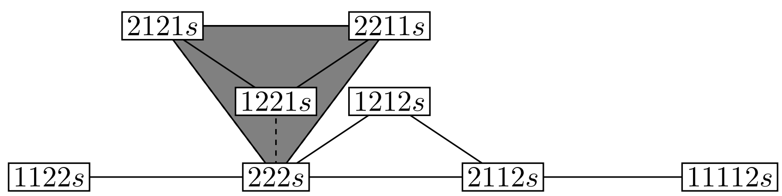



Note that the dashed edge may not be the only edge(s) of $P_{[n-1,n]}$ that goes from $\Gamma$ to $P \setminus \Gamma$. Depending on what $s$ is $222s$ could cover additional elements of $P_{n-1}$. Call these additional elements $x_1, \ldots, x_k$. Let $X_i = \{v \in P_n \mid v \text{ covers } x_i, v \neq 222s\}$. Note that $X_i \cap X_j = \emptyset$ for $i \neq j$ by Proposition 3.4. Similarly, Proposition 3.4 shows that $X_i \cap \Delta_n = \emptyset$.

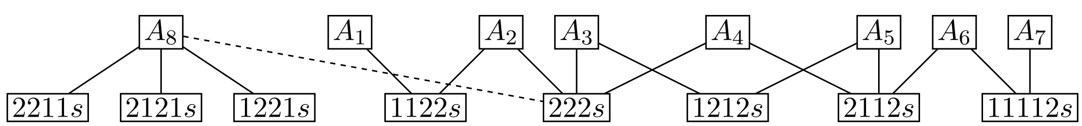
From the graph of $P_{[n-1,n]}$ restricted to $\Gamma \cup \Delta$ the adjacency relationships of $\Gamma_n \cup \Delta_n$ can be deduced. Thus, the skeleton of the hypergraph $G_n$ of $P$ restricted to $\Gamma \cup \Delta$ is:



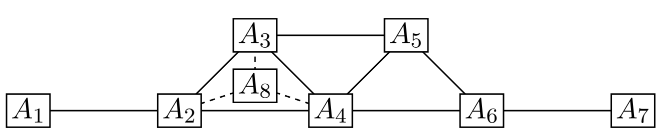

Since it has been assumed that there is a crown covering $\{222s, 1212s, 2112s\}$ we get that $\{222s, 1212s\}$, $\{222s, 2112s\}$, and $\{1212s, 2112s\}$ are maximal edges of $G_n(P|_{\Gamma\cup\Delta})$. Since $d_{up}(222s) = d_{G_n}(222s) = 4 + k$ and $\{222s, 1122s\}$ is a maximal edge, $\{222s, 2121s, 2211s, 1221s\}$ and $X_i$ for $1 \leq i \leq k$ must also be maximal edges. Thus, the hypergraph $G_n(P|_{\Gamma\cup\Delta})$ is:



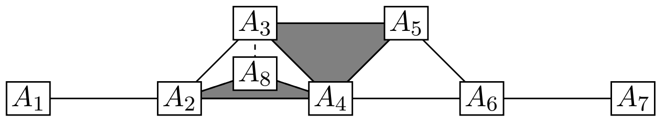

Let $\Gamma_{n+1} = \{A_i \mid 1 \leq i \leq 7\}$ and $\Delta_{n+1} = \{A_8\}$. So, the bipartite graph of $P_{[n,n+1]}$ restricted to $\Gamma \cup \Delta$ is:



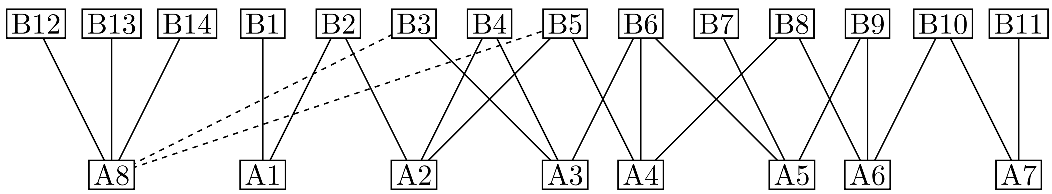

From the graph of $P_{[n,n+1]}$ restricted to $\Gamma \cup \Delta$ one can deduce the adjacency relationships of $\Gamma_{n+1} \cup \Delta_{n+1}$. Thus, the skeleton of the hypergraph $G_{n+1}(P|_{\Gamma\cup\Delta})$ is:



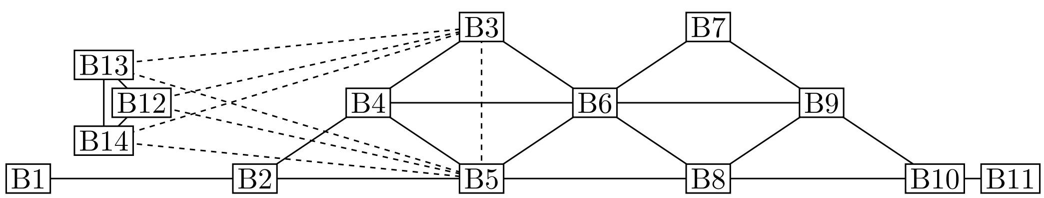

Lemma 6.1 tells us that $\{A_3, A_4, A_5\}$ must be a maximal edge. Then, since $d_{up}(A_4) = 3$, we get that $\{A_2, A_4, A_8\}$ is also a maximal face. So, $G_{n+1}(P|_{\Gamma\cup\Delta})$ is:




Let $\Gamma_{n+2} = \{Bi \mid 1 \le i \le 11\}$ and $\Delta_{n+2} = \{B12, B13, B14\}$. So, the bipartite graph of $P_{[n+1,n+2]}$ restricted to $\Gamma \cup \Delta$ is:



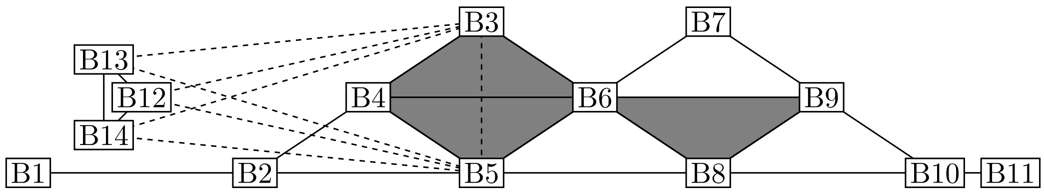

From the graph of $P_{[n+1,n+2]}$ restricted to $\Gamma \cup \Delta$ one can deduce the adjacency relationships of $\Gamma_{n+2} \cup \Delta_{n+2}$. Thus, the skeleton of $G_{n+2}(P|_{\Gamma\cup\Delta})$ is:




Lemma 6.1 shows that the following are subsets of edges of the hypergraph:

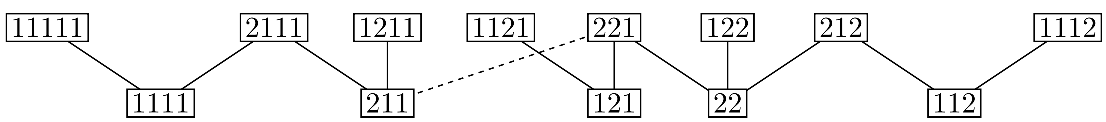
- $\{B6, B8, B9\}$
- $\{B3, B4, B5\}$
- $\{B3, B5, B6\}$

Since $\{B3, B4, B5\}$ and $\{B3, B5, B6\}$ are subsets of edges one gets that $\{B3, B4, B5, B6\}$ are all contained in one edge. So, an updated picture of the hypergraph is:



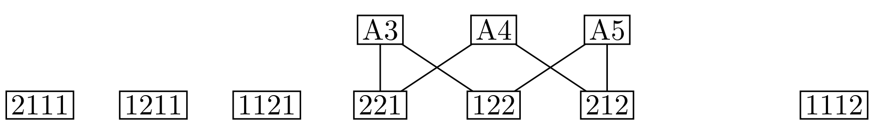

Since $d_{up}(B5) = 4$ one gets that $\{B5, B12, B13, B14\}$ is contained within an edge of the hypergraph. Now, $B3$ must belong to edges of $\{B3, B4, B5, B6\}$, $\{B3, B12\}$, $\{B3, B13\}$, and $\{B3, B14\}$. But, $d_{up}(B3) = 3$ and one cannot combine any of the faces containing $B3$. So, one gets a contradiction and $x$ cannot be $212s$.

Suppose $x = 22$. One gets the following bipartite graph of $P_{[4,5]}$:

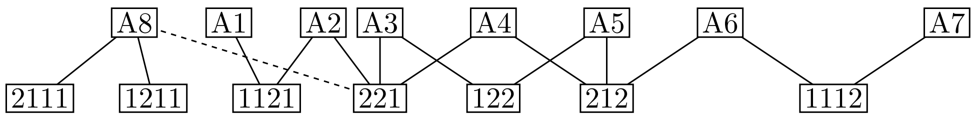



Let $\Gamma_4 = \{22, 121, 112\}$, $\Gamma_5 = \{1121, 221, 122, 212, 1112\}$, $\Delta_4 = \{211\}$, and $\Delta_5 = \{2111, 1211\}$.

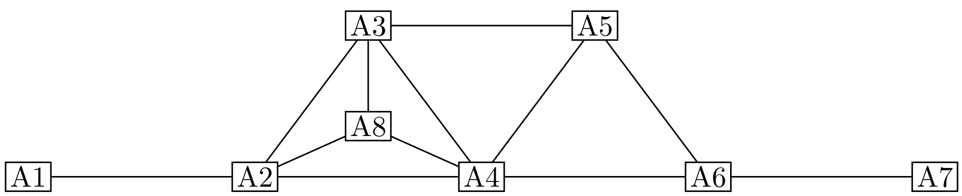
Recall that we are assuming that the covers of $x$ are covered by a crown. So, we get the following partial bipartite graph of $P_{[5,6]}$ restricted to $\Gamma \cup \Delta$:



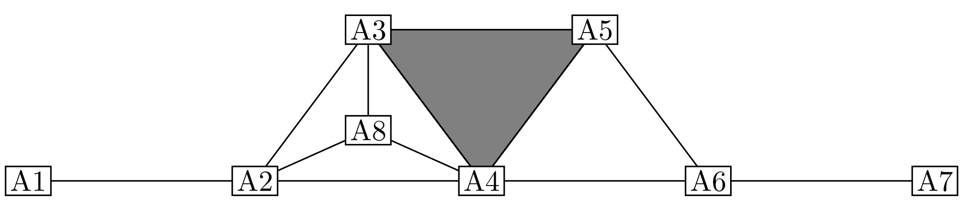

Adjacencies between edges and $d_{up}(221) = 4$ forces the following bipartite graph of $P_{[5,6]}$ restricted to $\Gamma \cup \Delta$:



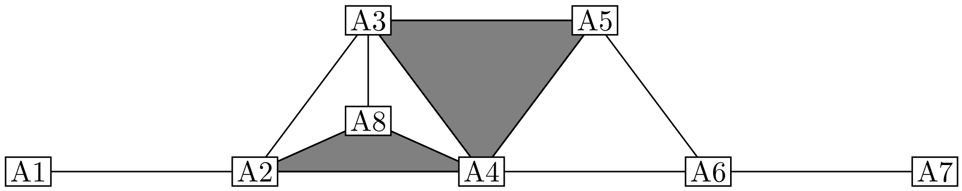

Let $\Gamma_6 = \{Ai \mid 1 \leq 1 \leq 7\}$ and $\Delta_6 = \{A8\}$. From the graph of $P_{[5,6]}$ restricted to $\Gamma \cup \Delta$ one can deduce the adjacency relationships amongst the vertices of $\Gamma_5 \cup \Delta_5$. This gives rise to the skeleton of $G_6(P|_{\Gamma \cup \Delta})$:



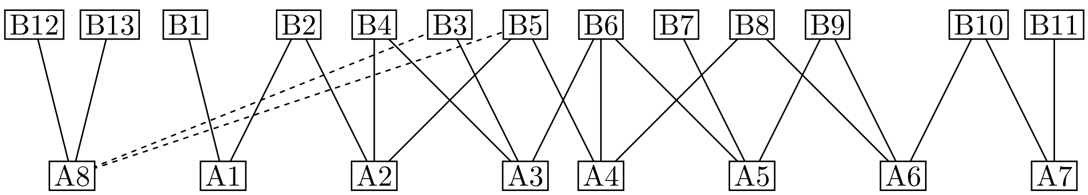

Lemma 6.1 forces $\{A3, A4, A5\}$ to be covered by a single vertex. This gives the following updated picture of the hypergraph $G_6$:



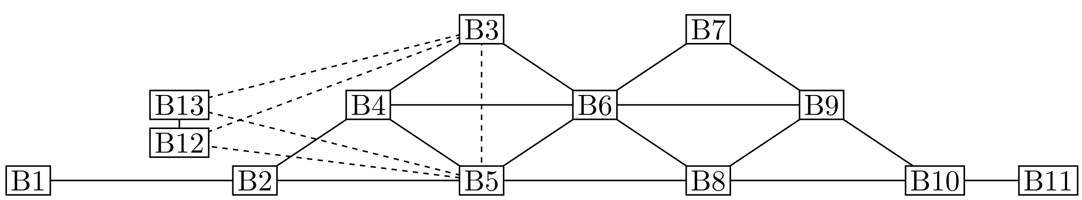

Since $d_{up}(A4) = 3$, $\{A2, A4, A8\}$ must be a subset of an edge of the hypergraph. So, one gets an updated picture of:



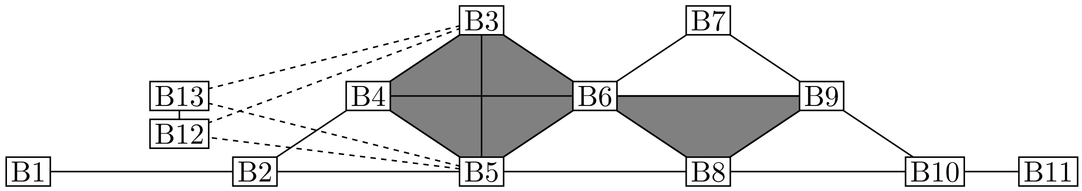

So, one gets a partial bipartite graph of $P_{[6,7]}$ of:



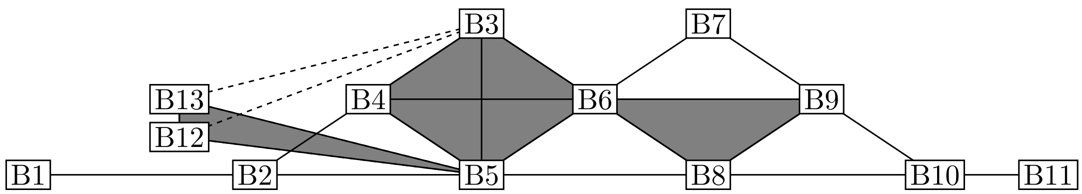

Let $\Gamma_7 = \{Bi \mid 1 \leq i \leq 11\}$ and $\Delta_7 = \{B12, B13\}$. From the graph of $P_{[6,7]}$ restricted to $\Gamma \cup \Delta$ one can deduce the adjacency relationships amongst the vertices of $\Gamma_7 \cup \Delta_7$. This gives rise to the skeleton of $G_7(P|_{\Gamma\cup\Delta})$:



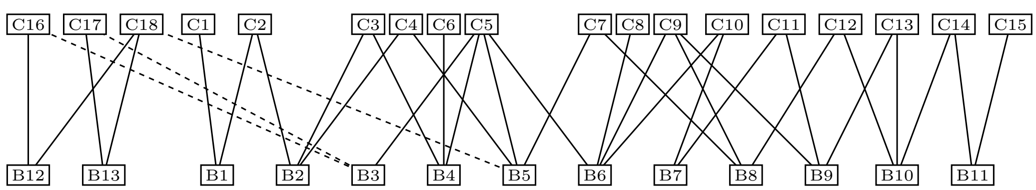

Lemma 6.1 forces $\{B6, B8, B9\}$ to be covered by a single vertex. Also, $\{B3, B4, B5\}$ and $\{B3, B5, B6\}$ are both covered by a single vertex. Thus, by Lemma 6.1 one gets that $\{B3, B4, B5, B6\}$ are covered by a single vertex. This updates $G_7$ as follows:



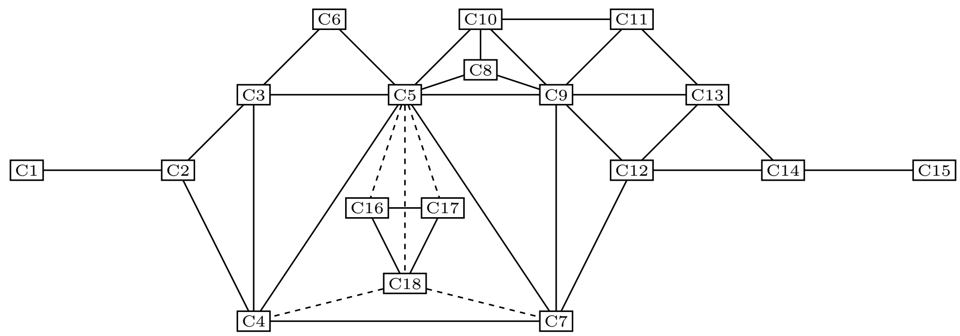

Now, $d_{up}(B5) = 4$. This forces $\{B5, B12, B13\}$ to be contained in an edge of the hypergraph. So, the final picture of $G_7(P|_{\Gamma\cup\Delta})$ is:




So, one gets a partial bipartite graph of $P_{[7,8]}$ of:



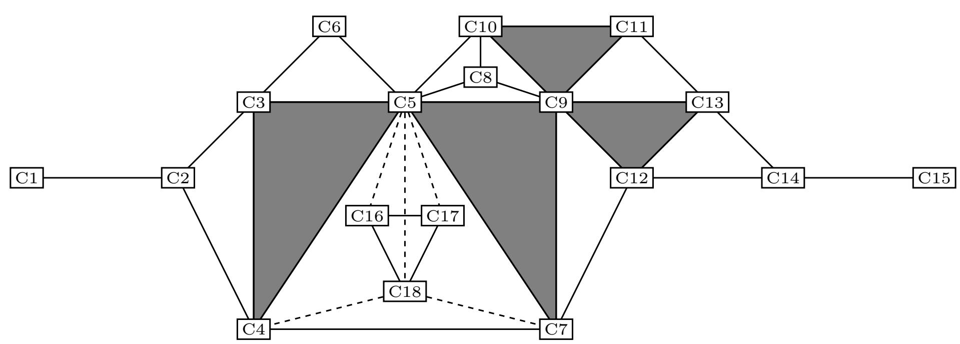

Let $\Gamma_8 = \{Ci \mid 1 \le i \le 15\}$ and $\Delta_8 = \{C16, C17, C18\}$. From the graph of $P_{[7,8]}$ restricted to $\Gamma \cup \Delta$ one can deduce the adjacency relationships amongst the vertices of $\Gamma_8 \cup \Delta_8$. This gives rise to the skeleton of $G_8(P|_{\Gamma\cup\Delta})$:




Crowns in $P_{[7,8]}$ give rise to the following subsets such that each subset must be contained in an edge of $G_8$:

- $\{C3, C4, C5\}$
- $\{C5, C7, C9\}$
- $\{C9, C10, C11\}$
- $\{C9, C12, C13\}$

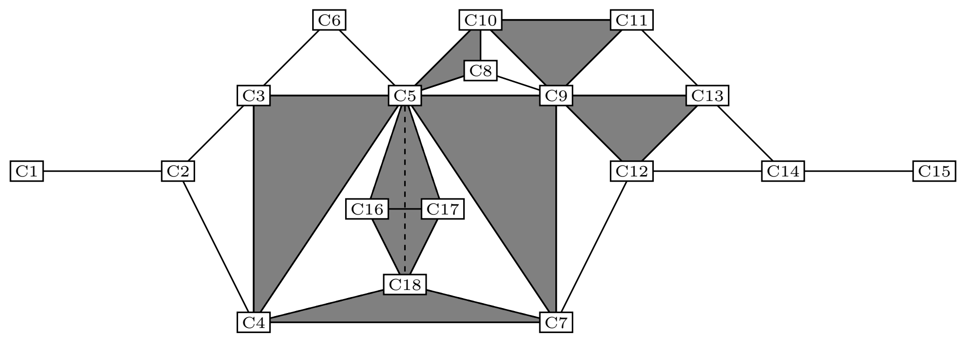
So, one gets an updated $G_8$ of:



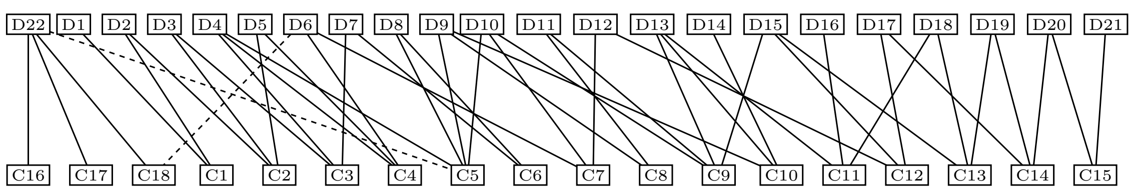

One gets the following additional subsets of edges by considering up degrees:

- $d_{up}(C8) = 2$ forces $\{C5, C8, C10\}$ to be contained in an edge.
- $d_{up}(C4) = 3$ forces $\{C4, C7, C18\}$ to be contained in an edge.
- $d_{up}(C5) = 5$ forces $\{C5, C16, C17, C18\}$ to be contained in an edge.

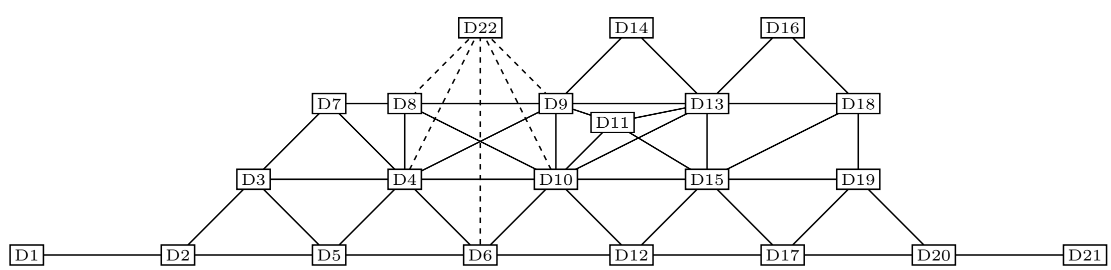
So, $G_8(P|_{\Gamma\cup\Delta})$ is:




So, one get a partial bipartite graph of $P_{[8,9]}$ of:




Let $\Gamma_9 = \{Di \mid 1 \le i \le 21\}$ and $\Delta_9 = \{D22\}$. From the graph of $P_{[8,9]}$ restricted to $\Gamma \cup \Delta$ one can deduce the adjacency relationships amongst the vertices of $\Gamma_9 \cup \Delta_9$. This gives rise to the skeleton of $G_9(P|_{\Gamma\cup\Delta})$:

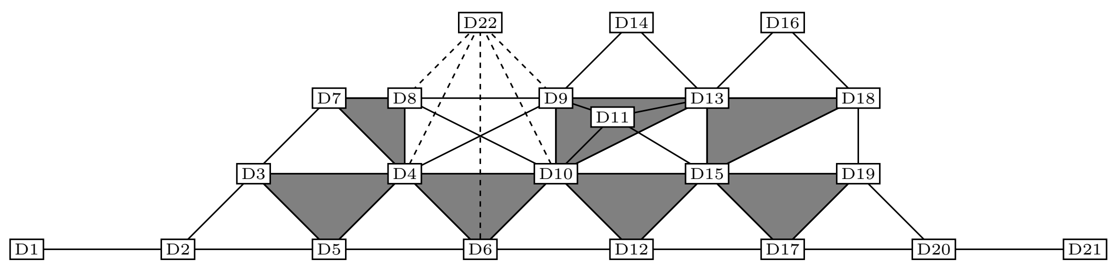



Crowns in $P_{[8,9]}$ force the following subsets to be contained in edges of $G_9$:

- $\{D3, D4, D5\}$
- $\{D4, D6, D10\}$

- $\{D4, D7, D8\}$
- $\{D10, D12, D15\}$
- $\{D15, D17, D19\}$
- $\{D13, D15, D18\}$
- $\{D9, D10, D11, D13\}$

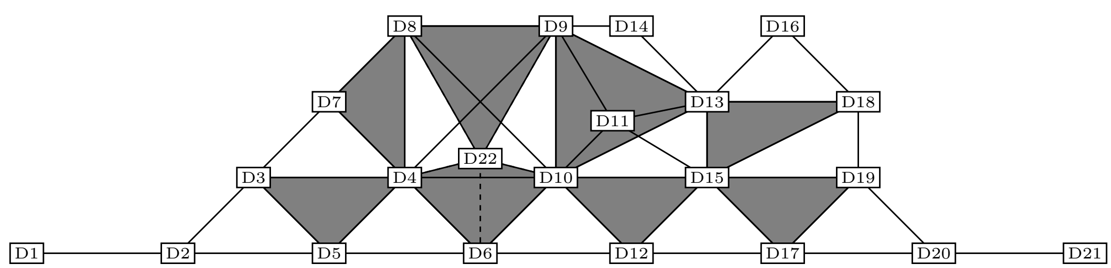
So, one gets an updated $G_9$ of:




Note that $D22$ is adjacent to each of the vertices in $\{D4, D6, D10\}$. If one takes $\{D4, D6, D10, D22\}$ as a subset of an edge of $G_9$, then up degree considerations give:

- $d_{up}(D8) = 3$ forces $\{D8, D9, D22\}$ to be contained in an edge.

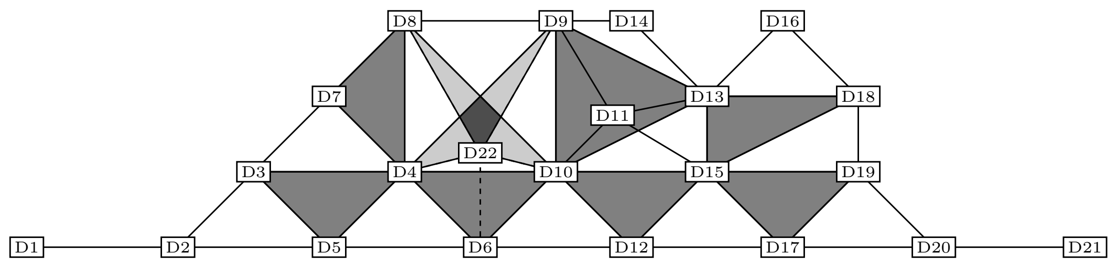
In this case the updated picture of the hypergraph $G_9$ is:



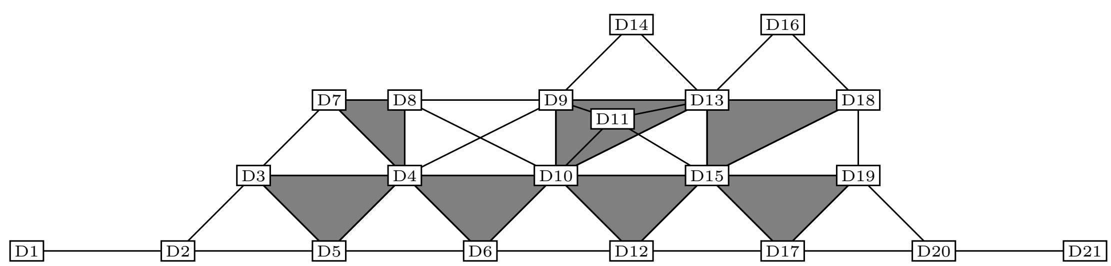

If one does not take $\{D4, D6, D10, D22\}$ to be contained in an edge of $G_9$, then up degree considerations give:

- $d_{up}(D4) = 4$ forces $\{D4, D9, D22\}$ to be contained in an edge.
- $d_{up}(D10) = 4$ forces $\{D8, D10, D22\}$ to be contained in an edge.

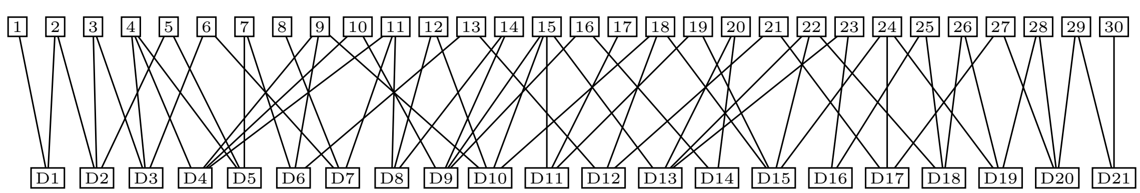
In this case $G_9$ is updated to:



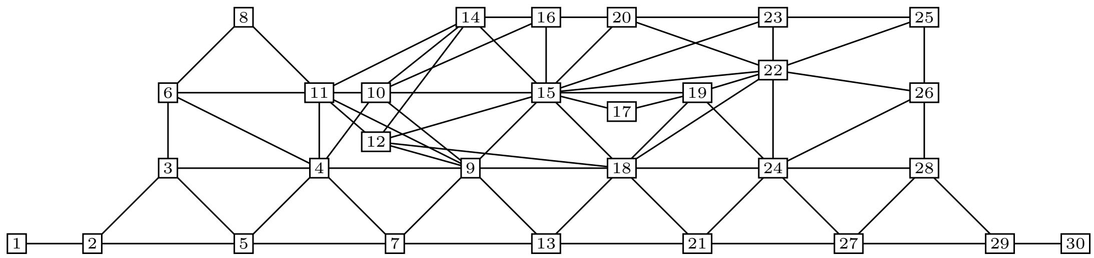

Note that in either case if one removes $D22$ from $G_9$ one gets the following picture:




So, one gets a partial bipartite graph of $P_{[9,10]}$ of (here 5 means $E5$):



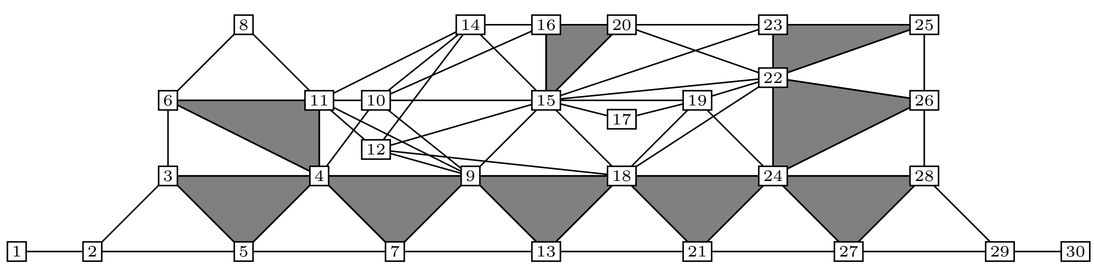

Let $\Gamma_{10} = \{i \mid 1 \leq i \leq 30\}$ and $\Delta_{10} = \emptyset$. From the graph of $P_{[9,10]}$ restricted to $\Gamma \cup \Delta$ one can deduce the adjacency relationships amongst the vertices of $\Gamma_{10}$. This gives rise to the skeleton of $G_{10}(P|_{\Gamma \cup \Delta})$:




Crowns in $P_{[9,10]}$ force the following subsets to be contained in edges of $G_{10}$:

- $\{E3, E4, E5\}$
- $\{E4, E6, E11\}$
- $\{E4, E7, E9\}$
- $\{E9, E13, E18\}$
- $\{E15, E16, E20\}$
- $\{E18, E21, E24\}$
- $\{E22, E23, E25\}$
- $\{E22, E24, E26\}$
- $\{E24, E27, E28\}$

So, one gets an updated $G_{10}$ of:




Consider $E22$. Since $d_{up}(E22) = 4$, $E22$ can be in at most 4 edges of $G_{10}$. The subsets of edges $\{E22, E23, E25\}$ and $\{E22, E24, E26\}$ cannot be extended. $\{E20, E22\}$

must also be an edge since there is no way it can be extended. Thus, in order for $E22$ to be in only 4 edges $\{E15, E18, E19, E22\}$ must be an edge of $G_{10}$.

Since $E15$ and $E19$ are both contained in an edge that does not contain $E17$ this means one also gets edges $\{E15, E17\}$ and $\{E17, E19\}$.

$\{E15, E23\}$ cannot be extended and is thus also an edge.

$\{E15, E16, E20\}$ also cannot be extended and is thus also an edge.

This gives us 4 edges containing $E15$:

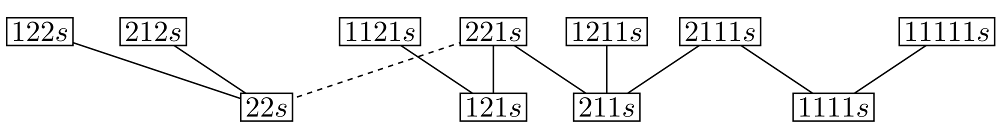
- $\{E15, E18, E19, E22\}$
- $\{E15, E17\}$
- $\{E15, E23\}$
- $\{E15, E16, E20\}$

Since $d_{up}(E15) = 5$, this means that one must get an edge of $\{E9, E10, E12, E14, E15\}$. But, $E9$ and $E14$ are not adjacent. So, one (finally) gets a contradiction.

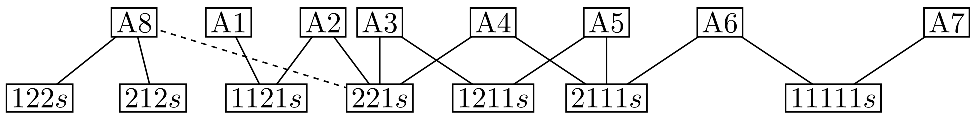
So, one now knows that $x$ could not have been 22.

Note: there is an example of a 1-differential poset that is isomorphic to the Fibonacci lattice up to rank 5 and is a meet semi-lattice up to rank 10. Thus, the number of ranks we considered is the minimal number necessary.

One can now continue with the third and last possibility for $x$ from the beginning of the proof. Suppose $x = 211s$ with $s \in \{1, 2\}^*$. Let $\Gamma_{n-1} = \{121s, 211s, 1111s\}, \Gamma_n = \{1121s, 221s, 1211s, 2111s, 11111s\}, \Delta_{n-1} = \{22s\}$, and $\Delta_n = \{122s, 212s\}$. The bipartite graph of $P_{[n-1,n]}$ restricted to $\Gamma \cup \Delta$ is:




Note that depending on $s$, $22s$ could be covered by additional vertices of $P_n$. But, Lemma 6.4 states that one will get the following partial bipartite graph of $P_{[n,n+1]}$ regardless of any additional vertices above $22s$:




Now, one has an isomorphic situation as to when $x = 22$. Thus, one knows that $P$ cannot extend to be a 1-differential lattice.

$\square$

Note that the proof of Theorem 1.3 is now complete.

## 6.3 Further directions and conjectures

Theorem 1.3 states that there are only two 1-differential lattices, but what about $r$-differential lattices for $r \geq 2$? Stanley has conjectured that the only $r$-differential lattices are direct products of $Z(k)$ and $Y$ (all such direct products will be $r$-differential lattices for some $r$). However, it is not clear how to extend the above proof in a simple manner. It is likely that even for 2-differential lattices the complicated part of the proof would become even more complicated (though, perhaps not). There would at least be more cases to consider. It should further be mentioned that the data from Chapter 4 is not nearly as good for 2-differential posets when compared to 1-differential posets. This opens the possibility that there's some not too complicated 2-differential poset that doesn't fit in Stanley's conjecture.

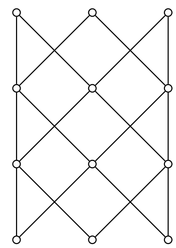
There are also other results that could be studied that generalize the idea of lattice for differential posets. Conjecture 6.6 is one step in that direction.

**Conjecture 6.6.** *An $r$-differential poset $P$ is a lattice if and only if $P$ does not contain a crown covering a crown.*

Note that Conjecture 6.6 is true for $r = 1$. Also, Lemma 6.1 shows that a differential lattice does not have a crown covering a crown.

Further, one could consider differential posets that have a crown covering a crown, but not a crown covering the upper of those two crowns.

**Question 6.7.** *What are all the* 1*-differential posets that do not contain the following induced subposet?*



If there were not too many 1-differential posets in the answer to Question 6.7, then perhaps such posets would be good candidates for which to try and find a "nice" combinatorial definition. The rationale here is that the only known 1-differential posets with a "nice" combinatorial definition are Young's lattice and the Fibonacci 1-differential poset.

# Chapter 7

# Variations on Differential Posets

There are several generalizations of differential posets that have been studied. This chapter will discuss several of these generalizations. This chapter will also include a proof of Theorem 1.4 which states that any quantized $r$-differential poset has the $r$-differential Fibonacci poset as its underlying differential poset.

## 7.1 Sequential differential posets

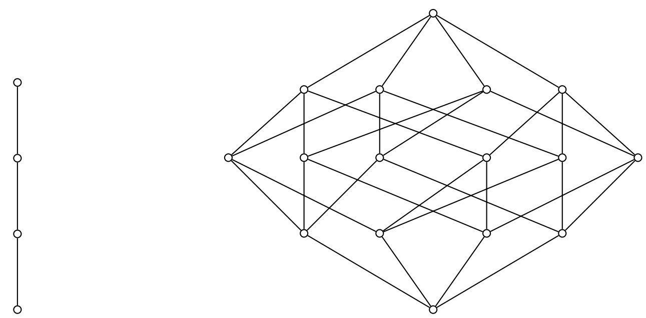
Sequential differential posets were introduced by Stanley in [20]. The idea is to allow different $r$'s at each rank. The following is a definition that closely resembles Definition 3.1

**Definition 7.1.** *Let $\vec{r} = (r_0, r_1, r_2, \ldots)$ be a sequence of integers.*

*A* sequential $\vec{r}$-differential poset *is a graded, locally finite poset with a $\hat{0}$ such that:*

1. *If $x$ is at rank $n$ and covers $k$ elements, then $x$ is covered by $k + r_n$ elements.*
2. *If $x \neq y$ are vertices that both cover exactly $t$ vertices, then there are exactly $t$ vertices that cover both $x$ and $y$.*

There are many more well known examples of sequential differential posets than differential posets. One example of a sequential differential poset is any finite chain such as that in Figure 7.1a. For a chain of $n$ elements one has $r_0 = 1$, $r_{n-1} = -1$, and $r_i = 0$ for all other $i$. Another example of a sequential differential poset is a boolean algebra such as that in Figure 7.1b. For a boolean algebra of rank $n$ one has that $r_k = n - 2k$ for $0 \leq k \leq n$ and $r_i = 0$ for $i > n$.



(a) Chain of length 4 (b) Boolean algebra of rank 4

Figure 7.1: Two examples of sequential differential posets

## 7.2 Dual graded graphs

Dual graded graphs were introduced by Fomin [2] concurrently with Stanley's introduction of differential posets. Basically, a dual graded graph has one vertex set and two different posets. One poset defines up maps. The other poset defines down maps. A differential poset is when the two posets coincide. A definition is as follows:

**Definition 7.2.** *A* dual graded graph *is a pair of graded, locally finite posets* $(P^1, P^2)$ *such that*

1. *If $x$ covers $k$ elements in $P^2$, then $x$ is covered by $k + r$ elements in $P^1$.*

2. *If $x \neq y$ are vertices that both cover exactly $t$ vertices in $P^2$, then there are exactly $t$ vertices that cover both $x$ and $y$ in $P^1$.*

In Definition 7.2 $P^1$ defines the up maps and $P^2$ determines down maps. If $P^1 \cong P^2$, then $P^1$ is an $r$-differential poset. Note that the $r$ in the term "dual graded graph" is suppressed.

Similarly to differential posets, there is an alternative definition of dual graded graphs involving up and down maps. Let $V$ be the common set of vertices of posets $P^1$

and $P^2$. Let $\mathbb{C}V$ be the complex vector space with basis $V$. Define linear maps $U$, $D$, and $I$ on $\mathbb{C}V$ by letting $U$, $D$, and $I$ act on $x \in V$ as:

$$Ux = \sum_{y \gtrdot_{P^1} x} y,$$

$$Dx = \sum_{y \lessdot_{P^2} x} y,$$

$$Ix = x.$$

**Definition 7.3.** *Let $P^1$ and $P^2$ be graded, locally finite posets with $\hat{0}$ and vertex set $V$. Let $r$ be a positive integer. Let $\mathbb{C}V$ be the complex vector space with basis $V$. The triple $(P^1, P^2, r)$ is is called a* dual graded graph *if:*

$$DU - UD = rI.$$

Let $P$ be an $r$-differential poset. Then, $(P, P, r)$ is a dual graded graph. So, the set of dual graded graphs contains all differential posets. There are also examples of dual graded graphs that are not differential posets.

## 7.3 Signed differential posets

Signed differential posets were introduced by Lam in [5]. The idea is to replace the relation $DU - UD = rI$ in Definition 3.10 with the relation $DU + UD = rI$. In order to make this work, one also needs to give a sign (either + or -) to each covering relation. Lam achieves this by putting a sign on each vertex and determining the signs on the cover relations from the vertex signs. The following is a slight generalization of Lam's definition.

**Definition 7.4.** *Let $r$ be a positive integer. Let $P$ be a locally finite poset with a $\hat{0}$. Let $V(P)$ be the vector space with basis $P$. Let $s : P \times P \to \{+1, -1\}$. Define linear operators $U$, $D$, and $I$ on $V(P)$ by:*

$$Ux = \sum_{y \gtrdot x} s(x, y)y,$$

$$Dx = \sum_{y \lessdot x} s(y, x)y,$$

$$Ix = x.$$

$(P, s)$ *is called a* signed $r$-differential poset *if:*

$$DU + UD = rI.$$

Definition 7.4 differs from Lam's definition in [5] in two ways. First, Lam fixed $r = 1$ and Definition 7.4 allows for arbitrary $r$. Second, Lam's definition actually requires that $s(x, y) = s(y, x)$. But, Lam also puts signs on the vertices of $P$. The vertex weights are then used in the definition of $U$ to mimic Definition 7.4.

One example of a signed 1-differential poset is a signed version of Young's lattice. In fact, Lam gives two different ways to create a signed version of Young's lattice in [5].

## 7.4 Quantized dual graded graphs

In [6] Lam introduced quantized dual graded graphs. The definition of quantized dual graded graphs differs from dual graded graphs by replacing the equation $DU - UD = rI$ with $DU - qUD = rI$ where $q$ is a parameter.

**Definition 7.5.** *A* quantized dual graded graph *is a quadruple* $(P_1, P_2, w_1, w_2)$ *where* $P_1$ *and* $P_2$ *are graded, locally finite posets that both have vertex set* $V$ *and* $w_i : V \times V \to \mathbb{N}[q]$ *such that* $DU - qUD = rI$*. Here,*

$$Ux = \sum_{y \gtrdot_{P_1} x} w_1(x, y) \cdot y,$$

*and*

$$Dx = \sum_{y \lessdot_{P_2} x} w_2(y, x) \cdot y.$$

There are many examples of quantized dual graded graphs. For instance, Lam gives examples on plane binary trees, Young tableaux, and permutations.

## 7.5 Quantized differential posets

Lam also introduced quantized differential posets in [6]. A quantized differential poset is a quantized dual graded graph where the two posets in question are isomorphic.

**Definition 7.6.** *A* quantized $r$-differential poset *is a triple:* $(P, m, m')$ *where* $P$ *is an* $r$*-differential poset,* $m\colon E(P) \to \mathbb{R}^+[q^\alpha \mid \alpha \in \mathbb{R}]$*, and* $m'\colon E(P) \to \mathbb{R}^+[q^\alpha \mid \alpha \in \mathbb{R}]$*; such that* $DU - qUD = rI$ *and* $m(x,y)|_{q=1} = m'(x,y)|_{q=1} = 1$ *for all* $(x,y) \in E(P)$*. Here* $E(P)$ *denotes the edges of the Hasse diagram of* $P$*,*

$$Ux = \sum_{y \text{ covering } x} m(x,y) \cdot y,$$

*and*

$$Dx = \sum_{y \text{ covered by } x} m'(y,x) \cdot y.$$

Definition 7.6 is more general than the definition of quantized differential posets in [6]. Lam only allows $m(x,y)$ and $m'(x,y)$ to be $q^i$ for $i$ a non-negative integer.

Note that if one sets $q = 1$ in $(P, m, m')$, then one gets the $r$-differential poset $P$. So in the definition of a quantized $r$-differential poset, it could have been assumed that $P$ was locally finite, graded, with a $\hat{0}$, and with finitely many elements of each rank (see Theorem 2.2 in [1]). Further, setting $q = -1$ in $(P, m, m')$ gives a signed differential poset.

Much of the terminology for differential posets extends to quantized differential posets. Given a quantized differential poset $(P, m, m')$, $(P_{[0,n]}, m, m')$ will be called a *partial quantized differential poset up to rank* $n$ (or sometimes just a *quantized differential poset up to rank* $n$).

In [6] Lam showed that any partial quantized $r$-differential poset $(P, m, m')$ up to rank $n$ can be extended to a partial quantized $r$-differential poset up to rank $n+1$ by *$q$-reflection extension* as follows:

- $P_{n+1} = \{x' \mid x \in P_{n-1}\} \cup \{y_1, \ldots, y_r \mid y \in P_n\}$
- for $y \in P_n$, $m(y, x') = qm'(x,y)$ and $m'(y,x') = m(x,y)$
- for $y \in P_n$, $m(y, y_i) = m'(y, y_i) = 1$

Note that $q$-reflection extension mimics reflection-extension for (non-quantized) differential posets except that $m$ and $m'$ need to be updated for the quantized case. There are other such definitions that could also have been chosen. For instance, it could have been chosen to have $m(y,x') = m'(x,y)$ and $m'(y,x') = qm(x,y)$. However, one needs to fix one choice in order to precisely define quantized Fibonacci $r$-differential posets.

Starting with $P_0 = \{0\}$, $P_1 = \{x_1, \dots, x_r\}$, $m(0,x_i) = m'(0,x_i) = 1$ one can build a quantized $r$-differential poset by continually applying $q$-reflection extensions. Call the resulting poset the *quantized Fibonacci $r$-differential poset*. Note that the quantized Fibonacci $r$-differential poset is a quantization of the Fibonacci $r$-differential poset (setting $q = 1$ gives a Fibonacci $r$-differential poset with up-weight and down-weight 1 on each edge).

A somewhat surprising result is that there is not nearly as much flexibility in building quantized $r$-differential posets as might be hoped.

**Theorem 1.4.** *If $(P,m,m')$ is a quantized $r$-differential poset, then $P$ is isomorphic to the Fibonacci $r$-differential poset.*

Theorem 1.4 solves a problem of Lam. Actually, Lam asked for a quantization of Young's lattice allowing weights of edges in $\mathbb{N}[q^{1/2}, q^{-1/2}]$. Theorem 1.4 shows that no quantization exists in a more general setting. And further, Theorem 1.4 shows that no quantization exists for any $r$-differential poset other than the Fibonacci $r$-differential poset even in this more general setting.

A natural question arising from Theorem 1.4 is if the definition of quantized $r$-differential posets is "wrong" in some sense. More precisely

**Question 7.7.** *Is there an alternative definition for a $q$ version of a differential poset with the following properties:*

- *Evaluating at $q = 1$ gives a differential poset.*
- *Evaluating at $q = -1$ gives a signed differential poset.*
- *Every differential poset arises by evaluating at $q = 1$ for some $q$ version of a differential poset.*

- *The commutator relationship $DU - UD = rI$ is replaced with something similar involving adding some $q$'s.*

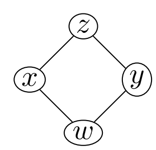
The remainder of this section is devoted to a proof of Theorem 1.4. One of the key ideas of the proof is to replace the edge weights $m$ and $m'$ with their product on each edge. This greatly reduces the possibilities to be considered and turns out to have enough power to complete the proof.

In order to use the idea of looking at the product of edge weights, the following definition of $n(x, y)$ will be used for the remainder of this section. Given a quantized $r$-differential poset $(P, m, m')$, let $n(x, y) = m(x, y) \cdot m'(x, y)$.

The following lemma will be useful for the proof of Theorem 1.4.

**Lemma 7.8.** *Suppose that $(P, m, m')$ is a quantized $r$-differential poset with $w, x, y, z \in P$, $w$ covered by $x$ and $y$, and $z$ covering $x$ and $y$. Then, $n(x, z)n(y, z) = q^2 n(w, x)n(w, y)$.*

*Proof.* The picture looks like this:




The $y$ coordinate of $DUx$ is: $m(x, z) \cdot m'(y, z)$. Note that there are no other vertices in $P$ that cover both $x$ and $y$ by Proposition 3.4. The $y$ coordinate of $(qUD + rI)x$ is: $q \cdot m'(x, z) \cdot m(y, z)$. Since $DU = qUD + rI$ this means that $m(x, z) \cdot m'(y, z) = q \cdot m'(x, z) \cdot m(y, z)$.

The $x$ coordinate of $DUy$ is: $m'(x, z) \cdot m(y, z)$. The $x$ coordinate of $(qUD + rI)y$ is: $q \cdot m(x, z) \cdot m'(y, z)$. Since $DU = qUD + rI$ this means that $m'(x, z) \cdot m(y, z) = q \cdot m(x, z) \cdot m'(y, z)$.

Multiplying the corresponding sides of the given equations gives: $n(x, z) \cdot n(y, z) = q^2 \cdot n(w, x) \cdot n(w, y)$. □

The next lemma shows that even though $m(x, y)$ and $m'(x, y)$ can take values in $\mathbb{R}^+[q^\alpha \mid \alpha \in \mathbb{R}]$ it turns out that $n(x, y)$ can only take values in $\mathbb{R}^+[q]$.

**Lemma 7.9.** *Let $(P, m, m')$ be a quantized $r$-differential poset. Then, $n(x, y) \in \mathbb{R}^+[q]$ for all $x, y \in V(P)$.*

*Proof.* Suppose not. Choose $x, y \in V(P)$ with $n(x,y) \notin \mathbb{R}^+[q]$ and with $\rho(x)$ minimal (here $\rho(x)$ is the rank of $x$ in $P$).

Now, the $x$ coordinate of $DUx$ is:

$$\sum_{z \text{ covering } x} n(x,z).$$

The $x$ coordinate of $(qUD + rI)x$ is:

$$r + \sum_{w \lessdot x} n(w,x).$$

Since $DU = qUD+rI$, $\sum_{z \gtrdot x} n(x,z) = r+\sum_{w \lessdot x} n(w,x)$. Note that $r+\sum_{w \lessdot x} n(w,x) \in \mathbb{R}^+[q]$ since $x$ has minimum rank amongst all elements of $P$ such that $n(a,b) \notin \mathbb{R}^+[q]$. However, $\sum_{z \gtrdot x} n(x,z) \notin \mathbb{R}^+[q]$ since $n(x,y) \notin \mathbb{R}^+[q]$ and all coefficients are nonnegative.

This is a contradiction since the right-hand-side is in $\mathbb{R}^+[q]$, but the left-hand-side is not in $\mathbb{R}^+[q]$. So, no such $x$ exists which completes the proof.

□

The next step in the proof of Theorem 1.4 is to create an order on $\mathbb{R}^+[q]$. Define an order $\succ$ on $\mathbb{R}^+[q]$ by $f(q) \succ g(q)$ if and only if the leading coefficient of $f(q) - g(q)$ is positive. An equivalent formulation is as follows.

Let $f(q) = \sum_{i=0}^{k} a_i q^i$ and $g(q) = \sum_{i=0}^{k} b_i q^i$. Then, $f \succ g$ if and only if there exists $t$ with $a_i = b_i$ for $i < t$ and $a_t > b_t$.

Some technical lemmas for dealing with $\succ$ will be useful.

**Lemma 7.10.** *Let $f, g, h \in \mathbb{R}^+[q]$. If $g \succ h$, then $fg \succ fh$.*

*Proof.* Note that $fg - fh = f(g-h)$. So, the leading coefficient of $fg - fh$ is the product of the leading coefficient of $f$ and the leading coefficient of $g - h$. Since $f \in \mathbb{R}^+[q]$, the leading coefficient of $f$ is positive. Since $g \succ h$, the leading coefficient of $g - h$ is positive. Thus, the leading coefficient of $fg - gh$ is positive. □

**Lemma 7.11.** *Let $f_1, f_2, g_1, g_2 \in \mathbb{R}^+[q]$. If $f_1 \succ g_1$ and $f_2 \succ g_2$, then $f_1 f_2 \succ g_1 g_2$.*

*Proof.* By Lemma 7.10 $f_1 f_2 \succ f_1 g_2$. Using Lemma 7.10 again gives $f_1 g_2 \succ g_1 g_2$. Thus, $f_1 f_2 \succ g_1 g_2$. □

**Lemma 7.12.** *Let $f_1, f_2, g_1, g_2 \in \mathbb{R}^+[q]$. If $f_1 \succ g_1$ and $f_2 \succeq g_2$, then $f_1 + f_2 \succ g_1 + g_2$.*

*Proof.* $(f_1 + f_2) - (g_1 + g_2) = (f_1 - g_1) + (f_2 - g_2)$. The result follows since the leading coefficient of $f_1 - g_1$ is nonnegative and the leading coefficient of $f_2 - g_2$ is positive. □

**Lemma 7.13.** *If $f \in \mathbb{R}^+[q]$ and $f(1) = 1$, then $f \succeq 1$.*

*Proof.* If $f$ is constant, then $f = 1$. If $f$ is not constant, then the leading coefficient of $f - 1$ is the leading coefficient of $f$. □

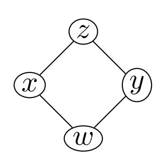
With the technical lemmas out of the way, it is now time to discuss the general idea behind the proof of Theorem 1.4. The main idea is to assume that $(P_{[0,n]}, m, m')$ is a partial quantized $r$-differential poset up to rank $n$. Then, one can show that $P$ must, in fact, be isomorphic to $Z(r)$.

The following are lemmas for the proof of Theorem 1.4.

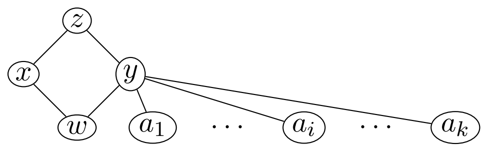
**Lemma 7.14.** *Let $(P, m, m')$ be a quantized $r$-differential poset such that $P_{[0,n]} \cong Z(r)_{[0,n]}$. Let $w \in P_{n-1}$, $x \in P_n$, $y \in P_n$, and $z \in P_{n+1}$ with $x$ and $y$ covering $w$ and $z$ covering both $x$ and $y$. Suppose that $n(y, z) \succ qn(w, y)$. Then, there exists $y', w', z' \in P$ such that:*

- *$z'$ covers both $y$ and $y'$.*
- *$w'$ is covered by both $y$ and $y'$.*
- *$n(y', z') \succ qn(w', y')$.*

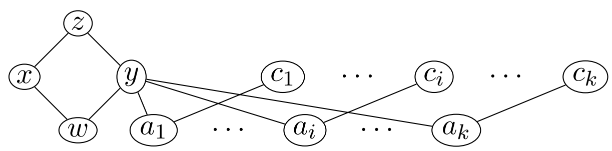
*Proof.* For clarity, here is a picture of the situation:




Let $w, a_1, \ldots, a_k$ be the complete set of elements of $P_{n-1}$ that are covered by $y$. An updated picture is as follows:

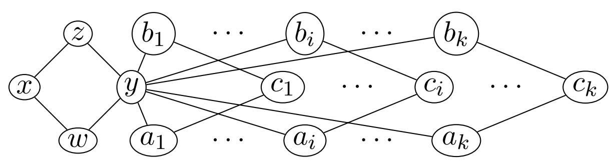



For $1 \leq i \leq k$, choose $c_i \in P_n$ with $a_i$ covered by $c_i$ and $a_i \neq y$ (such $a_i$ can be chosen since each element, except for $\hat{0}$, of an $r$-differential poset is covered by at least 2 elements). The picture now looks like:




Note that for $i \neq j$, $c_i \neq c_j$ by Proposition 3.4 since $y$ covers both $a_i$ and $a_j$.

For $1 \leq i \leq k$ choose $b_i \in P_{n+1}$ with $b_i$ covering $y$ and $b_i$ covering $c_i$. Such an element exists since $P$ is a differential poset and since $y$ and $c_i$ both cover $a_i$. The picture now looks like:




**Claim 7.15.** *If $i \neq j$, then $b_i \neq b_j$.*

*(Proof of Claim 7.15).* Suppose not. Let $b = b_i = b_j$. Than, $b$ covers all of $y$, $c_i$, and $c_j$. So, $y$, $c_i$, and $c_j$ are mutually adjacent. $a_i$ is covered by both $c_i$ and $y$. $a_j$ is covered by both $y$ and $c_j$. Since $a_i \neq a_j$ this means we mush get a crown covered by $\{y, c_i, c_j\}$ in $P_{[n-1,n]}$. But, $P_{[0,n]} \cong Z(r)_{[0,n]}$ and $Z(r)$ has no crowns. □

(Continuing the proof of Lemma 7.14) For $k+1 \leq i \leq k+r$ choose $b_i \in P_{n+1}$ such that $b_i$ covers $y$ and $b_i \notin \{z, b_1, \ldots, b_{i-1}\}$.

Now, the $y$ coordinate of $DUy$ is:

$$n(y,z) + \sum_{i=1}^{k} n(y,b_i) + \sum_{i=k+1}^{k+r} n(y,b_i).$$

The $y$ coordinate of $(qUD + rI)y$ is:

$$qn(w,y) + \sum_{i=1}^{k} qn(a_i, y) + r.$$

Since $n(y, b_i)|_{q=1} = 1$, we get that $n(y, b_i) \succeq 1$. Also, by assumption $n(y,z) \succ qn(w,y)$.

Suppose that $n(y, b_j) \succeq qn(a_j, y)$ for all $j$ with $1 \leq j \leq k$. Then,

$$\begin{aligned} DUy &= n(y,z) + \textstyle\sum_{i=1}^{k} n(y,b_i) + \sum_{i=k+1}^{k+r} n(y,b_i) \\ &\succ qn(w,y) + \sum_{i=1}^{k} n(y,b_i) + \sum_{i=k+1}^{k+r} n(y,b_i) \\ &\succeq qn(w,y) + \sum_{i=1}^{k} n(y,b_i) + r \\ &\succeq qn(w,y) + \sum_{i=1}^{k} qn(a_i,y) + r \\ &= (qUD + rI)y \end{aligned}$$

Since $DU = qUD + rI$ this is a contradiction. So for some $j$ with $1 \leq j \leq k$, $qn(a_j, y) \succ n(y, b_j)$.

Taking $z' = b_j$, $y' = c_j$, and $w' = a_j$ completes the proof.

□

**Lemma 7.16.** *Let $(P, m, m')$ be a quantized $r$-differential poset such that $P_{[0,n]} \cong Z(r)_{[0,n]}$. Let $w \in P_{n-1}$, $x \in P_n$, $y \in P_n$, and $z \in P_{n+1}$ with $x$ and $y$ covering $w$ and $z$ covering both $x$ and $y$. Then, $n(x,z) = qn(w,x)$ and $n(y,z) = qn(w,y)$.*

*Proof.* Suppose the lemma is false. By Lemma 7.8, $n(x,z) \cdot n(y,z) = q^2 \cdot n(w,x) \cdot n(w,y)$. Thus, it can be assumed without loss of generality that $n(y,z) \succ qn(w,y)$. Note that $x$, $y$, $w$, and $z$ satisfy the conditions of Lemma 7.14. The goal is to build sequences $\{x_i\}$, $\{y_i\}$, $\{z_i\}$, and $\{w_i\}$ such that for any $i$, $x_i$, $y_i$, $z_i$, and $w_i$ satisfy the conditions of Lemma 7.14.

Let $x_0 = x$, $y_0 = y$, $w_0 = w$, $z_0 = z$. Given $x_i$, $y_i$, $z_i$, and $w_i$ satisfying the conditions of Lemma 7.14 let $x_{i+1} = y_i$, $y_{i+1} = y'$, $z_{i+1} = z'$, and $w_{i+1} = w'$. Here, $y'$, $z'$, and $w'$ are the outputs of Lemma 7.14 given the inputs $x_i$, $y_i$, $z_i$, and $w_i$.

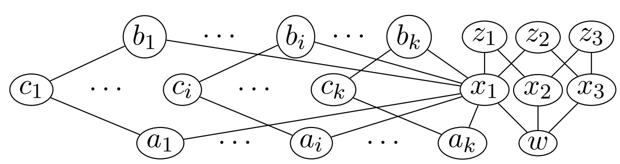
Since $P_n$ is finite, the $x_i$'s must eventually repeat. But, this means there is a cycle in the bipartite graph formed by $P_{[n-1,n]}$. This is a contradiction by Proposition 3.7 since $P_{[0,n]} \cong Z(r)_{[0,n]}$.

□

All of the necessary lemmas for the proof of Theorem 1.4 are now in place.

*Proof of Theorem 1.4.* Let $\{z_1, z_2, z_3\}$ be a crown over $\{x_1, x_2, x_3\}$ of minimal rank in $P$. Without loss of generality, let $z_1$ and $z_2$ cover $x_1$. Let $w$ be the element of $P$ that is covered by all of $x_1, x_2, x_3$. Let $\{w, a_1, \ldots, a_k\}$ be the complete set of elements of $P$ that are covered by $x_1$. Choose $b_1, \ldots, b_k$ analogously to as chosen in the proof of Lemma 7.14 with $x_1$ taking the role of $y$ and $z_1$ taking the role of $z$. Let $b_{k+1}, \ldots, b_{k+r-1}$ be the remaining elements of $P$ that cover $x_1$.

A picture is as follows:




Now, the $x_1$ coordinate of $DUx_1$ is:

$$n(x_1, z_1) + n(x_1, z_2) + \sum_{i=1}^{k} n(x_1, b_i) + \sum_{i=k+1}^{k+r-1} n(x_1, b_i).$$

The $x_1$ coordinate of $(qUD + rI)x_1$ is:

$$r + qn(w, x_1) + \sum_{i=1}^{k} qn(a_i, x_1).$$

Lemma 7.16 shows that $n(x_1, z_1) = n(x_1, z_2) = qn(w, x_1)$ and $n(x_1, b_i) = qn(a_i, x_1)$

for $1 \leq i \leq k$. Thus, from $DU = qUD + rI$:

$$qn(w, x_1) + \sum_{i=k+1}^{k+r-1} n(x_1, b_i) = r.$$

But, the degree of the left-hand-side is at least 1 and the degree of the right-hand-side is 0. This is a contradiction. So, $P$ must not have any crowns which means that $P \cong Z(r)$. $\square$

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