CRITICAL GROUPS OF GRAPHS

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ABSTRACT. In this thesis, we solve two open problems concerning the structure of critical groups of graphs.

Firstly, we compute explicitly the critical groups of threshold graphs, completing work started by Christianson and Reiner [3]. We also provide bounds on the number of invariant factors of these groups.

Secondly, we prove a conjecture of Kuperberg about critical groups of planar graphs. Propp and others [5] have found a bijection between spanning trees of a planar graph G and perfect matchings of a related bipartite graph H. These numbers are equal to the determinants of the reduced Laplacian matrix $\overline{L(G)}$ of G and the Kasteleyn-Percus matrix $M(H)$ of H. Kuperberg [6] conjectured further that the critical group $K(G)$ of G is isomorphic to the cokernel of $M(H)$. We prove this using different presentations of the critical group.

1. INTRODUCTION

Let G be a loopless, undirected graph on vertex set V and edge set E. The *Laplacian matrix* $L(G)$ of G is the $|V| \times |V|$ matrix satisfying

$$L(G)_{v,w} := \begin{cases} \deg(v) & \text{if } v = w, \\ -e_{v,w} & \text{otherwise,} \end{cases}$$

where $\deg(v)$ is the degree of vertex v and $e_{v,w}$ is the number of edges from v to w. Considering $L(G) : \mathbb{Z}^V \to \mathbb{Z}^V$ as a homomorphism of abelian groups, the *critical group* $K(G)$ of G is the torsion subgroup of the cokernel of $L(G)$, $\mathrm{coker}_{\mathbb{Z}}(L(G))$. (For a thorough discussion of critical groups, see [4].) Although the critical group is an interesting isomorphism invariant of a graph, there have been few results either relating the structures of G and $K(G)$ or determining the critical group explicitly for families of graphs (see e.g. [2, 7]). In this thesis, we solve two open problems about $K(G)$. Firstly, we compute explicitly the critical groups of threshold graphs and provide bounds on the number of invariant factors of these groups. Secondly, we prove a conjecture of Kuperberg refining a bijection between spanning trees of a planar graph G and perfect matchings of a related bipartite graph H.

In sections 2–3 we will discuss threshold graphs and some known facts related to its critical group structure, follow with a theorem (almost) explicitly describing its structure, and end with a proof of the theorem. In sections 4–7, we will discuss another interpretation of the critical group, explain the bijection between spanning trees of G and perfect matchings of H, and conclude with a proof of Kuperberg's conjecture.

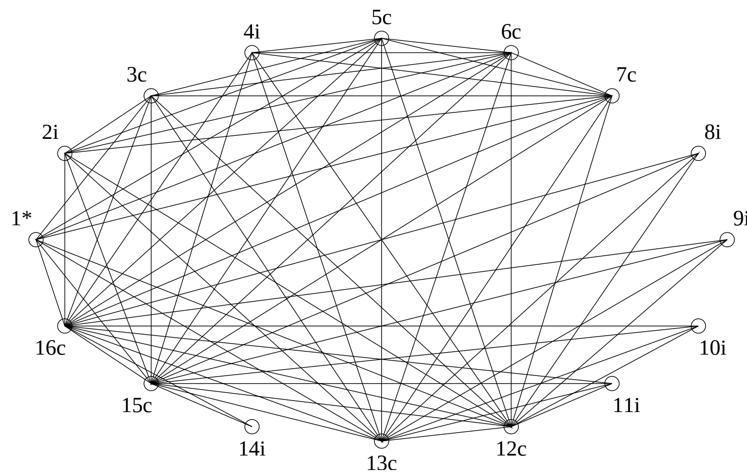

FIGURE 1. A threshold graph on 16 vertices with building sequence (isolated, cone, isolated, cone, cone, cone, isolated, isolated, isolated, isolated, cone, cone, isolated, cone, cone) and degree sequence (15,15,14,14,10,10,10,9,8,8,7,4,4,4,4,2). Here, the vertices are named $\{1*, 2i, 3c, \ldots\}$ in the order of the building sequence, where $*$ corresponds to the initial vertex, i to an isolated vertex and c to a cone vertex.

2. Threshold Graphs

Christianson and Reiner [3] provide motivation for studying threshold graphs and give some results. Here we will include only the results necessary for the proof of our theorem. We start with a characterization of threshold graphs. A graph G is threshold if and only if it can be obtained from a single vertex by iterating the operations of adding a new vertex that is either connected to no other vertex (an *isolated* vertex) or connected to every other vertex (a *cone* vertex). Call this sequence of operations the *building sequence* of G. For example, complete graphs are threshold graphs given by the building sequence (cone, cone, ..., cone). A threshold graph on 16 vertices is shown in Figure 1.

The order of $K(G)$, $\kappa(G)$, is equal to the number of spanning forests in G (or spanning trees if G is connected). The following is a version of Kirchoff's Matrix-Tree Theorem which relates $\kappa(G)$ to the eigenvalues of $L(G)$ (called the *Laplacian eigenvalues* of G).

Theorem 1. [2] *Let G be a connected graph. If $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n = 0$ are the eigenvalues of $L(G)$, then*

$$\kappa(G) = \frac{\lambda_1 \cdots \lambda_{n-1}}{n}.$$

Merris [8] found a beautiful expression for the Laplacian eigenvalues of a threshold graph.

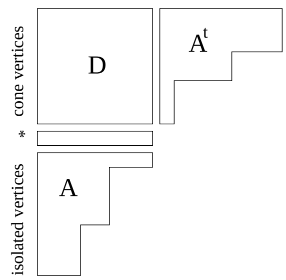

FIGURE 2. Ferrers diagram of the degree sequence d of a threshold graph. D is the *Durfee square* of d; A and A^t are conjugate partitions.

Theorem 2. [8] *Let G be a threshold graph. If $d = (d_1, \ldots, d_n)$ is the degree sequence of G, and $\lambda = (\lambda_1, \ldots, \lambda_r)$ are the non-zero eigenvalues of $L(G)$ listed in weakly decreasing order, then d and λ are conjugate partitions.*

In particular, if G is connected then $\lambda_1 = n$ and one concludes from Theorem 1 that $\kappa(G) = \lambda_2 \cdots \lambda_{n-1}$. There is another formulation of this idea related to the building sequence of G.

Proposition 3. *Let G be a connected threshold graph with vertices indexed in the order of the building sequence of G (see e.g. Figure 1). Let the sequence $c = (0, c_2, \ldots, c_n)$ be so that $c_i = \deg(v_i)$ if v_i is an isolated vertex and $c_i = \deg(v_i) + 1$ if v_i is a cone vertex. Then c is a list of the Laplacian eigenvalues of G. Also, $\kappa(G) = c_2 \cdots c_{n-1}$.*

In the graph in Figure 1, one has

$$c = (0, 8, 10, 7, 11, 11, 11, 4, 4, 4, 4, 15, 15, 2, 16, 16).$$

Proof. In light of Theorem 2, it is enough to show that $c_2, \ldots, c_n$ are the column lengths of the Ferrers diagram of d, the degree sequence of G. This is immediate once we have another characterization of threshold graphs, that the Ferrers diagram of their degree sequence has the form in Figure 2 (a proof is given in [9]). The second assertion follows from Theorem 2 since $c_n = n$. □

Notice some simple properties of the sequence c resulting from the building sequence of G. Suppose that there is a set of consecutive vertices $\{v_{l+1}, v_{l+2}, \ldots, v_{l+k}\}$ that are of the same *type*; that is, either each vertex is an isolated vertex, or each vertex is a cone vertex. Inspection shows that these vertices have the same degree, so in particular $c_{l+1} = c_{l+2} = \cdots = c_{l+k}$. On the other hand, suppose that $\{v_{l+1}, v_{l+2}, \ldots, v_{l+k}\}$ is a set of consecutive vertices of the same type and suppose that v_l and v_{l+k+1} have different type than v_{l+1}. Then the degrees of v_l and v_{l+k+1} differ by exactly k, so that $c_{l+k+1} = c_l \pm k$, where the sign is positive if v_l is a cone vertex and negative if v_l is an isolated vertex.

We are now ready to state the main theorem. Notice that if G is a disconnected threshold graph, then it contains at most one component not consisting of a single vertex. The critical groups of G and its non-trivial component are easily seen to be equal, so one loses no generality by assuming the graph is connected.

Theorem 4. *Let G be a connected threshold graph with vertices indexed in the order of the building sequence of G, and c be defined as in Proposition 3. Define a partition μ of the sequence $(c_2, \dots, c_{n-1})$ by placing c_i and c_{i+1} in the same block if they are unequal but not relatively prime. Then*

$$K(G) \cong \bigoplus_{\text{blocks } B \text{ of } \mu} \operatorname{coker}_{\mathbb{Z}}(M_B),$$

where if $B = (c_{l+1}, c_{l+2}, \dots, c_{l+k})$, then

$$M_B := \begin{pmatrix} c_{l+1} & 0 & 0 & \cdots & 0 \\ c_{l+1} \pm 1 & c_{l+2} & 0 & \cdots & 0 \\ 0 & c_{l+2} \pm 1 & c_{l+3} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & c_{l+k-1} \pm 1 & c_{l+k} \end{pmatrix},$$

where the sign in $c_{l+i} \pm 1$ is positive if v_{l+i} is a cone vertex and negative if v_{l+i} is an isolated vertex.

In particular, if $B = (c_{l+1}, c_{l+2}, \dots, c_{l+k})$ arises from Theorem 4, then the vertices $\{v_{l+1}, v_{l+2}, \dots, v_{l+k}\}$ alternate in type. Since $c_{l+i+2} = c_{l+i} \pm 1$, one has

$$M_B := \begin{pmatrix} c_{l+1} & 0 & 0 & \cdots & 0 \\ c_{l+3} & c_{l+2} & 0 & \cdots & 0 \\ 0 & c_{l+4} & c_{l+3} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & c_{l+k+1} & c_{l+k} \end{pmatrix}.$$

The matrices M_B and hence groups $\operatorname{coker}_{\mathbb{Z}}(M_B)$ are not difficult to determine in practice; in particular, if B has at most three elements, then $\operatorname{coker}_{\mathbb{Z}}(M_B) \cong \mathbb{Z}/(\prod_{c \in B} c)\mathbb{Z}$. The graph in Figure 1 has μ consisting of blocks of single elements except for vertices 2 and 3 (since $\gcd(8,10) = 2$) and vertices 14 and 15 (since $\gcd(2,16) = 2$). Thus its critical group is isomorphic to

$$\mathbb{Z}/(8 \cdot 10)\mathbb{Z} \oplus \mathbb{Z}/7\mathbb{Z} \oplus (\mathbb{Z}/11\mathbb{Z})^3 \oplus (\mathbb{Z}/4\mathbb{Z})^4 \oplus (\mathbb{Z}/15\mathbb{Z})^3 \oplus \mathbb{Z}/(2 \cdot 16)\mathbb{Z}.$$

A note should be made about the similarities between Theorem 4 and Christianson and Reiner's conjecture in [3]. They conjectured (in different terminology) that $\operatorname{coker}_{\mathbb{Z}}(M_B) \cong \mathbb{Z}/(\prod_{c \in B} c)\mathbb{Z}$ for all blocks B of μ. This turns out to be false, although the smallest counterexample to their claim is a graph with 21 vertices. This graph has degree sequence $(20, \dots, 20, 19, 15, 15, 15, 15, 15, 14)$ and it can be verified that it has the decomposition $\mu = \{(15), (15), (15), (15, 20, 14, 21), (21), \dots, (21)\}$. For $B = (15, 20, 14, 21)$, one has

$$M_B = \begin{pmatrix} 15 & 0 & 0 & 0 \\ 14 & 20 & 0 & 0 \\ 0 & 21 & 14 & 0 \\ 0 & 0 & 13 & 21 \end{pmatrix}.$$

The product $15 \cdot 20 \cdot 14 \cdot 21 = 88400$, however,

$$\operatorname{coker}_{\mathbb{Z}}(M(B)) \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/29400\mathbb{Z} \not\cong \mathbb{Z}/88400\mathbb{Z}.$$

3. Proof of Theorem 4

The proof of Theorem 4 is very specific to threshold graphs, although the techniques used are common to most problems in determining critical groups. We start by describing a more hands-on approach to the critical group. The critical group of a (connected) graph G was originally defined as the torsion subgroup of the cokernel of $L(G) : \mathbb{Z}^V \to \mathbb{Z}^V$. Note that the image of $L(G)$ actually lies in the subspace of $\mathbb{Z}^V$ orthogonal to the vector $\mathbf{1} = (1, \ldots, 1)^t$, and that $\mathbf{1}$ corresponds to the free copy of $\mathbb{Z}$ in the cokernel. Therefore we have $K(G) = \mathbf{1}^{\perp}/\operatorname{im} L(G)$. An alternate definition of the critical group is as follows.

Recall that the Smith Normal Form (SNF) of an integer matrix M is the unique diagonal matrix D with entries $d_1, d_2, \ldots, d_n$ satisfying $d_i | d_{i+1}$, where $D = PMQ$ for some unimodular matrices P and Q. Constructing P and Q is equivalent to performing integer row and column operations, defined as

- Interchanging two rows (columns),
- Adding a multiple of one row (column) to another row (column), and
- Multiplying a row (column) by ± 1.

Suppose G is a connected graph and let $d_1, \ldots, d_n$ be the diagonal entries of D, the SNF of $L(G)$. Notice that d_n is the only zero diagonal entry of D. Since P and Q are unimodular, one has

$$K(G) = \mathbf{1}^{\perp}/\operatorname{im} L(G) \cong \bigoplus_{i=1}^{n-1} \mathbb{Z}/d_i\mathbb{Z}.$$

Note that in the previous statement, it was only necessary for D to be in diagonal form. We are now ready to begin the proof of Theorem 4, which will consist of two lemmas.

Lemma 5. *Let G be a connected threshold graph with vertices indexed in the order of the building sequence of G, and let c be defined as in Proposition 3. Define a partition μ of the sequence $(c_2, \ldots, c_{n-1})$ by placing c_i and c_{i+1} in the same block if they are unequal (but possibly relatively prime). Then*

$$K(G) \cong \bigoplus_{\text{blocks } B \text{ of } \mu} \operatorname{coker}_{\mathbb{Z}}(M_B),$$

where M_B is defined in Theorem 4.

Lemma 6. *Consider $B = (c_{l+1}, \ldots, c_{l+k})$ and M_B as defined in Theorem 4. If some c_{l+j} and c_{l+j+1} are relatively prime, then*

$$\operatorname{coker}_{\mathbb{Z}}(M_B) \cong \operatorname{coker}_{\mathbb{Z}}(M_{B_1}) \oplus \operatorname{coker}_{\mathbb{Z}}(M_{B_2}),$$

where $B_1 = (c_{l+1}, \ldots, c_{l+j})$ and $B_2 = (c_{l+j+1}, \ldots, c_{l+k})$.

Proof of Lemma 5. It will be enough to show that we can achieve the matrix

$$1 \oplus 0 \bigoplus_{\text{blocks } B \text{ of } \mu} M_B$$

from $L(G)$ by a sequence of integer row and column operations. The proof will consist of six steps that bring $L(G)$ to the required form. At each stage we will fully decribe the resulting intermediate matrix, but will not provide the tedious proof that these descriptions are correct. We will also demonstrate the steps using the graph G in Figure 1. To begin, $L(G)$ is given below (with zeros suppressed).

	$1*$	$2i$	$3c$	$4i$	$5c$	$6c$	$7c$	$8i$	$9i$	$10i$	$11i$	$12c$	$13c$	$14i$	$15c$	$16c$
$1*$	8		−1		−1	−1	−1					−1	−1		−1	−1
$2i$		8	−1		−1	−1	−1					−1	−1		−1	−1
$3c$	−1	−1	9		−1	−1	−1					−1	−1		−1	−1
$4i$				7	−1	−1	−1					−1	−1		−1	−1
$5c$	−1	−1	−1	−1	10	−1	−1					−1	−1		−1	−1
$6c$	−1	−1	−1	−1	−1	10	−1					−1	−1		−1	−1
$7c$	−1	−1	−1	−1	−1	−1	10					−1	−1		−1	−1
$8i$								4				−1	−1		−1	−1
$9i$									4			−1	−1		−1	−1
$10i$										4		−1	−1		−1	−1
$11i$											4	−1	−1		−1	−1
$12c$	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	14	−1		−1	−1
$13c$	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	14		−1	−1
$14i$														2	−1	−1
$15c$	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	15	−1
$16c$	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	15

Step 1. One can make every entry in the last row and first column zero by first adding each row to the last row and then each column to the first column. This produces the 0 diagonal entry in the SNF of $L(G)$. The last column then has entries of -1 except in the last row (because our graph is connected). Subtract this column from each column corresponding to a cone vertex (those columns which have a -1 in the first row). Then use the -1 entry in the first row and last column to eliminate the other entries in the last column. Finally, multiply this column by -1 to produce a 1 as diagonal entry in the SNF of $L(G)$. One now has a lower triangular matrix $(m_{i,j})_{i,j=2,\ldots,n-1}$ with diagonal entries corresponding to c_i as described in Proposition 3. A full description is as follows.

$$m_{i,j} = \begin{cases} 0 & \text{if } i > j, \\ c_i & \text{if } i = j, \\ 1 & \text{if } v_i \text{ is isolated, } v_j \text{ is cone,} \\ -1 & \text{if } v_i \text{ is cone, } v_j \text{ is isolated,} \\ 0 & \text{otherwise.} \end{cases}$$

	$2i$	$3c$	$4i$	$5c$	$6c$	$7c$	$8i$	$9i$	$10i$	$11i$	$12c$	$13c$	$14i$	$15c$
$2i$	8													
$3c$	−1	10												
$4i$		1	7											
$5c$	−1		−1	11										
$6c$	−1		−1		11									
$7c$	−1		−1			11								
$8i$		1		1	1	1	4							
$9i$		1		1	1	1		4						
$10i$		1		1	1	1			4					
$11i$		1		1	1	1				4				
$12c$	−1		−1				−1	−1	−1	−1	15			
$13c$	−1		−1				−1	−1	−1	−1		15		
$14i$		1		1	1	1					1	1	2	
$15c$	−1		−1				−1	−1	−1	−1			−1	16

Step 2. Let $v_{j'}$ denote the next vertex after vertex v_j of the same type. Starting with the first column, subtract column j' from column j, whenever j' is defined. Then one has

$$m_{i,j} = \begin{cases} c_i & \text{if } i = j, \\ -c_i & \text{if } i = j', \\ 1 & \text{if } j < i < j' \text{ and } v_j \text{ is cone,} \\ -1 & \text{if } j < i < j' \text{ and } v_j \text{ is isolated,} \\ 0 & \text{otherwise, i.e. } i \notin [j, j']. \end{cases}$$

	$2i$	$3c$	$4i$	$5c$	$6c$	$7c$	$8i$	$9i$	$10i$	$11i$	$12c$	$13c$	$14i$	$15c$
$2i$	8													
$3c$	−1	10												
$4i$	−7	1	7											
$5c$		−11	−1	11										
$6c$			−1	−11	11									
$7c$			−1		−11	11								
$8i$			−4			1	4							
$9i$						1	−4	4						
$10i$						1		−4	4					
$11i$						1			−4	4				
$12c$						−15				−1	15			
$13c$										−1	−15	15		
$14i$										−2		1	2	
$15c$												−16	−1	16

Step 3. Notice that almost every column sums to zero. Starting with the last row, add to each row each of the rows above it. Then one has

$$m_{i,j} = \begin{cases} c_i & \text{if } i = j, \\ c_j + i - j & \text{if } j < i < j' \text{ and } v_j \text{ is cone}, \\ c_j - i + j & \text{if } j < i < j' \text{ and } v_j \text{ is isolated}, \\ 0 & \text{otherwise, i.e. } i \notin [j, j'). \end{cases}$$

	$2i$	$3c$	$4i$	$5c$	$6c$	$7c$	$8i$	$9i$	$10i$	$11i$	$12c$	$13c$	$14i$	$15c$
$2i$	8													
$3c$	7	10												
$4i$		11	7											
$5c$			6	11										
$6c$			5		11									
$7c$			4			11								
$8i$						12	4							
$9i$						13		4						
$10i$						14			4					
$11i$						15				4				
$12c$										3	15			
$13c$										2		15		
$14i$												16	2	
$15c$													1	16

Steps 4–6 will resolve local problems stemming from consecutive vertices of the same type. For example, in column 7, we have entries 12, 13, 14, 15 which correspond to the four isolated vertices 8-11. The general procedure begins with the right-most set of consecutive vertices of the same type. In our case, this is the two cone vertices 12 and 13.

Step 4. Let $v_i, \ldots, v_j$ be a (maximal) set of more than two consecutive vertices of the same type. (Here maximal means that both v_{i-1} and v_{j+1} are of different type than $v_i, \ldots, v_j$.) Note that the other entry in row $i-1$ (if there is another) is exactly c_i. Therefore when one subtracts row $i-1$ from row i one gets ± 1 in the entry $(i, i-1)$ and $-c_i$ somewhere else. Since c_i is the only entry in column i, one can use it to eliminate the $-c_i$. At the end of Step 6 one reverses this procedure to change the ± 1 back to $c_{i-1} \pm 1$. An example with a consecutive set in G is shown below.

	$7c$	$11i$	$12c$	$13c$
$11i$	15	4	0	0
$12c$	0	3	15	0
$13c$	0	2	0	15

→

	$7c$	$11i$	$12c$	$13c$
$11i$	15	4	0	0
$12c$	−15	−1	15	0
$13c$	0	2	0	15

→

	$7c$	$11i$	$12c$	$13c$
$11i$	15	4	0	0
$12c$	0	−1	15	0
$13c$	0	2	0	15

(1) Subtract row 11 from row 12. **(2)** Add column 12 to column 7.

Step 5. For each vertex in the middle of the group, i.e. v_k for $i < k < j$, one can eliminate the off-diagonal entry in its row. Since one has the ± 1 entry in row i, one can add some multiple of it to row k to eliminate the off-diagonal entry, introducing a multiple of c_i in column i. But $c_k = c_i$ is the only entry in column k so one can use it to eliminate the multiple of c_i.

Step 6. Proceed in the same way as Step 5 for v_j, adding to row j some multiple of row i. Since there is another entry in column j, when one adds the multiple of column j to column i, one ends up with a multiple of the other entry in column i. Notice that this multiple is equal to the next diagonal entry c_{j+1} except possibly for a sign (it can be shown that this *always* happens because of the structure of the matrix after Step 3). Therefore, one can use that diagonal to eliminate the entry in the first column, introducing a multiple of the next diagonal entry in the next row of the first column. This process stops eventually, when one gets to a column with only one nonzero entry. This occurs when either one is at the end of the matrix or when one is at another set of consecutive vertices of the same type. (Remember that this procedure started with the rightmost such set, so the first vertex in the next group encountered will be the only entry in its column.) The example is shown below.

	$12c$	$13c$	$14i$	$15c$
$12c$	15	0	0	0
$13c$	$-2\cdot 15$	15	0	0
$14i$	0	16	2	0
$15c$	0	0	1	16

$\rightarrow$

	$12c$	$13c$	$14i$	$15c$
$12c$	15	0	0	0
$13c$	0	15	0	0
$14i$	$16\cdot 2$	16	2	0
$15c$	0	0	1	16

$\rightarrow$

	$12c$	$13c$	$14i$	$15c$
$12c$	15	0	0	0
$13c$	0	15	0	0
$14i$	0	16	2	0
$15c$	$-1\cdot 16$	0	1	16

$\rightarrow$

	$12c$	$13c$	$14i$	$15c$
$12c$	15	0	0	0
$13c$	0	15	0	0
$14i$	0	16	2	0
$15c$	0	0	1	16

(1) Add 2 times column 13 to column 12. **(2)** Subtract 16 times column 14 from column 12. **(3)** Add column 15 to column 12.

After Step 6 one uses Step 4 to change the ± 1 entry at $(i, i-1)$ back to $c_{i-1} \pm 1$. One next performs Steps 4–6 with the next rightmost set of consecutive vertices of the same type. The final matrix one obtains with G is shown below.

	$2i$	$3c$	$4i$	$5c$	$6c$	$7c$	$8i$	$9i$	$10i$	$11i$	$12c$	$13c$	$14i$	$15c$
$2i$	8													
$3c$	7	10												
$4i$		11	7											
$5c$			6	11										
$6c$					11									
$7c$						11								
$8i$						12	4							
$9i$								4						
$10i$									4					
$11i$										4				
$12c$										3	15			
$13c$												15		
$14i$												16	2	
$15c$													1	16

When finished, one has exactly the matrix

$$1 \oplus 0 \bigoplus_{\text{blocks } B \text{ of } \mu} M_B$$

This concludes the proof.

□

Proof of Lemma 6. To prove this lemma, we will show that the SNFs of the matrices M_B and $M_{B_1} \oplus M_{B_2}$ are equal. This uses another characterization of the SNF.

Proposition 7. *Let Δ_i be the greatest common divisor (gcd) of the determinants of all $i \times i$ submatrices of an integer matrix M. Then the SNF of M has diagonal entries $d_1, \ldots, d_n$ satisfying $\Delta_i = \prod_{j=1}^{i} d_j$, for each i.*

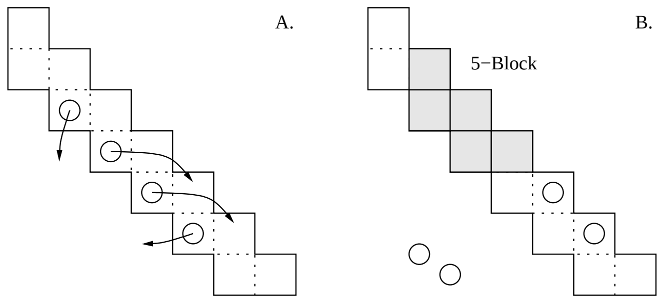

FIGURE 3. Creating choices in $M_{B_1} \oplus M_{B_2}$ from one in M_B.

Thus it will be enough to show that $\Delta_i = \Delta_i'$, where Δ_i corresponds to M_B and Δ_i' corresponds to $M_{B_1} \oplus M_{B_2}$. Notice that the nonzero determinants of $i \times i$ submatrices of M_B correspond exactly to the product of i nonzero entries, with no two in the same row or column. (Call such a selection of entries and its product a *choice*.) Since M_B and $M_{B_1} \oplus M_{B_2}$ differ in only the $(j+1, j)$ entry (which we name e), it is clear that every choice in $M_{B_1} \oplus M_{B_2}$ will also be a choice in M_B. Therefore Δ_i divides Δ_i'. To prove that these values are equal, it will be enough to show that for every choice C involving e, there are some choices not involving e, whose gcd divides C.

Given a choice C involving e, consider the string S of off-diagonal entries in C that contains e; that is, the set of entries

$$S = \{(m+1, m), (m+2, m+1), \ldots, (n+1, n)\} \subseteq C$$

such that $(j+1, j) \in S$ but $(m, m-1) \notin C$ and $(n+2, n+1) \notin C$. If this string consists only of e, consider the two choices which instead of e have either c_j or c_{j+1}. This is a valid choice, since there are no entries in the adjacent off-diagonals. By hypothesis c_j and c_{j+1} are relatively prime, so the gcd of these two choices will be exactly C/e. Therefore there are two choices whose gcd divides C.

If the set S consists of more than one element, one can find choices by the following procedure. First, remove both the first and last elements in the set. Then instead of selecting the off-diagonal entries in the set, choose those entries two columns to the right and one row down (see Figure 3). The structure of M_B guarantees that these entries are the same, and this is a valid choice because the last element of S has been removed. Notice that there is now at least a *5-block* (three diagonal entries and two off-diagonal entries, as shown in Figure 3) of legal positions. One needs two entries in this 5-block to create a choice. If it is possible to find some valid choices in this 5-block which have a gcd equal to 1, then there will be some choices that divide C and the proof will be complete. This is possible, and breaks into three cases, illustrated in Figure 4.

Case I: e is in the left off-diagonal. The choices (A, C) and (A, D) imply that the gcd divides A (since $C = D \pm 1$). The choices (A, C) and (C, E) imply that the gcd divides C (since $A = E \pm 1$). By hypothesis A and C are relative prime, so the gcd equals 1.

Case II: e is in the right off-diagonal. The choices (A, C) and (C, E) imply that the gcd divides C (since $A = E \pm 1$). The choices (A, E) and (B, E) imply that the

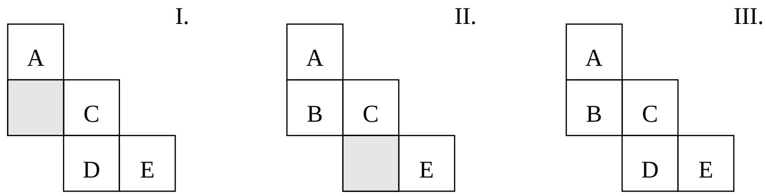

FIGURE 4. Three cases in Lemma 6. The shaded square represents e.

gcd divides E (since $A = B \pm 1$). By hypothesis C and E are relatively prime, so the gcd equals 1.

Case III: e is in neither off-diagonal. The choices (A, C) and (C, E) imply that the gcd divides C (since $A = E \pm 1$). The choices (A, D) and (B, D) imply that the gcd divides D (since $A = B \pm 1$). The gcd divides both C and D, and since $C = D \pm 1$ the gcd equals 1.

□

We conclude this section with some discussion of consequences of Theorem 4. We first point out that one cannot expect a more precise statement about the structure of $\text{coker}_{\mathbb{Z}}(M_B)$ in general, without encountering interactions of prime factorizations of the entries $B = (c_{l+1}, \ldots, c_{l+k})$. If B has at most three elements, M_B is at most a 5-block, so we conclude that $\text{coker}_{\mathbb{Z}}(M_B) \cong \mathbb{Z}/(\prod_{c \in B} c)\mathbb{Z}$. However, when B has more than three elements, we are not guaranteed that $\Delta_3 = 1$. In the 21 vertex counterexample mentioned earlier, there is a block B which results in $\Delta_3 = 3$. This gives $\text{coker}_{\mathbb{Z}}(M_B) \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/29400\mathbb{Z}$ instead of $\mathbb{Z}/88200\mathbb{Z}$. On the other hand, Theorem 4 does have the following general corollary.

Corollary 8. *The number of invariant factors of $K(G)$ for a connected threshold graph G is bounded below and above by the number of invariant factors of*

$$\bigoplus_{\text{blocks } B \text{ of } \mu} \mathbb{Z}/\left(\prod_{c \in B} c\right)\mathbb{Z} \quad \text{and} \quad \bigoplus_{i=2}^{n-1} \mathbb{Z}/c_i\mathbb{Z},$$

respectively, where μ and $c = (c_2, \ldots, c_{n-1})$ are defined in Theorem 4.

Proof. It is enough to show that for each $B = (c_{l+1}, \ldots, c_{l+k})$, the number of invariant factors of $\text{coker}_{\mathbb{Z}}(M_B)$ is bounded below and above by the number of invariant factors of

$$\mathbb{Z}/\left(\prod_{i=1}^{k} c_{l+i}\right)\mathbb{Z} \quad \text{and} \quad \bigoplus_{i=1}^{k} \mathbb{Z}/c_{l+i}\mathbb{Z},$$

respectively. The first bound is obvious, since there is just one invariant factor. Note that if M is a non-singular $k \times k$ matrix with Δ_i defined in Proposition 7, then the number of invariant factors of $\text{coker}_{\mathbb{Z}}(M)$ is equal to $k - i$ where

$$i = \max\{j \mid \Delta_j = 1\}.$$

Let M'_B be the matrix with nonzero entries $m_{i,i} = c_{l+i}$. If Δ and Δ' correspond to M_B and M'_B, respectively, then Δ_i divides Δ'_i for each i, since every choice in M'_B is also a choice in M_B. Thus if $k - i$ is the number of invariant factors of $\text{coker}_{\mathbb{Z}}(M'_B)$, then $\Delta'_i = 1$, so $\Delta_i = 1$ and $\text{coker}_{\mathbb{Z}}(M_B)$ has at most $k - i$ invariant factors. □

4. Laplacian and Kasteleyn-Percus Matrices

We now shift our attention towards understanding the conjecture of Kuperberg. We start by recalling another version of Kirchoff's Matrix-Tree theorem.

Theorem 9. [2] *Let $\kappa(G)$ denote the number of spanning trees in a connected graph G and let the* reduced Laplacian $\overline{L(G)}^v$ *be $L(G)$ with row and column v removed. Then for any $v \in V$,*

$$\kappa(G) = \det(\overline{L(G)}^v).$$

Since this result does not depend on the choice of vertex v, one often writes $\overline{L(G)}$ instead of $\overline{L(G)}^v$. Considering $L(G) : \mathbb{Z}^V \to \mathbb{Z}^V$ as a homomorphism of abelian groups, we had defined the *critical group* $K(G)$ to be the torsion subgroup of the cokernel of $L(G)$, $\operatorname{coker}_{\mathbb{Z}}(L(G))$. If G is connected, then $K(G) \cong \operatorname{coker}_{\mathbb{Z}}(\overline{L(G)})$.

Let H be a bipartite graph with vertex set $V_1 \sqcup V_2$. Assign a weight of ± 1 to each edge (these weights are sometimes thought of as signs $+$ or $-$). The *bipartite adjacency matrix* $M(H)$ is a $|V_1| \times |V_2|$ matrix with nonzero entries

$$M(H)_{v,w} = \sum \text{weight}(e),$$

where the sum is taken over all edges e with endpoints v and w. The following theorem is due to Percus.

Theorem 10. [6] *Let $\rho(H)$ denote the number of (perfect) matchings in a simple, planar, bipartite graph H. Then H admits a weight assignment so that*

$$\rho(H) = \det(M(H)).$$

Such a $M(H)$ is called a *Kasteleyn-Percus matrix* for H. Propp and others [5] have found a nice bijection between spanning trees in any connected, planar graph G and perfect matchings in a related simple, planar, bipartite graph H. Kuperberg [6] has further conjectured that $\operatorname{coker}_{\mathbb{Z}}(M(H)) \cong K(G)$. In this rest of this paper, we will further discuss the critical group, explain the bijection found by Propp, and conclude by proving Kuperberg's conjecture.

5. Spaces and Lattices of Cuts and Flows

We start by discussing the critical group of a graph. It has another description in terms of the cut and flow space of the graph. (For a more thorough discussion, see [1] or [4].) Inside these spaces are relavant lattices, so we begin by stating a few results about lattices in general.

Consider $\mathbb{R}^m$ with the usually inner product $\langle \cdot, \cdot \rangle$. Let $C_{\mathbb{R}}$ be a rational subspace of $\mathbb{R}^m$ (meaning $C_{\mathbb{R}}$ has a basis in $\mathbb{Q}^m$ or equivalently $\mathbb{Z}^m$), and let $F_{\mathbb{R}}$ be its orthogonal complement. Define the rational lattices

$$\begin{aligned} C &:= C_{\mathbb{R}} \cap \mathbb{Z}^m \\ C^\sharp &:= \{x \in C_{\mathbb{R}} \mid \langle x, y \rangle \in \mathbb{Z}, \forall y \in C\} \end{aligned}$$

and F and $F^\sharp$ similarly. An example is shown in Figure 5, where

$$C_{\mathbb{R}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \mathbb{R}, \quad C = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \mathbb{Z}, \quad \text{and} \quad C^\sharp = \begin{bmatrix} 2/5 \\ 1/5 \end{bmatrix} \mathbb{Z}.$$

Since $F \subseteq F^\sharp$, one can consider the *determinant group* $F^\sharp / F$.

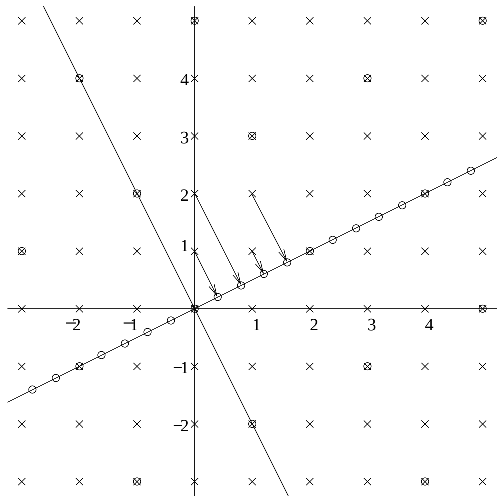

FIGURE 5. An example of a rational lattice.

Theorem 11. [1] *For rational lattices C and F defined above, one has*

$$\mathbb{Z}^m/(C \oplus F) \cong F^\sharp/F(\cong C^\sharp/C).$$

Proof. Notice that the natural projection map $\pi_{F_\mathbb{R}} : \mathbb{R}^m \to F_\mathbb{R}$ restricts to $\pi_{F_\mathbb{R}} : \mathbb{Z}^m \to F^\sharp$, since for any $x \in \mathbb{Z}^m$ and $y \in F \subset \mathbb{Z}^m$ we have $\langle \pi_{F_\mathbb{R}}(x), y\rangle = \langle x, y\rangle \in \mathbb{Z}$. One can show that the composite

$$\mathbb{Z}^m \xrightarrow{\pi_{F_\mathbb{R}}} F^\sharp \to F^\sharp/F$$

has kernel $F \oplus C$. Note that $x \in \mathbb{Z}^m$ is in the kernel if and only if $\pi_{F_\mathbb{R}}(x) \in F$. This happens if and only if

$$x = \underbrace{x - \pi_{F_\mathbb{R}}(x)}_{\in C_\mathbb{R} \cap \mathbb{Z}^m = C} + \underbrace{\pi_{F_\mathbb{R}}(x)}_{\in F},$$

that is, if and only if $x \in C \oplus F$. Thus one has an injection $\mathbb{Z}^m/C \oplus F \hookrightarrow F^\sharp/F$. This map can also be shown to be a surjection (see [1]), which completes the proof. □

Theorem 12. *Suppose L is a rank r sublattice of $\mathbb{Z}^m$ inside $\mathbb{R}^m$ and $\{z_1, \ldots, z_r\}$ is an integer basis for L. Then*

$$L^\sharp/L \cong \operatorname{coker}_\mathbb{Z} \left(\langle z_i, z_j\rangle\right)_{i,j=1,\ldots,r}.$$

Proof. The matrix $\left(\langle z_i, z_j\rangle\right)_{i,j=1,\ldots,r}$ is known as the *Gram matrix* of $\{z_1, \ldots, z_r\}$. Since $\{z_1, \ldots, z_r\}$ is an integer basis for L, there is an integer basis for $L^\sharp$ given by the unique $w_1, \ldots, w_r \in L_\mathbb{R}$ satisfying $\langle w_i, z_j\rangle = \delta_{ij}$. Therefore when expressing one basis in terms of the other,

$$z_i = \sum_{j=1}^{r} c_{i,j} w_j,$$

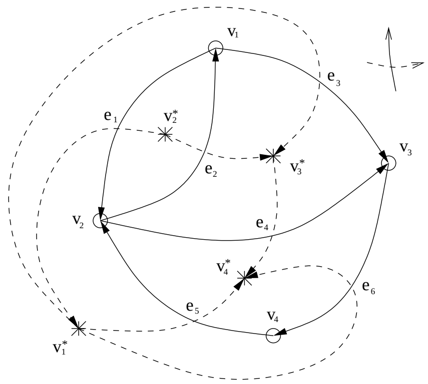

FIGURE 6. An example of consistent orientations. The edges of G^* always point just clockwise of their counterparts in G.

and one concludes that $c_{i,j} = \langle z_i, z_j \rangle$. Thus

$$\begin{aligned} L^\sharp/L &= (\mathbb{Z}w_1 \oplus \cdots \oplus \mathbb{Z}w_r)/(\mathbb{Z}z_1 \oplus \cdots \oplus \mathbb{Z}z_r) \\ &= \mathbb{Z}^r/\mathrm{im}\,(\langle z_i, z_j \rangle)_{i,j=1,\ldots,r} \\ &= \mathrm{coker}_{\mathbb{Z}}\,(\langle z_i, z_j \rangle)_{i,j=1,\ldots,r}\,. \end{aligned}$$

□

We now are ready to apply this theory to graphs. Define an *orientation* on a graph G by assigning a direction on each edge. The *incidence matrix* ∂ of G is the $|V| \times |E|$ matrix with nonzero entries

$$\partial_{v,e} = \begin{cases} 1 & \text{if } v \text{ is the head of edge } e, \\ -1 & \text{if } v \text{ is the tail of edge } e. \end{cases}$$

The *cutspace* C of G is the rowspace of ∂ in $\mathbb{R}^E$; the *flowspace* F is the orthogonal complement of C, which is the nullspace of ∂. The reason for this terminology is that these two spaces are spanned by vectors corresponding to cuts (or bonds) and flows (or cycles) in the graph, respectively. An example is given in Figure 6. The incidence matrices are given below.

$$\partial = \begin{pmatrix} -1 & 1 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}, \ \partial^* = \begin{pmatrix} 1 & 0 & -1 & 0 & -1 & -1 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

For any vertex v, let $\overline{\partial}^v$ be ∂ with row v removed. Then for any $v \in V$, the rows of $\overline{\partial}^v$ will be an integer basis for C. Note that $L(G) = \partial\,\partial^t$ for any orientation on G. One concludes from Theorem 12 that

Corollary 13. *Let ∂ be an incidence matrix of a connected graph G. Then*

$$C^\sharp/C \cong \mathrm{coker}_{\mathbb{Z}}(\overline{\partial}\,\overline{\partial}^\perp) \cong \mathrm{coker}_{\mathbb{Z}}(\overline{L(G)}) \cong K(G).$$

Let G be a planar graph and let G^* be its dual graph. It is not hard to see that a set of edges is a cut (flow) in G if and only if it is a flow (cut) in G^*. Therefore, if consistent orientations are put on G and G^* (see Figure 6), then the

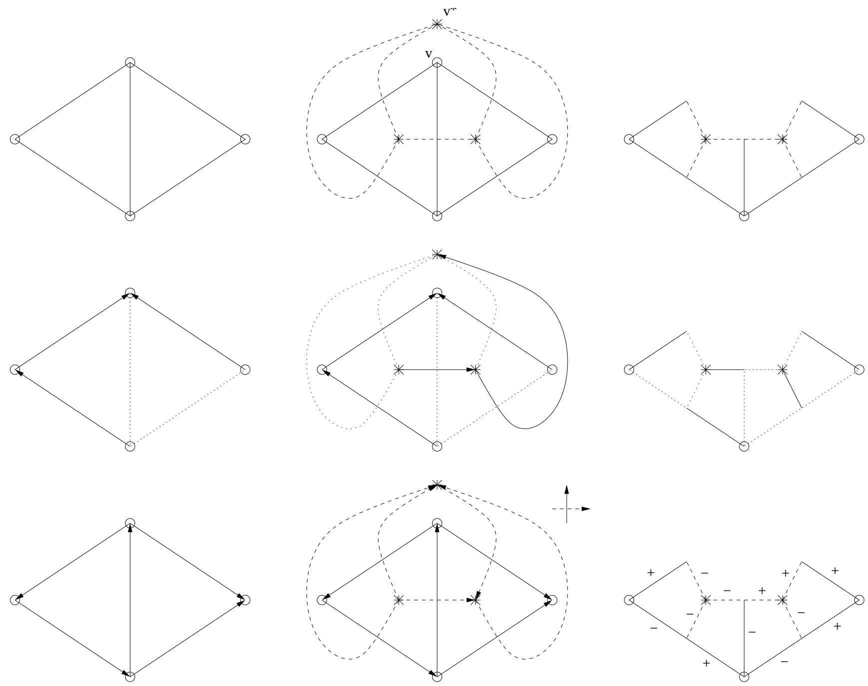

FIGURE 7. In each row, the first column is a graph G, the second column is G and G^*, and the third column is the resulting graph H. The first row illustrates the construction of H. The second row gives an example of finding a matching in H from a spanning tree in G. The third row shows how to assign weights to the edges of H based on an orientation on G.

cutspace (flowspace) of G will be the flowspace (cutspace) of G^*. From Theorem 11, $F^\sharp/F \cong C^\sharp/C$, so then $K(G) \cong K(G^*)$.

6. TEMPERLEY BIJECTION

We are now ready to explain the bijection between spanning trees in a planar graph G and perfect matchings in a related planar, bipartite graph H. The bijection was originally due to Temperley, although Propp and others [5] have generalized his argument to weighted, directed trees and weighted matchings. However, we will only need the original bijection for the proof of Kuperberg's conjecture.

Let G be a connected, planar graph. We will now describe a method for creating a simple, planar, bipartite graph H so that $\kappa(G) = \rho(H)$.

Graph Construction. Let G^* be the dual graph to G. Create the simple graph H' on vertex set $V \sqcup V^* \sqcup E$ by forming an edge (v, e) in H' whenever $v \in V \sqcup V^*$ is incident to edge e (here the edges of G and G* are identified). This process is illustrated in the first row of Figure 7. Let $v \in V$ and $v^* \in V^*$ be two vertices incident to the same edge. Then define the graph H to be H' with the vertices v and v^* (and all edges incident to either vertex in H') removed.

Theorem 14. *If G and H are the graphs described above, then $\kappa(G) = \rho(H)$.*

Proof. We will show a bijection between spanning trees of G and perfect matchings in H. Given a spanning tree T in G, let T^* be the graph in G^* consisting exactly of the edges not in T. It is not hard to see that T^* is a spanning tree of G^*. Direct the edges of T and T^* so that the edges point towards v and v^*, respectively. Create the set of edges M in H by taking the half–edges corresponding to tail ends of edges in T or T^*. (This process is illustrated in Figure 7.) Then M is a perfect matching. On the other hand, given a perfect matching M in H, we can construct spanning trees T and T^* of G and G^*, respectively, in a similar manner. (A more detailed proof can be found in [5].) □

7. Kuperberg's Conjecture

We now come to the conjecture of Kuperberg, which we present as a theorem.

Theorem 15. *Let G be a connected, planar graph and let H be the associated bipartite graph in Temperley's bijection. Let $K(G)$ be the critical group of G and let $M(H)$ be the Kasteleyn-Percus matrix of H. Then $K(G) \cong \mathrm{coker}_{\mathbb{Z}}(M(H))$.*

Proof. Begin by putting consistent orientations on G and G^*. Each edge of H corresponds to a half–edge in either G or G^*. Give an edge of H a weight of 1 if it is on the head end of its corresponding edge in G or G^*, and -1 if it is on the tail end. Let ∂ and ∂^* be the incidence matrices for G and G^*, respectively. If v and v^* are the removed vertices, then $M(H)$ has rows indexed by E and columns indexed by $(V - \{v\}) \sqcup (V^* - \{v^*\})$. It is not hard to see that in fact

$$M(H) = \left(\overline{\partial}^t : \overline{\partial^*}^t\right),$$

the side-by-side concatentation of the two matrices $\overline{\partial}^t$ and $\overline{\partial^*}^t$. Let F and C be the flowspace and cutspace of G. Then F is the rowspace of ∂ and C is the rowspace of ∂^*. In light of Theorem 11 and Corollary 13, one has

$$\mathrm{coker}_{\mathbb{Z}}(M(H)) = \mathbb{Z}^E/\mathrm{im}\,(M(H)) = \mathbb{Z}^E/(F \oplus C) \cong K(G).$$

□

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