Combinatorics of Bulgarian Solitaire

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Senior Thesis

Abstract

The Bulgarian Solitaire rule induces a finite dynamical system on the set of integer partitions of n. Brandt [3] characterized and counted all cycles in its recurrent set for any given n, with orbits parametetrized by necklaces of black and white beads. However, the transient behavior within each orbit has been almost completely unknown. The only known case is when $n = \binom{k}{2}$ is a triangular number, in which case there is only one orbit. Eriksson and Jonsson [6] gave an analysis for convergence of the structure as k grows, and to what extent the limit applied to the finite case.

In this thesis, we generalize the convergent structure for any n and provide first results about the size of any orbits, corresponding to various different types of necklaces, including those whose beads alternate $BWBWBW\cdots = (BW)^k$, and also $BWWBWW\cdots = (BWW)^k$ and $BBWBBW\cdots = (BBW)^k$. For necklaces of the form $(BW)^k$ the orbit size is the Chebyshev polynomial $T_k(x)$ evaluated at $x = 2$. Furthermore, we derive a generating function counting the transients in these orbits according to their distance from the periodic cycle. A similar analysis shows that the orbits corresponding to the necklace $(BWW)^k$ and $(BBW)^k$ have sizes 5^k and $7 \cdot 5^{k-1}$. respectively.

We will also give some properties for the partitions in each orbit and conjecture some general formulas for the sizes of Bulgarian Solitaire orbits.

Contents

Acknowledgement

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Part I
Background and results

1 Preliminaries

1.1 Integer partitions

A partition of a positive integer n is a way to write it as a sum of integers. In partition theory, order of the parts does not matter, e.g. $1+2+4, 1+4+2$ and $2+4+1$ are the same partition of 7. We write $p(n)$ to be the number of partitions of n and denote the set of partitions on n by $\mathcal{P}(n)$. Integer partitions have interested mathematicians since the 18th century with applications in computer science, statistical mechanics, algebra and other branches of math. A lot of interesting classes of integer partitions and their properties have been discovered. Moreover, although the number of partitions of a given number n was found as a limit of a series (the *Hardy-Ramanujan-Rademacher expansion* [2]) or as coefficients of the generating functions

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{n=1}^{\infty} \frac{1}{1-x^n},$$

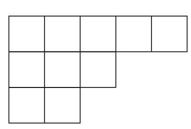
mathematicians are still interested in finding a simpler, explicit formula for it. One result is a recurrence formula [2] involving the *pentagonal numbers* $g_k = k(3k-1)/2$:

$$p(n) = \sum_k (-1)^{k-1} p(n-g_k).$$

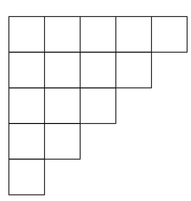
Recently, Bruinier and Ono proved a formula for $p(n)$ as a finite sum of algebraic numbers lying in the usual discriminant $-24n+1$ ring class field [4]. In this thesis, we work with a dynamical system on $\mathcal{P}(n)$ and hope that our structure might give interesting properties of integer partitions.

1.2 Young diagrams

The *Young diagram* is a visualization of an integer partition. For example, the partition $10 = 5+3+2$ is drawn as



with the rows being the parts in non-strict descending order. The $k-$staircase partition of $n = \binom{k+1}{2}$ is denoted by $\Delta_k = (k, k-1, \ldots, 1, 0)$. For example of Δ_5 is



2 Introduction to Bulgarian Solitaire

2.1 History

The game of Bulgarian Solitaire was introduced by Martin Gardner in 1983. The original game starts with 45 cards divided into a number of piles. Now keep repeating the *Bulgarian Solitaire moves*: in each turn, take one card from each pile and form a new pile. The game ends when the sizes of the piles are not changed by performing the moves. Surprisingly, it turns out that regardless of initial state of the game, it must end in a finite number of moves at the state with one pile of one card, one pile of two cards, ..., and one pile of nine cards. Then Gardner stated the problem for any triangular number $n = 1+2+\ldots+m = \binom{m+1}{2}$. But how about other arbitrary n?

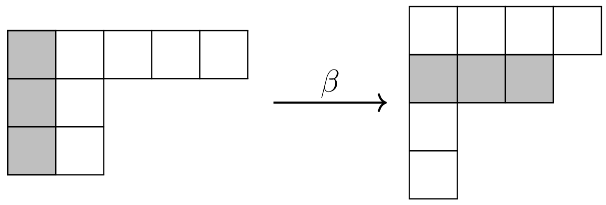


Figure 1: Bulgarian Solitaire on Young diagram $\beta((5,2,2)) = (4,3,1,1)$

Now let n be the number of cards we start the game with and represent a configuration of piles of cards by a partition of n, with the piles being the parts. Thus we have made the Bulgarian Solitaire game into a dynamical system on $\mathcal{P}(n)$ with the operation: in each step, take one from each part, form a new part and put the parts in weakly decreasing order. In addition, the game can be described in terms of Young diagrams: in each turn, remove the longest column and reinsert it as a new row into the diagram. An example is shown in Figure 1.

2.2 Notation and terminology

We write a partition as $\lambda = (\lambda_1, \ldots, \lambda_n)$ where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n > 0$. We use $l(\lambda)$ as the length of the partition λ, e.g. in this case $l(\lambda) = n$. The symbol $|\lambda| = \lambda_1 + \lambda_2 + \ldots + \lambda_n$ is the number of which λ is an integer partition.

Let β be the Bulgarian Solitaire operation on $\mathcal{P}(n)$, described in subsection 2.1. For example, $\beta((5,3,2)) = (4,3,2,1)$. The *game graph* of Bulgarian Solitaire system is a directed graph whose nodes are partitions of n with directed edges connecting $\lambda \to \beta(\lambda)$.

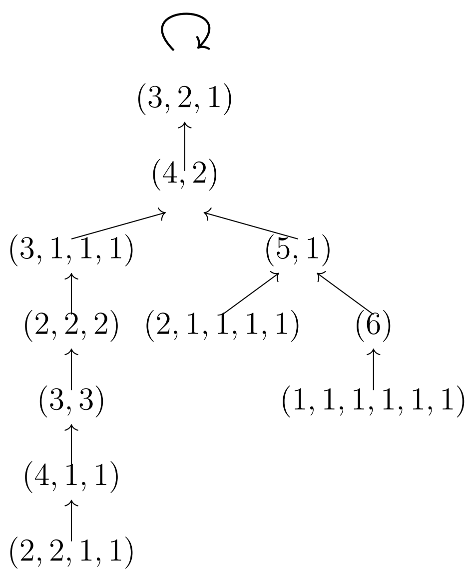


Figure 2: Bulgarian Solitaire game graph for $n = 6$.

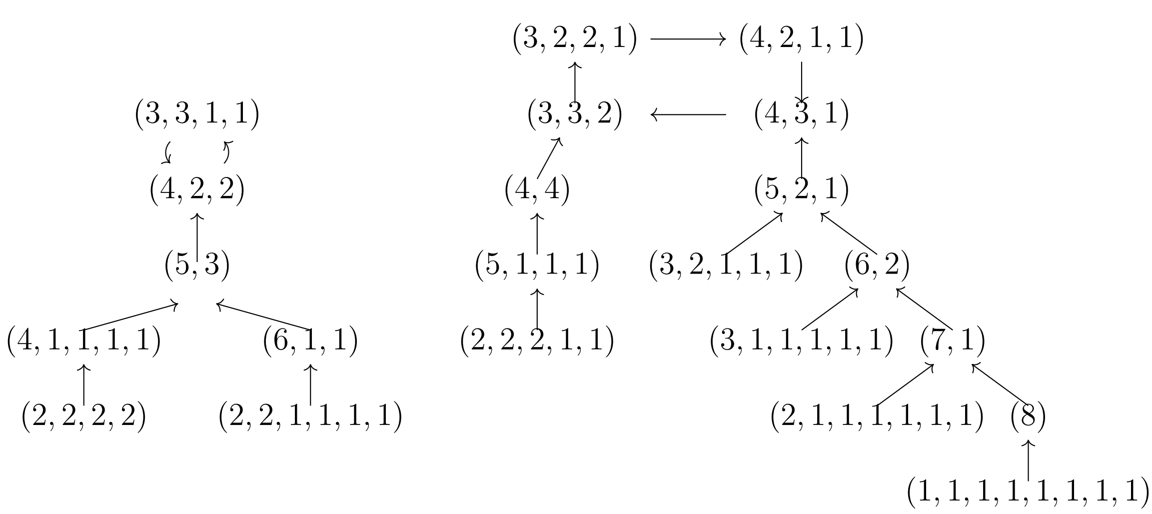


Figure 3: Bulgarian Solitaire game graph for $n = 8$.

Definition 2.2.1. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a finite sequence of letters $\{B, W\}$. Define the cyclic rotation ω by

$$\omega(\alpha_j) = \alpha_{(j+1)\ mod\ n}$$

A *necklace* N of black and white beads is an equivalence class of sequences of letters $\{B, W\}$ under cyclic rotation ω. We call N a *primitive necklace* if it cannot be written as a concatenation $N = P^k = PP\cdots P$ of copies of another necklace P. We will reserve P for *primitive necklaces.*

Example 2.2.2. The sequences $BWWW = WBWW = WWBW = WWWB$ all represent the same necklace P, which is primitive. The same is true for the sequences $BBWW = WBBW = WWBB = BWWB$ representing a different primitive necklace P'. However the sequences $BWBW = WBWB$ represent a non-primitive necklace $N = (BW)^2 = (WB)^2$. The three necklaces W, $BW = WB$ and $BWW = WBW = WWB$ are all primitive.

For a necklace N, we will denote by $b(N)$ the number of black beads and by $w(N)$ the number of white beads in any sequence representing N. The number of beads in necklace N is denoted $|N| = b(N) + w(N)$.

Let $\mathcal{BS}$ be the set of orbits of Bulgarian Solitaire systems and $\mathcal{N}$ be the set of necklaces with at least one white bead. We denote $\psi(\lambda)$ to be the Bulgarian Solitaire orbit that contains λ, that is, λ, μ lie in the same BS orbit $\psi(\lambda) = \psi(\mu)$ if there exists integers $a, b \geq 0$ for which $\beta^a(\lambda) = \beta^b(\mu)$. By [3] we have a bijection

$$\mathcal{O} : \mathcal{N} \longrightarrow \mathcal{BS} \tag{1}$$

that maps a necklace to the orbit of Bulgarian Solitaire system which has the unique recurrent cycle represented by the necklace. Specifically, for each necklace $N = N_1N_2 \ldots N_l$, the map is defined as $\mathcal{O}_N = \mathcal{O}(N) = \psi((\Delta_{l-1}, 0)+\sigma)$, where $\sigma = (\sigma_1, \ldots, \sigma_l)$ and for each $j = 1, \ldots, l$, $\sigma_j = 0$ if $N_j = W$, otherwise $\sigma_j = 1$. **Figure 4** and **Figure 5** are examples to illustrate the map $\mathcal{O}$. For convenience, we use $\mathcal{O}_N$ for the BS game graph restricted to the orbit $\mathcal{O}_N$. The map $\mathcal{O}$ satisfies that for any $P^k \in \mathcal{N}$, where P is a primitive necklace, $|P|$ is equal to the size of the recurrent subset in $\mathcal{O}_{P^k}$. That recurrent subset exists because for a given number n, the set $\mathcal{P}(n)$ is finite. The following is a rephrasing of Brandt's result [3], along with an enumeration corollary from Drensky [5] using Polyá's enumeration theorem, elaborated by Akin and Davis [1]:

Theorem 2.2.3. *Uniquely express $n = \binom{m}{2} + r$ for some $0 < r \leq m - 1$ and let $\lambda \in \mathcal{P}(n)$. Then the orbits of the Bulgarian Solitaire system on $\mathcal{P}(n)$ biject with necklaces N with $b(N) = r$ black beads and $w(N) = m - r$ white beads. The partitions λ within the recurrent cycle of orbit $\mathcal{O}_N$ consist of the triangular partition along with an extra square in each row indexed by a black bead from a necklace in N.*

Therefore the number of the components of the game graph associated with the partitions of n is equal to the number of necklaces consisting of r black beads and $m - r$ white beads having this formula

$$\frac{1}{m} \sum_{d | gcd(r,m)} \varphi(d) \binom{m/d}{r/d},$$

where $\varphi(d)$ is the Euler φ function, i.e., the number of positive integers $\leq d$ and relatively prime to d.

Let $\mathcal{C}_{P^k}$ denote the unique cycle of partitions contained in the recurrent subset inside the BS orbit indexed by the necklace P^k, with P primitive. One can easily prove the proposition below by using the bijection $\mathcal{O}$ described above:

Proposition 2.2.4. *For any $P^k \in \mathcal{N}$ with P primitive and $|P| = p$, the orbit $\mathcal{O}_{P^k}$ has partitions of size*

$$n = \binom{pk}{2} + kb(P).$$

Conversely, given $n = \binom{k}{2} + b$, where $0 \leq b \leq k - 1$, then BS orbits on $\mathcal{P}(n)$ correspond to necklaces N of length k with b black beads, and thus

$$\mathcal{P}(n) = \bigsqcup_{\substack{|N|=k \\ b(N)=b}} \mathcal{O}_N. \tag{2}$$

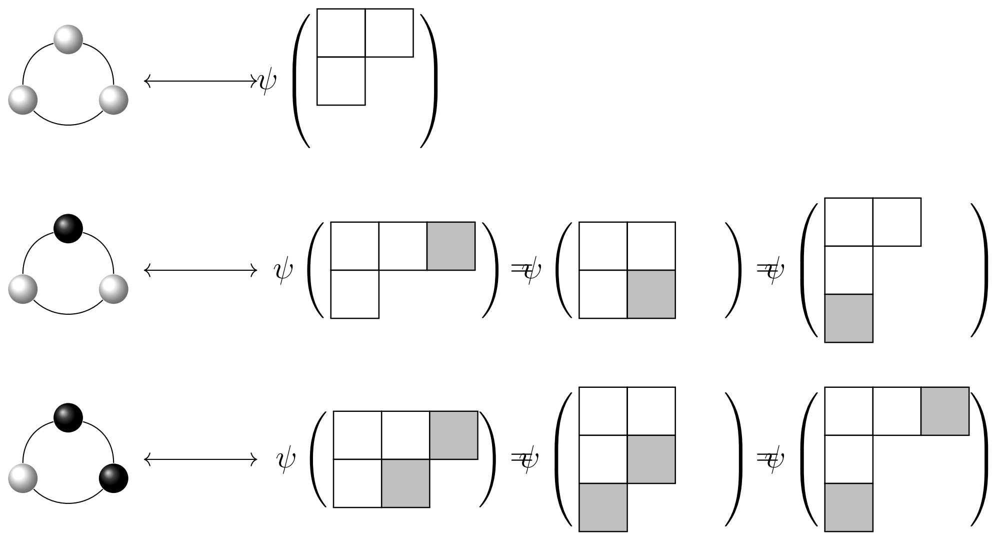


Figure 4: The map $\mathcal{O}$ for primitive necklaces of length 3, which are WWW, BWW, BBW.

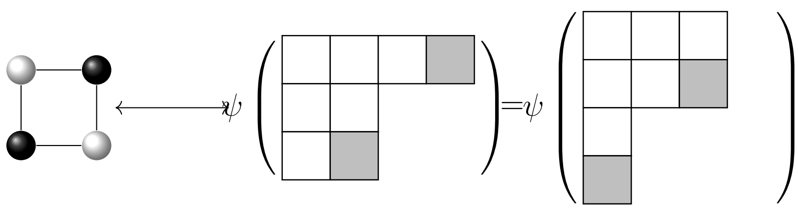


Figure 5: The map $\mathcal{O}$ for non-primitive necklaces of length 4. The recurrent set in $\mathcal{O}_{(BW)^2}$ has only 2 elements, shown above.

Remark 2.2.5. The Bulgarian Solitaire system on partitions of a triangular number $n = \binom{k}{2}$ has a unique orbit, corresponding to the necklace W^k. On the other hand, we will occasionally wish to think of it as also corresponding to the all-black necklace B^{k-1}, which was deliberately excluded from the domain of the map map $\mathcal{O}$, to make it a bijection.

For any primitive necklace $P \in \mathcal{N}$ and any $\lambda \in \mathcal{O}_{P^k}$, we denote by $D_{P^k}(\lambda)$ the minimum number of moves to reach the recurrent cycle starting from λ, that is, define $D_{P^k} : \mathcal{O}_{P^k} \to \mathbb{N}$ by

$$D_{P^k}(\lambda) = \min\{d \in \mathbb{N} : \beta^d(\lambda) \in \mathcal{C}_{P^k}\}$$

Roughly speaking, the *reversed Bulgarian Solitaire* is done by reversing all arrows in the directed game graph. For Young diagrams, the reversed rule is: in each turn, take out a row no shorter than the number of rows minus 1 and insert it again as the leftmost column. For a partition λ, the operation is: in each turn, take a part $\lambda_j \geq l(\lambda) - 1$, then distribute it into other parts, one for each. We use $\xrightarrow{j}$ to describe the move that takes out the row j. For example $(4, 2, 2) \xrightarrow{1} (3, 3, 1, 1)$ and $(4, 2, 2) \xrightarrow{2} (5, 3)$. **Figure 6** illustrates the reverse rule.

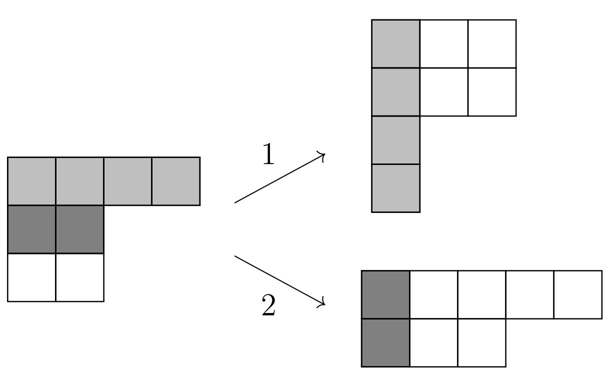


Figure 6: Reverse BS on a Young diagram

We denote the reversed operation by $R_j(\lambda)$, where λ is the partition to perform the rule on the part λ_j. We also use $\mathbf{0}$ as a result of an invalid move, that is, the removed row is too short to be the leftmost - or the longest - column. Obviously, $R_j(\lambda)$ is valid (or legal) if and only if $\lambda_j \geq l(\lambda) - 1$.

Example 2.2.6. *For example,*

$$\begin{aligned} R_1((4,3,2,1)) &= (4,3,2,1), \\ R_2((4,3,2,1)) &= (5,3,2), \\ R_4((4,3,2,1)) &= \mathbf{0}. \end{aligned}$$

A *(reverse BS) playing sequence* is a sequence of parts that are played legally. For example, with $(BW)^2$ and $n = 8$, the playing sequence [211] starting from $(4,2,2)$ is

$$\begin{matrix} \langle 4\rangle \\ \langle 2\rangle \\ \langle 2\rangle \end{matrix} \xrightarrow{2} \begin{matrix} \langle 5\rangle \\ \langle 3\rangle \end{matrix} \xrightarrow{1} \begin{matrix} \langle 4\rangle \\ 1 \\ 1 \\ 1 \\ 1 \end{matrix} \xrightarrow{1} \begin{matrix} 2 \\ 2 \\ 2 \\ 2 \end{matrix}$$

Figure 7: Reverse BS

where the parts λ_j which are playable are enclosed in angle brackets as $\langle \lambda_j \rangle$. Extend the notation R for playing sequences $\sigma = [\sigma_1, \ldots, \sigma_m]$, that is, define

$$R_\sigma(\lambda) := R_{\sigma_m}(\cdots (R_{\sigma_2}(R_{\sigma_1}(\lambda))\cdots)$$

E.g. $R_{[211]}((4,2,2)) = (2,2,2,2)$. Now $\mathcal{O}_N^{op}$ is used for the directed graph $\mathcal{O}_N$ with reversed arrows - that is, the directed edges go $\beta \to R_j(\lambda)$ and such an edge is labelled R_j. In other words, $\mathcal{O}_N^{op}$ is the directed graph of the orbit $\mathcal{O}_N$ under the reversed BS operation R.

We also need the *difference reversed game graph* for an orbit of partitions. If the partitions λ in the orbit have size $|\lambda| = n = \binom{k}{2} + r$ with $0 \leq r \leq k-1$, then the difference game graph is obtained from the game graph by subtracting the staircase Δ_{k-1} out of each partitions in the orbit. For example, the part of the difference game graph for the playing sequence in **Figure 7** is

$$\begin{matrix} \langle 1\rangle \\ \langle 0\rangle \\ \langle 1\rangle \end{matrix} \xrightarrow{2} \begin{matrix} \langle 2\rangle \\ \langle 1\rangle \\ -1 \end{matrix} \xrightarrow{1} \begin{matrix} \langle 1\rangle \\ -1 \\ 0 \\ 1 \\ 1 \end{matrix} \xrightarrow{1} \begin{matrix} -1 \\ 0 \\ 1 \\ 2 \end{matrix}$$

Figure 8: Difference BS

2.3 Previously known results

The game graph when n is a triangle number, which turned out to be a tree, has been studied by Igusa [9], Etienne [7], Griggs and Ho [8] and Eriksson and Jonsson [6]. Recall that the Bulgarian solitaire system on $\mathcal{P}(n)$ for $n = \binom{k+1}{2}$ has only one orbit $\mathcal{O}_{W^{k+1}} = \psi(\Delta_k)$, converging to a unique fixed point in at most $k^2 - k$ moves. Eriksson and Jonsson prove [6] that, in the limit as k grows, the sequence of level sizes $(D^{-1}_{W^k}(1), D^{-1}_{W^k}(2), \ldots)$ converges to the subsequence of evenly-indexed Fibonacci numbers $(F_{2d})_{d=0}^{\infty}$, with the generating function

$$H_W(x) = \frac{(1-x)^2}{1-3x+x^2}. \tag{3}$$

Eriksson and Jonsson also showed that for $n = \binom{k+1}{2}$, the sizes of levels $0, 1, \ldots, \lfloor \frac{k}{2} \rfloor$ in the reversed Bulgarian solitaire game tree coincide with those of an object that they called the *quasi-infinite game tree*, but the next level, $\lfloor \frac{k}{2} \rfloor + 1$, has fewer elements, 1 less for odd k and $1 + \frac{k}{2}$ less for even k.

Figure 9 display some initial difference BS game graphs up to some levels and **Figure 10** is the *quasi-infinite game tree*. When we remove the left branch of the quasi-infinite game tree, which turns out to be an entire copy of the tree (corresponding to the loop in the finite graphs), we can see a containment relationship between the quasi-infinite tree and the finite graphs up to some levels.

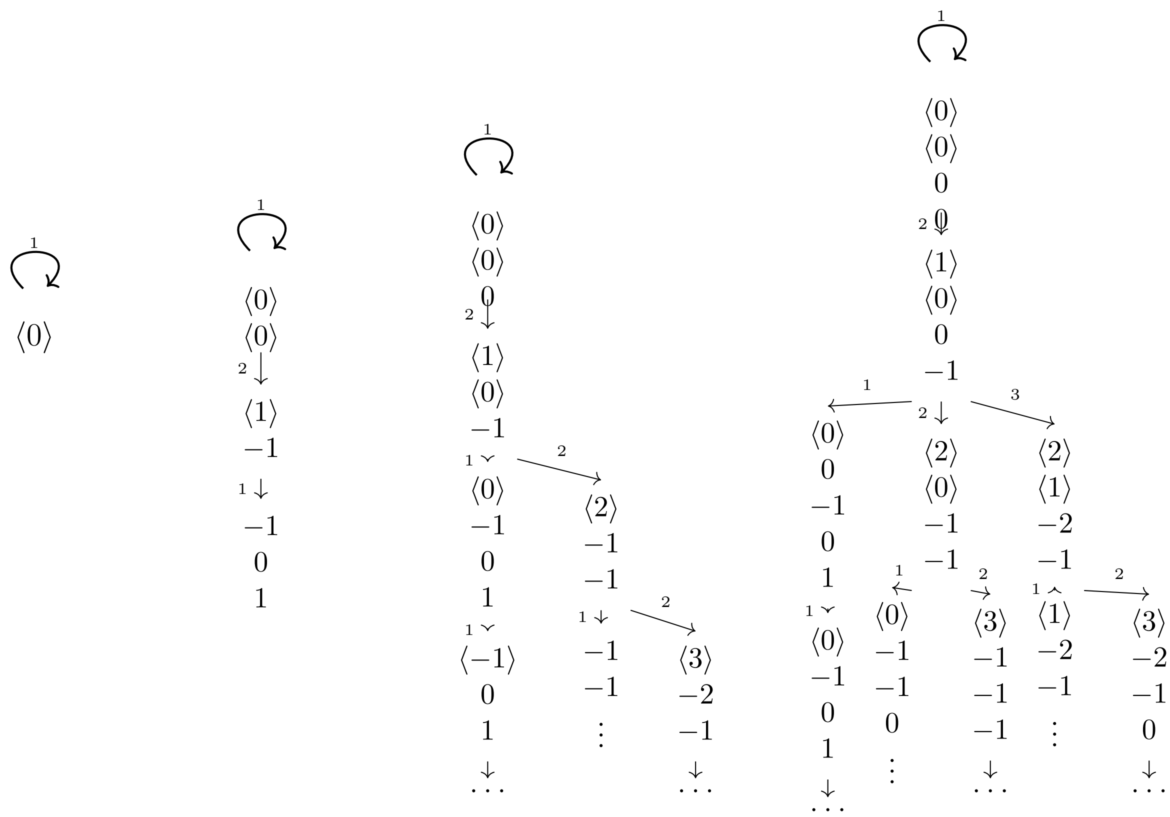


Figure 9: Difference reversed BS game graph for $n = \binom{k+1}{2}$ with $k = 1, 2, 3, 4$ up to level $\lfloor \frac{k+1}{2} \rfloor + 1$.

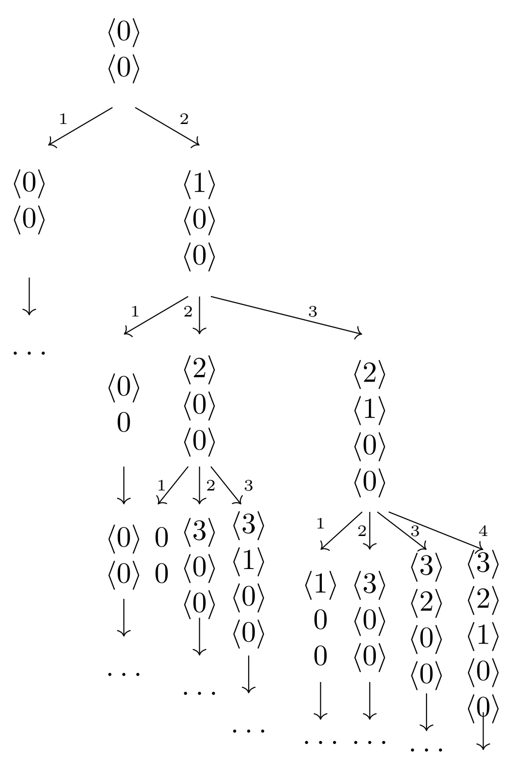


Figure 10: The *quasi-infinite game tree* for triangle numbers.

The rules for $\xrightarrow{i}$ in the *quasi-infinite tree* [6, §3] are described below:

1. Delete the bracketed ith number.
2. Increase all numbers above it by 1 and make them bracketed.
3. Bracket the new ith number (if there is one) if it differs at most 1 from the old one.
4. If a zero was played, add zeros at the end so that there are two, and make them bracketed.

Note that the triangular number $n = \binom{m+1}{2}$ corresponds to necklaces W^m. Let $\mathcal{O}_{W^\infty}^{op}$ denote the quasi-infinite tree for triangular numbers above without the left branch. Then Eriksson and Jonsson [6] showed

$$\mathcal{O}_{W^\infty}^{\mathrm{op}} = \lim_{m\to\infty} \mathcal{O}_{W^m}^{\mathrm{op}}.$$

We wish to generalize this idea for a *quasi-infinite forest* to find the limit of the level sizes for arbitrary n. For each primitive necklace P, we have a finite number of elements in the recurrent set $\mathcal{C}_{P^k}$ for any k, so we can build trees in the *quasi-infinite forest* $\mathcal{F}_P$ rooting at each of those elements. The modified rules will be discussed in **Part II**.

3 Data, new results and conjectures

Here we briefly present the main results and conjectures of this thesis, along with some of the data that suggested them.

3.1 Orbit sizes and distance generating function

Since Bulgarian solitaire orbits on $\mathcal{P}(n)$ are parametrized by necklaces N, it is natural to ask for their sizes. When n is a triangular number, there is only one component of size $p(n)$. However, other orbits' sizes have remained unknown. Here is some data on the orbit sizes corresponding to some small necklaces N. Recall from Proposition 2.2.4 that if $N = P^k$ for some primitive necklaces P, the size of the partition n is given by

$$n = \binom{pk}{2} + kb(P),$$

where $b(P)$ is the number of black beads in P.

N	$\lvert\mathcal{O}_N\rvert$
W	1
W^2	3
W^3	11
W^4	42

N	$\lvert\mathcal{O}_N\rvert$
BW	2
$(BW)^2$	7
$(BW)^3$	26
$(BW)^4$	97

N	$\lvert\mathcal{O}_N\rvert$
BWW	5
$(BWW)^2$	25
$(BWW)^3$	125
$(BWW)^4$	625

N	$\lvert\mathcal{O}_N\rvert$
BBW	7
$(BBW)^2$	35
$(BBW)^3$	175
$(BBW)^4$	875

N	$\lvert\mathcal{O}_N\rvert$
$BWWW$	15
$(BWWW)^2$	225
$(BWWW)^3$	3375
$(BWWW)^4$	50625

N	$\lvert\mathcal{O}_N\rvert$
$BBWW$	15
$(BBWW)^2$	150
$(BBWW)^3$	1500
$(BBWW)^4$	15000

N	$\lvert\mathcal{O}_N\rvert$
$BBBW$	30
$(BBBW)^2$	450
$(BBBW)^3$	6750
$(BBBW)^4$	101250

N	$\lvert\mathcal{O}_N\rvert$
$BBWWW$	45
$(BBWWW)^2$	1215
$(BBWWW)^3$	32805

N	$\lvert\mathcal{O}_N\rvert$
$BBBWW$	67
$(BBBWW)^2$	1809
$(BBBWW)^3$	48843

N	$\lvert\mathcal{O}_N\rvert$
$WBWBW$	32
$(WBWBW)^2$	544
$(WBWBW)^3$	9248

N	$\lvert\mathcal{O}_N\rvert$
$BWBWB$	34
$(BWBWB)^2$	578
$(BWBWB)^3$	9826

N	$\lvert\mathcal{O}_N\rvert$
$BWWWW$	56
$(BWWWW)^2$	2464

N	$\lvert\mathcal{O}_N\rvert$
$BBBBW$	135
$(BBBBW)^2$	5940

The data in the table for $(BW)^k$ necklaces suggested our first main result, Theorem 3.1.1 below, involving the sequence of *Chebyshev polynomials of the first kind* $\{T_k(x)\}_{k=0}^{\infty}$, with initial conditions $T_0(x) = 1, T_1(x) = x$ and recurrence relation $T_k(x) = 2xT_{k-1}(x) - T_{k-2}(x)$ for $k \geq 2$. In particular, we will need their specialization at $x = 2$, satisfying:

$$\begin{aligned} T_0(2) &= 1 \\ T_1(2) &= 2 \\ T_k(2) &= 4T_{k-1}(2) - T_{k-2}(2) \quad \text{for } k \geq 2. \end{aligned} \tag{4}$$

Theorem 3.1.1. *For each $k = 1, 2, 3, \ldots$, one has*

$$\left|\mathcal{O}_{(BW)^k}\right| = T_k(2).$$

Moreover, if we define the generating functions for distance to the recurrent cycle $\mathcal{C}_{(BW)^k}$

$$\mathcal{D}_N(x) := \sum_{\lambda \in \mathcal{O}_N} x^{D_N(\lambda)} = \sum_{d=0}^{\infty} D^{-1}_{(BW)^k}(d)\ x^d$$

then this sequence of generating functions satisfies the following generalization of the recurrence (4)*:*

$$\begin{aligned}
&\mathcal{D}_{(BW)^0}(x) := 1 \quad \textit{by convention,}\\
&\mathcal{D}_{(BW)^1}(x) = 2,\\
&\mathcal{D}_{(BW)^k}(x) = x(3x+1)\mathcal{D}_{(BW)^{k-1}}(x) - x^3\mathcal{D}_{(BW)^{k-2}}(x) + (x-1)^2(3x+2) \quad \textit{for } k \geq 2.
\end{aligned} \tag{5}$$

The data in the tables for the necklaces $(BWW)^k$ and $(BBW)^k$, suggested our next main result.

Theorem 3.1.2. *For each $k = 1, 2, 3, \ldots$, one has*

$$\begin{aligned}
\left|\mathcal{O}_{(BWW)^k}\right| &= 5^k,\\
\left|\mathcal{O}_{(BBW)^k}\right| &= 7 \cdot 5^{k-1}.
\end{aligned}$$

The data in the last two tables suggest the following conjecture.

Conjecture 3.1.3. *For each $k = 2, 3, 4, \ldots$, one has*

$$|\mathcal{O}_{P^k}| = (c_P)^{k-1}|\mathcal{O}_P|$$

where

$$c_P = \begin{cases} 15 \textit{ for both } P = BWWW, BBBW \\ 10 \textit{ for } \qquad P = BBWW \\ 17 \textit{ for both } P = WBWBW, BWBWB \\ 27 \textit{ for both } P = BBWWW, WWBBB \\ 44 \textit{ for both } P = BWWWW, WBBBB \end{cases}.$$

All of the preceding results and data then suggest a general conjecture.

Conjecture 3.1.4. *For any primitive necklace P with $|P| \geq 3$, there is an integer c_P such that for $k \geq 2$,*

$$|\mathcal{O}_{P^k}| = (c_P)^{k-1}|\mathcal{O}_P|$$

for some constant c_P that depends only on P. Moreover, if P and P' are obtained from each other by swapping black beads to white beads and vice versa, then $c_P = c_{P'}$.

Remark 3.1.5. If one could prove Conjecture 3.1.4 and provide explicit formulas for the constants c_P and $|\mathcal{O}_P|$ appearing there, it would lead to an interesting formula for the partition function $p(n) = |\mathcal{P}(n)|$, as a sum of $|\mathcal{O}_N|$ over necklaces N, corresponding to the BS orbit decomposition of $\mathcal{P}(n)$ in (2).

Remark 3.1.6. In contrast to Conjecture 3.1.4, for the two cases $P = W$ and $P = BW$, one doesn't have exact geometric growth with some ratio c_P. But they still grow approximately geometrically. Specifically:

- For $P = W$, one has $|\mathcal{O}_{W^k}| = p\left(\frac{k(k-1)}{2}\right)$ with $p(n) = |\mathcal{P}(n)|$. The Hardy-Ramanujan asymptotic says

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right),$$

  yielding this asymptotic:

$$|\mathcal{O}_{W^k}| \sim \frac{1}{2k(k-1)\sqrt{3}} \exp\left(\pi\sqrt{\frac{k(k-1)}{3}}\right) \sim \left(\exp\left(\frac{\pi}{\sqrt{3}}\right)\right)^k,$$

  whose geometric ratio is $\exp\left(\frac{\pi}{\sqrt{3}}\right) \approx 6.1337...$

- For $P = BW$, Theorem 3.1.1 says that $\left|\mathcal{O}_{(BW)^k}\right| = T_k(2)$, satisfying the recurrence $T_{k+1}(2) = 4T_k(2) - T_{k-1}(2)$, which leads to an explicit formula and asymptotic

$$\left|\mathcal{O}_{(BW)^k}\right| = T_k(2) = \frac{1}{2}\left((2-\sqrt{3})^k + (2+\sqrt{3})^k\right) \sim (2+\sqrt{3})^k,$$

  whose geometric ratio is $2+\sqrt{3} \approx 3.732...$

Together with Theorem 3.1.2 and Conjecture 3.1.3, we expect that for primitive necklaces of length greater than 1, the geometric ratio is increasing as the length increases.

Remark 3.1.7. More computations of larger primitive necklaces are needed to confirm Conjecture 3.1.4, however, in this thesis, with current access to Sage, we are not able to generate data for more necklaces P^k for either P of length greater than 6 or P of length 5 or 6 and $k > 3$. That is because $\mathcal{P}(n)$ grows exponentially as in Remark 3.1.6. Specifically, for necklace $(BWWWW)^4$, we need to work with the set of partitions of $n = 194$, which has approximately 2.45×10^{12} elements.

3.2 Convergence of level sizes in $\mathcal{O}_{P^k}$ as k grows

Jonsson raises the question in his thesis [10, I.§2.4] as to how one might generalize their convergence result and the generating function (3) for BS orbits on $\mathcal{P}(n)$ when n is not a triangular number. Here we examine such convergence results, as k grows, in the orbits $\mathcal{O}_{P^k}$ for each primitive necklace P.

For the necklaces of the form $N = (BW)^k$, here are some data on the level sizes in the orbit $\mathcal{O}_N$:

$d\backslash N$	BW	$(BW)^2$	$(BW)^3$	$(BW)^4$	$(BW)^5$	$(BW)^6$	$(BW)^7$	$(BW)^8$	$(BW)^9$	$(BW)^{10}$
0	2	2	2	2	2	2	2	2	2	2
1	0	1	1	1	1	1	1	1	1	1
2	0	2	3	3	3	3	3	3	3	3
3	0	2	6	7	7	7	7	7	7	7
4	0	0	8	14	15	15	15	15	15	15
5	0	0	6	24	32	33	33	33	33	33
6	0	0	0	28	60	70	71	71	71	71
7	0	0	0	18	92	142	154	155	155	155
8	0	0	0	0	96	248	320	334	335	335
9	0	0	0	0	54	344	614	712	728	729
10	0	0	0	0	0	324	996	1432	1560	1578

Table 1: $\left|D_N^{-1}(d)\right|$ Distribution by level sizes for necklaces $N = (BW)^k$ of alternating black-white beads.

The data in **Table 1** suggested that, as k grows, the sequence $(|D^{-1}_{(BW)^k}(d)|)_{d=0}^{\infty}$ converges to a sequence that starts $(2, 1, 3, 7, 15, 33, 71, 155, 335, \ldots)$. Our next main result proves this, and identifies the limit of the level size sequence as having a rational generating function.

Theorem 3.2.1. *There is a power series $H_{BW}(x)$ in $\mathbb{Z}[[x]]$ such that*

$$\lim_{k\to\infty} \mathcal{D}_{(BW)^k} = H_{BW}(x)$$

Moreover, $H_{BW}(x)$ is a rational function, given by

$$\begin{aligned} H_{BW}(x) &= \frac{(x-1)^2(3x+2)}{x^3 - 3x^2 - x + 1} \\ &= 2 + x + 3x^2 + 7x^3 + 15x^4 + 33x^5 + 71x^6 + 155x^7 + 335x^8 + \ldots \end{aligned}$$

We also prove an analogous result for the primitive necklaces $P = BBW, BWW$ with $|P| = 3$.

Theorem 3.2.2. *For $P = BWW, BBW$, as $k \to \infty$, the generating functions by level sizes $\mathcal{D}_{P^k}$ converge to the rational function*

$$\begin{aligned} H_{BWW}(x) = H_{BBW}(x) &= (1-x)\frac{x^3 - 3x^2 - 4x - 3}{2x^3 + x^2 - 1} \\ &= 3 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 17x^7 + 25x^8 + 39x^9 + 59x^{10} + \dots \end{aligned} \tag{6}$$

Remark 3.2.3. Although we do not include the proof here, we have also proven that for all primitive necklaces P having $|P| \leq 4$, the BS orbits $\mathcal{O}(P^k)$ have sequences of level sizes which converge as k grows. The cases where $|P| = 1, 2, 3$ were taken care of by Eriksson and Linusson's result (3), and our Theorems 3.2.1, 3.2.2, respectively. For $|P| = 4$, with a similar technique discussed in **Part II**, the generating functions for the limiting level sizes are as follows:

$$H_{BWWW}(x) = (1-x)\frac{x^5 + 8x^4 - 3x^3 - 8x^2 - 6x - 4}{6x^4 + 4x^3 + x^2 - 1}, \tag{7}$$

$$H_{BBBW}(x) = (1-x)\frac{2x^5 + 8x^4 - 5x^3 - 10x^2 - 7x - 4}{6x^4 + 4x^3 + x^2 - 1}, \tag{8}$$

$$H_{BBWW}(x) = (1-x)\frac{x^3 + x^2 + x + 1}{3x^4 + 2x^3 + x^2 - 1}. \tag{9}$$

Note that although the primitive necklaces $P = BWW, BBW$ have identical generating functions $H_P(x)$, $P = BBBW, BWWW$ do not, but their generating functions at least share the same denominator.

These results suggested the following theorem, which we have recently proven, but whose proof we omit in this thesis:

Theorem 3.2.4. *For primitive necklaces P with $|P| \geq 3$, there is a power series H_P in $\mathbb{Z}[[x]]$ such that the sequence of generating functions $(\mathcal{D}_{P^k})_{k=0}^{\infty}$ converges to H_P. Moreover, H_P is a rational function having denominator polynomial of degree at most $|P|$.*

Conjecture 3.2.5. *In the statement of Theorem 3.2.4, the denominator of $H_P(x)$ is of degree exactly $|P|$.*

3.3 Characterizing partitions within an orbit

It seems difficult to characterize exactly which partitions λ lie in the BS orbit $\mathcal{O}_N$ for a fixed necklace N. One approach is to consider the distribution by their length $l(\lambda)$, that is, their number of parts. For example, here is the data for the orbits corresponding to the necklaces $N = (BW)^k$:

N	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
BW	1	1																		
$(BW)^2$		1	2	2	1	1														
$(BW)^3$			1	3	5	5	3	3	3	3										
$(BW)^4$				1	4	9	13	12	8	9	10	11	11	9						
$(BW)^5$					1	5	14	26	33	29	22	25	28	34	38	41	39	27		
$(BW)^6$						1	6	20	45	72	83	72	60	68	79	95	109	130	144	151

Table 2: Distribution of $l(\lambda)$ for partitions λ in $\mathcal{O}_{(BW)^k}$.

Here are some properties that we observed and proved for the orbits $\mathcal{O}_{(BW)^k}$.

Proposition 3.3.1. *The partitions in the orbit $\mathcal{O}_{(BW)^k}$ satisfy these conditions:*

1. *The largest part size is $4k-2$. Thus partitions in $\mathcal{O}_{(BW)^k}$ have at most $4k-2$ parts.*
2. *The longest playing sequence is of length $2k$. The number of longest playing sequences is $2 \cdot 3^{k-2}$.*
3. *The number of partitions within the orbit having $4k-2$ parts is 3^{k-2}.*

We continued this analysis for the necklaces $N = (BWW)^k$ and proved these properties:

Proposition 3.3.2. *The partitions in the orbit $\mathcal{O}_{(BWW)^k}$ satisfy these conditions:*

1. *The largest part size is $9k-5$. Thus partitions in $\mathcal{O}_{(BWW)^k}$ have at most $9k-5$ parts.*
2. *The number of partitions within the orbit having $9k-5$ parts is 2^{k-2}.*

The previous results and data suggest the following conjectures.

Conjecture 3.3.3. *Let P be a primitive necklace with $|P| \geq 2$. Then there exists an integer γ_P, depending only on P, such that partitions λ in the orbit $\mathcal{O}_{P^k}$ have*

$$l(\lambda) \leq |P|^2 \cdot k + \gamma_P.$$

Conjecture 3.3.4. *Let P be a primitive necklace with $|P| \geq 2$. Then partitions λ in the orbit $\mathcal{O}_{P^k}$ have*

$$l(\lambda) \geq k.$$

Moreover, if P has consecutive black beads (that is, $P = B^s W^t$ for some s,t), there is exactly one such λ having $l(\lambda) = k$. Specifically, if $|P| = p$, λ is of the form

$$\lambda = \begin{array}{cc} (2p-1)k & +x \\ (2p-1)k - p & +x \\ \vdots & \\ (2p-1)k - (k-1)p & +x \end{array}$$

where

$$x = \frac{kp(p-3)}{2} - p + k + b(P).$$

Part II

Proofs

We will use some notations and rules discussed below in the proofs.
For a partition λ, we will use $\lambda + m$ to denote the partition $(\lambda_1 + m, \ldots, \lambda_n + m)$. Moreover, $\begin{pmatrix}\lambda \\ \mu\end{pmatrix}$ means a partition with the top entries from partition λ followed by partition μ - in this case, the last part of λ is no less than the first part of μ. To denote a prefix of a partition, we use $\lambda[: j] = (\lambda_1 \ldots, \lambda_j)$.
If a partition λ has some playable parts of the same size, we will use the smallest index in the playing sequence. For example, the playing sequence in **Figure 7** is [211] even though we can play [3] at the first move.
First of all, it is easy to see this property for partitions in $\mathcal{O}_N$ for general necklace N:

Claim 1. *Let λ be a partition in $\mathcal{O}_N$. If a part λ_j is playable, then its immediate following part λ_{j+1} is playable in the next state if and only if their difference is at most* 2.

Proof. Since playing other parts than λ_j and λ_{j+1} does not affect the gap between them, we only consider the moment when performing R_j. The result is a partition λ' with λ_j parts. Thus, a part is playable from that moment if and only if its size is at least $\lambda_j - 1$. Because R_j adds one to every other vertex, we obtain the claim. □

4 Analysis of $\mathcal{O}_{(BW)^k}$

4.1 On the reverse BS game graph for $\mathcal{O}_{(BW)^k}$

Proposition 2.2.4 implies that $n = |\lambda| = 2k^2$ for any partitions λ in the orbit $\mathcal{O}_{(BW)^k}$. **Figure 11** shows both the reversed game graphs $\mathcal{O}^{op}_{(BW)^k}$ and their difference labelings for $k = 1, 2$.

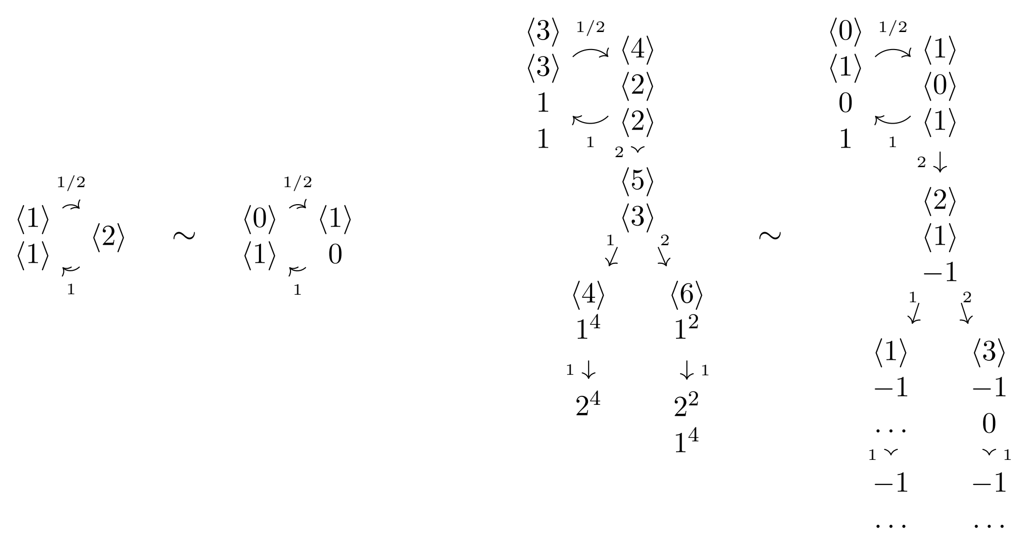


Figure 11: The reversed game graphs $\mathcal{O}^{op}_{(BW)^k}$ and their difference labelings for $k = 1, 2$.

The recurrent 2-cycle $\mathcal{C}_{(BW)^k}$ consists of two partitions $(2k-1, 2k-1, 2k-3, 2k-3, \ldots, 1, 1)$ and $(2k, 2k-2, 2k-2, \ldots, 2, 2)$ whose difference labels are $((01)^k)^T$ and $((10)^k)^T$, respectively. Here the symbol $(\alpha)^k$ means a vector of k α's stacking on each other. These two elements will be used as the roots of the trees in the *quasi-infinite forest* we analyze shortly. But first, we prove an important property for the elements in those $(BW)^k$ orbits:

Proposition 4.1.1. *Given a playing sequence σ in $\mathcal{O}^{op}_{(BW)^k}$. Then $\sigma_{j+1} \leq \sigma_j + 1$ for any $1 \leq j \leq |\sigma| - 1$.*

Proof. Due to the bijection described in **Section 2**, any partition λ in $\mathcal{C}_{(BW)^k}$ has pairs of consecutive parts of the same size. Moreover, if $\lambda_j = \lambda_{j+1}$ then $\lambda_{j+2k} = \lambda_{j+2k+1} = \lambda_j - 2k$, whichever exist. The difference between the consecutive part sizes are preserved unless one of them is played. If $R_k(\lambda) = \psi$, then $\psi_{k-1} - \psi_k \geq 2$, since $\lambda_{k-1} - \lambda_{k+1} \geq 2$. Equality holds if $\lambda_k = \lambda_{k+1}$.

If $R_{\sigma_j}(\lambda) = R_k(\lambda) = \psi$, only vertices up to ψ_{k+2} can be playable. But that ψ_{k+2} is playable means $\lambda_{k+1}, \lambda_{k+2}, \lambda_{k+3}$ have not been played before R_{σ_j}, so $\psi_{k+2} = \psi_{k+1}$. Therefore, we have the proposition. □

4.2 Limiting level sizes of $\mathcal{O}_{(BW)^k}$ as k grows

We consider the difference game graph as described in **Figure 8** but we take only the parts above the first negative part (or else take all the partition). We define the *BW quasi-infinite forest* $\mathcal{F}_{BW}$ of reverse Bulgarian Solitaire according to the rules below. Here *forest* is used instead of *tree* in **Section 2.3** because we build two trees, one of them rooted at each the two elements in the recurrent cycle.

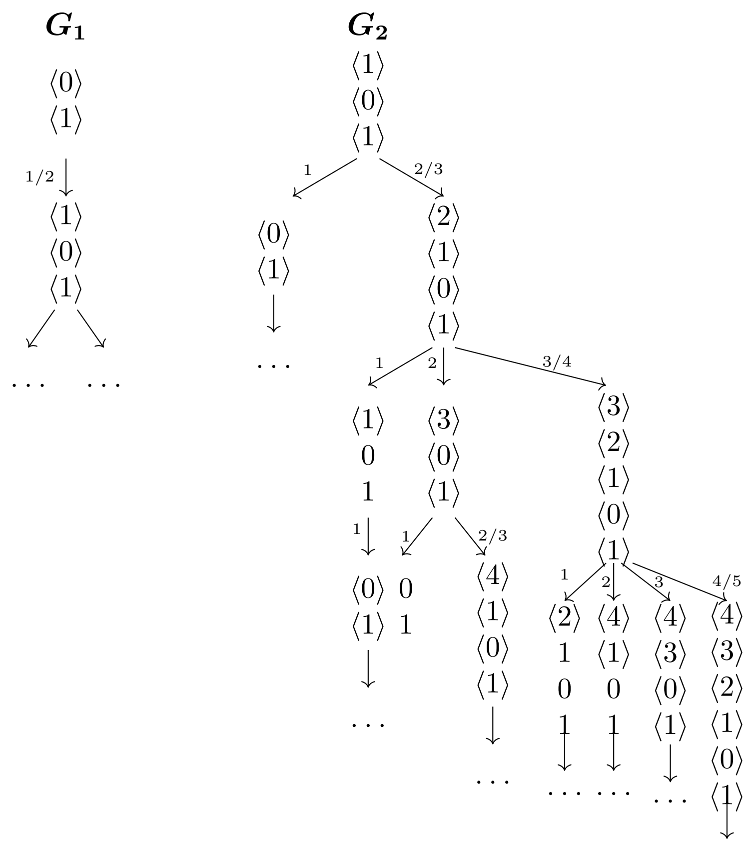


Figure 12: Quasi-infinite BW graph

Rules for $\xrightarrow{i}$ *in the BW quasi-infinite forest:*

1. Delete the bracketed entry in the i^{th} row. If the i^{th} bracket is $\langle 0\rangle$ followed immediately by a $\langle 1\rangle$, playing either vertex i or vertex $i+1$ results the same, so we label the play by $\xrightarrow{i/i+1}$.

2. Increase all entries in rows above it by 1 each, and bracket them.

3. Bracket the new i^{th} entry (if there is one) if it differs at most 1 from the old one. If there are two consecutive entries $\begin{smallmatrix}0\\1\end{smallmatrix}$ and 0 is bracketed, so is 1.

4. If the $\langle 1\rangle$ or $\begin{smallmatrix}\langle 0\rangle\\\langle 1\rangle\end{smallmatrix}$ at the lowest position is played, add $\begin{smallmatrix}\langle 0\rangle\\\langle 1\rangle\end{smallmatrix}$ at the end.

Let $g(x)$ be the generating function for the level sizes of the *BW* quasi-infinite forest. There are two trees rooted at the two elements in $\mathcal{C}_{(BW)^k}$: G_1 starts with $\gamma_1 = \begin{smallmatrix}\langle 0\rangle\\\langle 1\rangle\end{smallmatrix}$ and the other G_2 starts with $\gamma_2 = \begin{smallmatrix}\langle 1\rangle\\\langle 0\rangle\\\langle 1\rangle\end{smallmatrix}$. Let $g_1(x)$ and $g_2(x)$ denote the generating functions by level sizes of the two trees, G_1 and G_2 respectively, so $g(x) = g_1(x) + g_2(x)$. Starting from $\begin{smallmatrix}\langle 0\rangle\\\langle 1\rangle\end{smallmatrix}$, we play R_1

(the only playable part) and get back to the root of G_2. Thus the level generating functions of the two trees satisfy

$$g_1(x) = 1 + xg_2(x). \tag{10}$$

As a reminder, if we have $\begin{smallmatrix}\langle 0\rangle\\ \langle 1\rangle\end{smallmatrix}$ in a partition λ, adding a staircase makes their values the same. Thus playing either of them results in the same element. To represent this move in a playing sequence, we will use the vertex with $\langle 0\rangle$ on it. Moreover, we use the notation R_σ for playing sequences in the quasi-infinite game graph similarly to the normal BS game graph. e.g. $R_{[222]}\begin{pmatrix}\langle 1\rangle\\ \langle 0\rangle\\ \langle 1\rangle\end{pmatrix} = \begin{matrix}\langle 4\rangle\\ \langle 1\rangle\\ \langle 0\rangle\\ \langle 1\rangle\end{matrix}.$

Similarly to Jonsson and Erikson's paper [6, Proposition 3.1, p. 4], we have the following proposition:

Proposition 4.2.1. *The BW quasi-infinite reversed BS graph rooted at* $\begin{matrix}\langle 1\rangle\\ \langle 0\rangle\\ \langle 1\rangle\end{matrix}$ *has the following properties:*

i. Once R_1 is played, only R_1 can be played (until $\begin{smallmatrix}\langle 0\rangle\\ \langle 1\rangle\end{smallmatrix}$ or a leaf is reached).

ii. For $r \geq 2$, the playing sequence $[234\ldots r1^r]$ leads to $\begin{smallmatrix}\langle 0\rangle\\ \langle 1\rangle\end{smallmatrix}$.

Proof. i. When we play R_1, if the second part differs from the first part by at least 2, nothing is bracketed in the next state, thus we reach the leaf. Otherwise, according to the rule 3, either we reach the root $\begin{smallmatrix}\langle 0\rangle\\ \langle 1\rangle\end{smallmatrix}$ if the second and third parts are $\begin{smallmatrix}0\\ 1\end{smallmatrix}$ or the only bracketed part in the next state is the first part.

ii. It is easy to see that

$$R_{[23\ldots r]}\begin{pmatrix}\langle 1\rangle\\ \langle 0\rangle\\ \langle 1\rangle\end{pmatrix} = \begin{matrix}\langle r\rangle\\ \langle r-1\rangle\\ \vdots\\ \langle 1\rangle\\ \langle 0\rangle\\ \langle 1\rangle\end{matrix}$$

and thus playing $[1^r]$ consecutively deletes the first r rows and reaches $\begin{smallmatrix}\langle 0\rangle\\ \langle 1\rangle\end{smallmatrix}$. From property i, we see that this is the only way to get back to the roots.

□

4.3 Proof of Theorem 3.2.1

We begin by using **Proposition 4.2.1** to construct the *growth function*, or the generating function of g_2 by its level sizes. **Table 3** shows various playing sequences and their contributions to the generating function $g_2(x)$, explained below. We use $[\geq t]$ to denote the set of playing sequence with entries no less than t, including the empty sequence.

Sequence	Contribution	Sequence	Contribution
$[1\ldots]$	xg_1	$[2[\geq 2]]$	xg_2
$[21^2\ldots]$	x^3g_1	$[2[\geq 2]1]$	x^2g_2
$[231^3\ldots]$	x^5g_1	$[23[\geq 3]1^2]$	x^4g_2
$[2341^4\ldots]$	x^7g_1	$[234[\geq 4]1^3]$	x^6g_2

Table 3: Growth function for tree g_2.

By **Proposition 4.2.1.ii.**, the sequences $[23\ldots r1^r]$ lead to the root of G_1, so each of them contribute the whole G_1 tree at level $2r-1$, that is, $x^{2r-1}g_1(x)$. This is also true for playing sequence [1], since $R_1\begin{pmatrix}\langle 1\rangle\\ \langle 0\rangle\\ \langle 1\rangle\end{pmatrix} = \begin{matrix}\langle 0\rangle\\ \langle 1\rangle\end{matrix}.$

Next, if we play R_2, we reach $\begin{matrix}\langle 2\rangle\\ \langle 1\rangle\\ \langle 0\rangle\\ \langle 1\rangle\end{matrix} = \begin{matrix}\langle 2\rangle\\ \gamma_2\end{matrix}$. Thus if we leave the top part untouched, which means we only play parts of indices greater or equal than 2, then we have a subtree that is isomorphic to G_2. The isomorphism is defined by excluding the top part. Thus sequences $[2[\geq 2]]$ contribute xg_2. Similarly with $[23\ldots r[\geq r]1^{r-1}]$, each contributes $x^{2r-2}g_2$, since

$$R_{[23\ldots r]}(\gamma_2) = \begin{matrix}\langle r\rangle\\ \langle r-1\rangle\\ \vdots\\ \langle 1\rangle\\ \langle 0\rangle\\ \langle 1\rangle\end{matrix} = \begin{matrix}\langle r\rangle\\ \langle r-1\rangle\\ \vdots\\ \gamma_2\end{matrix}.$$

If we play $[23\ldots r[\geq r]]$, the top $r-1$ rows above $\begin{matrix}\langle 1\rangle\\ \langle 0\rangle\\ \langle 1\rangle\end{matrix}$ are always playable due to rule 2. Thus the playing sequences $[23\ldots r[\geq r]1^s]$ with any $s \leq r-1$ are legal. For each $r \geq 2$, we only count for $[23\ldots r[\geq r]1^r]$, since if $s < r$, $[23\ldots r[\geq r]1^s]$ are counted as $[23\ldots s[\geq s]1^s]$. Hence, the type $[23\ldots r[\geq r]1^{r-1}]$ contributes $x^{2r-2}g_2$.

Therefore, we obtain

$$\begin{aligned} g_2(x) &= 1 + (x + x^3 + \ldots)g_1(x) + (x + x^2 + x^4 + \ldots)g_2(x) \\ &= 1 + (x + x^3 + \ldots) + (x + 2x^2 + 2x^4 + \ldots)g_2(x) \qquad \text{(substituting for } g_1(x) \text{ using (10))} \\ &= \left(1 + \frac{x}{1-x^2}\right) + xg_2(x) + \frac{2x^2}{1-x^2}g_2(x) \end{aligned}$$

Bring all occurrences of $g_2(x)$ to the left side gives

$$\frac{x^3 - 3x^2 - x + 1}{1 - x^2} g_2(x) = \frac{-x^2 + x + 1}{1 - x^2}$$

and therefore

$$g_2(x) = \frac{-x^2 + x + 1}{x^3 - 3x^2 - x + 1}.$$

From this one concludes, again using (10), that

$$\begin{aligned} g(x) &= g_1(x) + g_2(x) = (1 + xg_2(x)) + g_2(x) = 1 + (1 + x)g_2(x) \\ &= \frac{-3x^2 + x + 2}{x^3 - 3x^2 - x + 1} = \frac{(1-x)(3x+2)}{x^3 - 3x^2 - x + 1}. \end{aligned}$$

However, we desire the generating function for the level sizes of $\mathcal{O}^{op}_{(BW)^k}$ in the limit as $k \to \infty$. As constructed, our quasi-infinite forest has an entire copy of itself after playing R_1, giving rise to the left branch $[1\ldots]$, which we wish to disregard. Letting $H_{BW}(x)$ denote the height generating function for the rest of the quasi-infinite forest, that is, the two roots and the elements in the branch $[2\ldots]$, one then has

$$g(x) = xg(x) + H_{BW}(x)$$

and therefore

$$H_{BW}(x) = (1-x)g(x) = \frac{(1-x)^2(3x+2)}{x^3 - 3x^2 - x + 1}.$$

Thus to complete the proof of **Theorem 3.2.1**, it only remains to show that the level sizes of $\mathcal{O}^{op}_{(BW)^k}$ actually converge to the coefficients given by $H_{BW}(x)$. This is a consequence of a somewhat more precise theorem.

Theorem 4.3.1. *The finite reverse Bulgarian solitaire graph$\mathcal{O}^{op}_{(BW)^k}$ coincides at least up to level k with the BW quasi-infinite forest after removing its left branch* $[1\ldots]$.

Proof. For each k, to get to level k in the quasi-infinite forest, we add at most k blocks $\begin{smallmatrix}\langle 0\rangle \\ \langle 1\rangle\end{smallmatrix}$.

Thus if we start with the root of k blocks of $\begin{smallmatrix}1 \\ 0\end{smallmatrix}$ as below

$$\begin{matrix}\langle 1\rangle \\ \langle 0\rangle \\ \langle 1\rangle \\ 0 \\ \vdots \\ 1 \\ 0\end{matrix}$$

we don't need to add new parts until using up those k blocks. Moreover, the rule of bracketing stays the same in the finite tree due to the observation at the beginning of the section. This implies the coincidence of the quasi-infinite forest with the finite graph. $\square$

4.4 Proof of Theorem 3.1.1

Let us recall the statement of the theorem.

Theorem 3.1.1. *Let $\{T_k(x)\}_0^\infty$ be Chebyshev polynomials of the first kind. For each $k = 1, 2, 3, \ldots$, one has*

$$\left|\mathcal{O}_{(BW)^k}\right| = T_k(2).$$

Moreover, the generating functions for distance to the recurrent cycle $\mathcal{C}_{(BW)^k}$

$$\mathcal{D}_{(BW)^k}(x) := \sum_{\lambda \in \mathcal{O}_N} x^{D_N(\lambda)} = \sum_{d=0}^{\infty} D^{-1}_{(BW)^k}(d)\ x^d$$

satisfy the following generalization of the $\{T_k(2)\}_{k=0}^\infty$ recurrence (4):

$$\begin{aligned}
&\mathcal{D}_{(BW)^0}(x) := 1 \qquad \textit{by convention,}\\
&\mathcal{D}_{(BW)^1}(x) = 2,\\
&\mathcal{D}_{(BW)^k}(x) = x(3x+1)\mathcal{D}_{(BW)^{k-1}}(x) - x^3\mathcal{D}_{(BW)^{k-2}}(x) + (x-1)^2(3x+2) \qquad \textit{for } k \geq 2.
\end{aligned}$$

Note that the assertion on the orbit sizes $\left|\mathcal{O}_{(BW)^k}\right|$ follows once we have recurrence for for $\mathcal{D}_{(BW)^k}(x)$, since this gives the orbit sizes at $x = 1$, and its initial conditions coincide with those for $T_k(2)$.

Figures 13, 14, 15 illustrate the recursive structural relationship that we will prove, relating the graphs of $\mathcal{O}_{(BW)^k}$. We will use $\mathcal{G}_k = \mathcal{O}^{op}_{(BW)^k}$ for short. In each $\mathcal{G}_k$, a node is labelled by the playing sequence leading to it. Also, the coloring used to group the nodes in $\mathcal{G}_k$ follow these rules:

(a) **Black** nodes come from an embedding of $\mathcal{G}_{k-1}$ into $\mathcal{G}_k$, by one of these two rules:

- (a1) Increment each entry in the playing sequences of $\mathcal{G}_{k-1}$ by 1 and prepend a 1 to the beginning.
- (a2) Replace the first 1 in each playing sequence of $\mathcal{G}_{k-1}$ by 12^2.

(b) Red nodes are another copy of $\mathcal{G}_{k-1}$ in $\mathcal{G}_k$ directly related to the black nodes from rule $(a1)$ by append 1 to the playing sequences representing them.

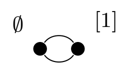
(c) Blue nodes are from an embedding of $\mathcal{G}_{k-2}$ into $\mathcal{G}_k$ by either:

- (c1) Replace the initial 1 from each of the playing sequences in $\mathcal{G}_{k-2}$ by 121^2, or
- (c2) Increment each value of the playing sequence of $\mathcal{G}_{k-2}$ by 2, replace the first entry of each playing sequence by [123] and append 1^2 at the end.

(d) Green nodes (if there are any) are from an embedding of $\mathcal{G}_{k-3}$ into $\mathcal{G}_k$ by replacing the first 1 by 121^4.

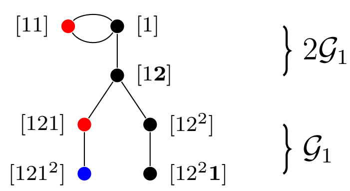
(e) Hollow diamonds ($\lozenge$) are actually not vertices of $\mathcal{G}_k$. They are vacated after modifying their playing sequences as follows: replace the initial 12^2 with 123, and erase a 2 in the fourth position, then append a final 1. One then moves this modified node to the position of the solid diamond node ($\blacklozenge$).

(f) This rule only applies to $\mathcal{G}_k$ where $k \geq 4$. The playing sequences in this group are of the form $[123^3[\geq 3]1^2]$. They are in bijection with the playing sequences of the form $[121^3 \ldots]$ in G_{k-1} by replacing the initial 121^3 with 123^3, incrementing each of the other entry by 2 and appending 1^2 at the end.



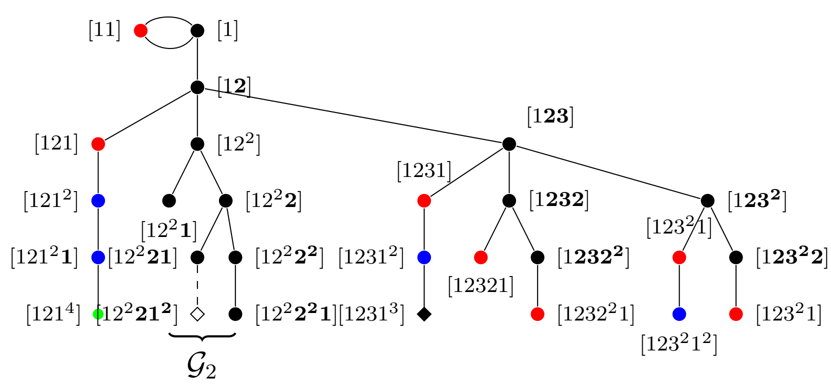
$\mathcal{G}_1 : \emptyset, [1]$

Figure 13: $\mathcal{G}_1 = \mathcal{O}^{op}_{BW}$. $|\mathcal{G}_1| = 2$.



$$\begin{cases} \mathcal{G}_1 : [1], [\mathbf{12}] & \text{rule (a1)} \\ \mathcal{G}_1 : [11], [121] & \text{rule (b)} \\ \mathcal{G}_1 : [12^2], [12^2\mathbf{1}] & \text{rule (a2)} \\ \mathcal{G}_0 : [121^2] & \text{rule (c1)} \end{cases}$$

Figure 14: $\mathcal{G}_2 = \mathcal{O}^{op}_{BWBW}$. The playing sequences are split into four groups that give $|\mathcal{G}_2| = 3|\mathcal{G}_1| + |\mathcal{G}_0| = 4|\mathcal{G}_1| - |\mathcal{G}_0|$.



$$\begin{cases} \mathcal{G}_2 : & [1], [\mathbf{12}], [\mathbf{123}], [\mathbf{1232}], [\mathbf{1232^2}], [\mathbf{123^2}], [\mathbf{123^22}] & \text{rule (a1)} \\ \mathcal{G}_2 : & \text{everything above and ends with 1} & \text{rule (b)} \\ \mathcal{G}_2 : & [12^2], [12^2\mathbf{1}], [12^2\mathbf{2}], [12^2\mathbf{2^2}], & \text{rule (a2)} \\ & [12^2\mathbf{2^21}], [12^2\mathbf{21}], [\mathit{12^2 21^2}] & \\ \mathcal{G}_1 : & [121^2], [121^2\mathbf{1}] & \text{rule (c1)} \\ \mathcal{G}_1 : & [1231^2], [123\mathbf{3}1^2] & \text{rule (c2)} \\ \mathcal{G}_0 : & [121^4] & \text{rule (d)} \\ \text{Extra} : & [1231^{\mathbf{2}}\mathit{1}] & \text{rule (e)} \end{cases}$$

Figure 15: $\mathcal{G}_3 = \mathcal{O}^{op}_{BWBWBW}$. The hollow diamond was modified and moved as in rule (e) to the position of the solid diamond. The playing sequence groups give $|\mathcal{G}_3| = 3|\mathcal{G}_2| + 2|\mathcal{G}_1| + |\mathcal{G}_0| = 4|\mathcal{G}_2| - |\mathcal{G}_1|$.

Proof. First of all, note that starting from $([01]^k)^T$, the playing sequences $\emptyset$ and [11] are the same. Let

$$G_k(x) = (\mathcal{D}_{(BW)^k}(x) - 2)x + 1 + x = 1 - x + x\mathcal{D}_{(BW)^k}(x). \tag{11}$$

so that

$$\mathcal{D}_{(BW)^k}(x) = \frac{G_k(x) - 1 + x}{x} = 2 + \frac{G_k(x) - 1 - x}{x} \tag{12}$$

We stretch out the cycle of the orbits, which means the element with the difference notation $((01)^k)^T$ is of level 0 and the one with the difference notation $((10)^k)^T$ is of level 1. This

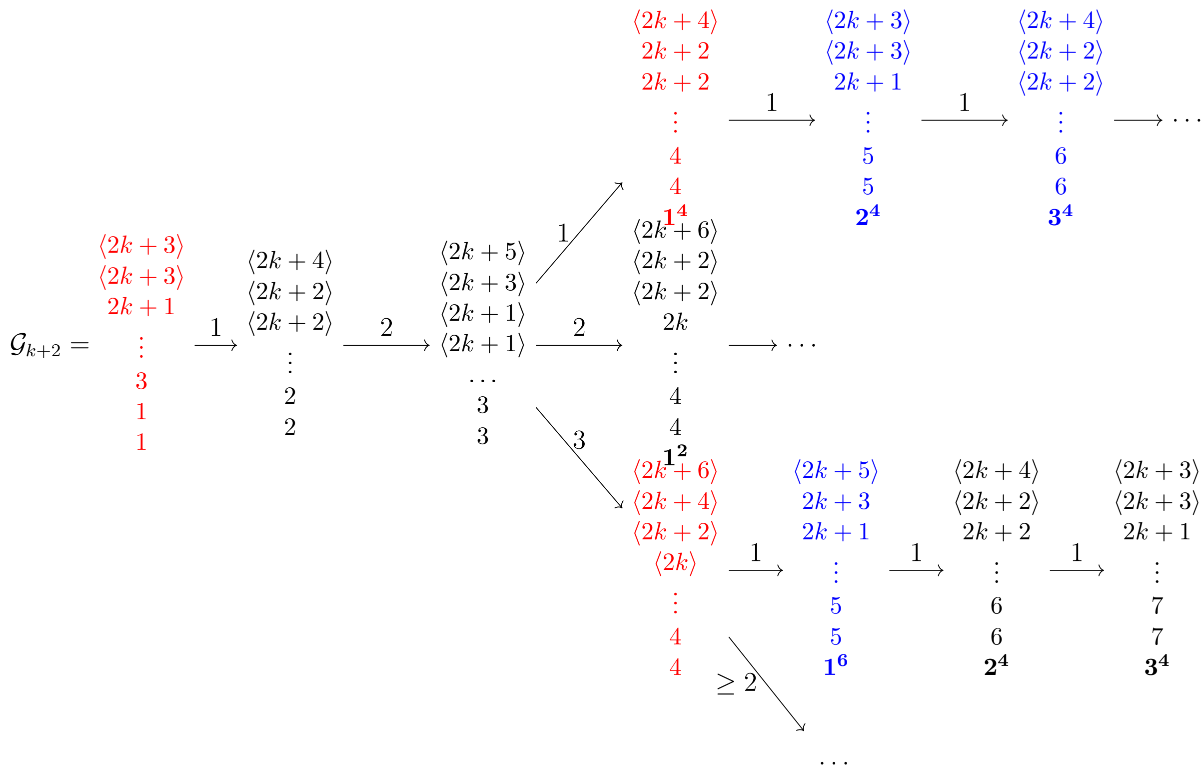


Figure 16: Diagram for $\mathcal{G}_{k+2}$

modification turns the graph $\mathcal{G}_k$ into a tree with root τ_k corresponding to the element at level 0. More specifically, $l(\tau_k) = 2k$ and

$$\tau_k = \begin{matrix} \langle 2k-1\rangle \\ \langle 2k-1\rangle \\ 2k-3 \\ \vdots \\ 1 \\ 1 \end{matrix}.$$

Consider $\mathcal{G}_{k+2}$, starting with $2k+4$ parts in the **Figure 16**. From the diagram, we see that any new parts (the bold entries) which are added at the bottom are not playable at any time due to **Claim 1**. Thus we can also investigate the quasi-infinite forest, where we have $k+2$ times to add the block $\begin{matrix}\langle 0\rangle \\ \langle 1\rangle\end{matrix}$ when performing the rule 3 (Subsection 4.2). We split the set of valid playing sequences in $\mathcal{G}_{k+2}$ into the following groups:

Group 1: $[1], [1\mathbf{2}], [1\mathbf{23}[\geq \mathbf{2}]]$: rule (a1)

We easily see that by playing R_1, we get to a partition which, if we ignore the first part and decrease each of the other parts by 1, turns out to be τ_{k-1}. Specifically:

$$R_1(\tau_{k+2}) = \begin{matrix}\langle 2k+4\rangle \\ \tau_{k+1}+1\end{matrix}.$$

Since new added parts do not have any impact, for each playing sequence σ in this group, if we erase $\sigma_1 = 1$ and decrease other indices by 1, we get a valid playing sequence in $\mathcal{G}_{k+1}$. That is because we have played only the lower part $\tau_{k+1}+1$ of $R_1(\tau_{k+2})$ if we keep playing parts of order at least 2. This map is clearly a bijection, and also induces a graph isomorphism between the elements that the playing sequences represent. Hence this group contribute $xG_{k+1}(x)$ to $G_{k+2}(x)$.

Group 2: $[11], [121], [123[\geq 2]1]$: rule (b)

Here $[11] = \emptyset$. These playing sequences are the results of playing R_1 at the end of the playing sequences in group 1, and are valid due to rule 2 in the quasi-infinite forest section. Each of them is exactly one level lower than the element in group 1 from which R_1 is played. However, level 0 (originally corresponding to $\emptyset$) is now at level 2 (corresponding to [11] in this group). Thus this group contributes $x^2G_{k+1}(x) - x^2 + 1$ to $G_{k+2}(x)$.

Group 3: $[121^2], [121^2\mathbf{1}], [121^2\mathbf{12...}]$: rule (c1)

It is easy to see from **Figure 16** that

$$R_{[121^2]} = \begin{matrix} \tau_n + 4 \\ \boldsymbol{\beta} \end{matrix}$$

where $\boldsymbol{\beta}$ has no playable parts. Thus this group contributes $x^4 G_k(x)$.

Group 4: $[121^4], [121^4\mathbf{11}], [121^4\mathbf{12...}]$: rule (d)

Similarly to group 3, the playing sequence $[121^4]$ leads to $\tau_{k-1}+6$ and some non-playable parts below, thus this group contributes $x^6 G_{k-1}(x)$.

A similar phenomenon happens for playing sequences starting with $[121^{2j}]$ for $j \leq k+1$, and each contributes $x^{2j+2} G_{k-j+1}(x)$.

Therefore, this branch of playing sequences $[121^2 \ldots]$ contributes

$$\sum_{j=1}^{k+1} x^{2j+2} G_{k-j+1} = \sum_{j=0}^{k} x^{2j+4} G_{k-j}.$$

Group 5: $[12^2 \ldots]$ and $[123[\geq 4]1^3 \ldots]$: rule (a2) together with rule (e)

As we can see in the diagram above, the top part of $R_{[12^2]}(\tau_{k+2})$ is 4 more than the second entry, so if we play R_1 any time after, any parts below are not playable anymore (**Claim 1**). On the other hand, this branch still performs like τ_{k+1} by mapping

$$\lambda = R_{[12^2]}(\tau_{k+2}) \longmapsto \begin{matrix} \lambda_1 - 4 \\ \lambda[2:(2k+1)] - 2 \end{matrix} = \begin{matrix} \langle 2k+2 \rangle \\ \langle 2k \rangle \\ \langle 2k \rangle \\ \vdots \\ 2 \\ 2 \end{matrix} = R_1(\tau_{k+1}) \tag{13}$$

except for the missing playing sequences of the form $[(12^2)2[\geq 3]1^2\rho]$, where $[12[\geq 3]1^2\rho]$ is a valid playing sequence in $\mathcal{G}_{k+1}$. We replace them by $[(123)[\geq 4]1^3\rho]$. Specifically, the map is as below

- First, replace the initial (12^2) by (123).
- Second, erase the fourth 2, increase the entries by 1 until 1^2 is reached.
- Third, add one more play 1, so 1^2 becomes 1^3. The rest of the playing sequence is kept.

The missing $[(12^2)2[\geq 3]1^2 \ldots]$ are bijectively the playing sequences $[12[\geq 3]1^2 \ldots]$ in $\mathcal{G}_{k+1}$ if we map $R_{[12^2]}(\tau_{k+2})$ as in (13) and apply rule (a2). They represent the partitions in bijection with the ones that are represented by the sequences $[(123)[\geq 4]1^3 \ldots]$. That can be seen from the diagram, where

$$R_{[123]}(\tau_{k+2}) = \begin{matrix} \langle 2k+6 \rangle \\ \langle 2k+4 \rangle \\ R_1(\tau_k) + 2 \end{matrix} \tag{14}$$

That is, if we ignore the top two parts and play R_σ where indices of σ are at least 4, we play only on $R_1(\tau_{k+1})$. Since

$$R_{[12]}(\tau_{k+1}) = \begin{matrix} \langle 2k+3 \rangle \\ R_1(\tau_k) + 2 \end{matrix} \tag{15}$$

From (15) and (14), we get the correspondence. In addition, the corresponding pairs of playing sequences represent partitions of the same level.

Hence, this group contributes $x^2(G_{k+1}(x) - 1 + x^2)$, because we start with the level 1 (as (13) shows) so the level 0 of $\mathcal{G}_{k+1}$ is now at level 2 instead.

Group 6: $[123\mathbf{1^2}], [123\mathbf{31^2}], [123\mathbf{4}[\geq \mathbf{3}]\mathbf{1^2}]$: rule (a2)

Those have $[1^2]$ at the end, and are valid by **Proposition 4.2.1**. From the diagram, we have

$$R_{[123]}(\tau_{k+2}) = \begin{matrix} \langle 2k+6 \rangle \\ \langle 2k+4 \rangle \\ R_1(\tau_n)+2 \end{matrix},$$

and

$$R_{[123^2]}(\tau_{k+2}) = \begin{matrix} \langle 2k+7 \rangle \\ \langle 2k+5 \rangle \\ \tau_k+2 \end{matrix}.$$

Since $R_{[123]}(\tau_{k+2})$ corresponds to the level 1 of $\mathcal{G}_k$, for which the next move is [2]. We introduce a bijection between playing sequences of $\mathcal{G}_k$ and those of group 6:

- First, replace the initial 1 by 123.
- Second, increase each of the following indices by 2.
- Third, append 1^2 to the end.

More specifically, the playing sequences $[1], [11], [12[\geq 1]]$ of $\mathcal{G}_k$ are mapped to $[123\mathbf{1^2}], [123\mathbf{31^2}], [123\mathbf{4}[\geq \mathbf{3}]\mathbf{1^2}]$.

Hence group 6 of playing sequences represent a part that is almost the same as $\mathcal{G}_k$, except that the level 0 ($\emptyset$ in $\mathcal{G}_k$) is moved to level 2 ([11] is mapped to $[123\mathbf{31^2}]$). This group's contribution is $x^4(G_k(x) - 1 + x^2)$.

Group 7: $[123^3[\geq \mathbf{3}]\mathbf{1^2}]$: rule (f)

They represent a part that is isomorphic to the part that is reached from playing $[121^3]$ in $\mathcal{G}_{k+1}$ and at the same level, since

$$R_{[123^3]}(\tau_{k+2}) = \begin{matrix} \langle 2k+8 \rangle \\ \langle 2k+6 \rangle \\ \langle 2k+2 \rangle \\ \langle 2k \rangle \\ \vdots \\ 6 \\ 6 \\ \mathbf{1^2} \end{matrix} = \begin{matrix} \langle 2k+8 \rangle \\ \langle 2k+6 \rangle \\ R_1(\tau_{k-1})+4 \\ \mathbf{1^2} \end{matrix}.$$

This part is similar to group 3 and 4, except that in each portion that is isomorphic to $\mathcal{G}_j$, the level 0 (sequence $\emptyset$) is moved to level 2 (sequence [11]). In that way, we don't count $[121^{2k+1}]$ leading to $\mathcal{G}_0$. Hence, this group 7 contributes

$$x^6 \left(\sum_{j=0}^{n-2} x^{2j}(G_{n-1-j} - 1 + x^2) \right).$$

Below is a summary of those groups of playing sequences in $\mathcal{G}_{k+2}$:

$$\begin{aligned}
\mathcal{G}_{k+1} &: [1], [1\mathbf{2}], [\mathbf{123}[\geq \mathbf{2}]] \\
\mathcal{G}_{k+1} &: [11], [121], [123[\geq 2])1] \\
\mathcal{G}_{k+1} &: [12^2\ldots], \underbrace{[\mathit{12^2}\mathbf{2}[\geq \mathbf{3}]\mathbf{1^2}\ldots]}_{\text{Missing part (*)}} \\
\mathcal{G}_k &: [121^2], [121^2\mathbf{1}], [121^2\mathbf{12\ldots}] \\
\mathcal{G}_k &: [1231^2], [123\mathbf{31}^2], [123\mathbf{4}[\geq \mathbf{3}]1^2] \\
\mathcal{G}_{k-1} &: [121^4], [121^4\mathbf{1}], [121^4\mathbf{12\ldots}] \\
&\ldots \\
\mathcal{G}_1 &= [121^{2n}], [121^{2n}\mathbf{1}] \\
\mathcal{G}_0 &= [121^{2(n+1)}] \\
\text{Extra} &: \underbrace{([\mathit{123}[\geq \mathit{4}]\mathit{1^3}\ \boldsymbol{\ldots}]}_{\text{Replacing for part (*)}}, ([123^3[\geq \mathbf{3}]\mathbf{1^2}]
\end{aligned} \tag{16}$$

Together we have the recurrence below:

$$\begin{aligned}
G_{k+2} =& xG_{k+1} + (x^2G_{k+1} - x^2 + 1) + x^2(G_{k+1} - 1 + x^2) + x^4(G_k - 1 + x^2) + \\
&+ \sum_{j=0}^{k} x^{2j+4} G_{k-j}(x) + x^6 \left(\sum_{j=0}^{k-2} x^{2j}(G_{k-1-j} - 1 + x^2) \right) \\
=& 1 - 2x^2 + x^6 + (x + 2x^2)G_{k+1} + x^4G_k + \\
&+ \sum_{j=0}^{k} x^{2j+4} G_{k-j}(x) + x^6 \left(\sum_{j=0}^{k-2} x^{2j}(G_{k-1-j} - 1 + x^2) \right)
\end{aligned}$$

This last formula implies:

$$\begin{aligned}
& G_{k+2}(x) - xG_{k+1}(x) = 2x^2 \sum_{j=0}^{k-1} x^{2j} G_{k-j}(x) + 1 - 2x^2 + 2x^{4+2k} \\
\text{together with} \quad & G_{k+1}(x) - xG_k(x) = 2x^2 \sum_{j=1}^{k-1} x^{2j-2} G_{k-j}(x) + 1 - 2x^2 + 2x^{2+2k} \\
\text{we get} \quad & G_{k+2}(x) - (x + x^2)G_{k+1}(x) + x^3G_k(x) = 2x^2G_{k+1}(x) + 1 - 3x^2 + 2x^4 \\
\text{Therefore} \quad & G_{k+2}(x) = (3x^2 + x)G_{k+1}(x) - x^3G_k(x) + 1 - 3x^2 + 2x^4.
\end{aligned}$$

Making the substitution from (11) for $G_k(x)$ in terms of $\mathcal{D}_{(BW)^k}(x)$ yields the recurrence in Theorem 3.1.1. □

4.5 More properties of partitions in $\mathcal{O}_{(BW)^k}$ and proof of Proposition 3.3.1

Proposition 4.5.1 contained three assertions, which we prove as three propositions (and a corollary) below.

Proposition 4.5.1. *The largest part size of a partition in $\mathcal{O}_{(BW)^k}$ is $4k-2$.*

Proof. Starting with the root of $\mathcal{G}_k$ which has largest part $2k-1$, if we play any parts not 1, the top entry gets 1 added into it. Since any new parts created by performing reverse BS are not playable at any time, as observed in the diagram **Figure 16** above, the number of operations R_j we can play are at most the number of parts in τ_k. One of the longest playing sequences is then $[12^{2k-2}1]$. Thus, the largest part we can get is right before we play the last R_1, which is

$$2k - 1 + 2k - 1 = 4k - 2.$$

□

Corollary 4.5.2. *Partitions in $\mathcal{O}_{(BW)^k}$ have at most $4k-2$ parts.*

Proof. Easily deduced by performing one reverse BS operation on elements with the largest part size. □

Proposition 4.5.3. *The longest playing sequence in $\mathcal{O}_{(BW)^k}$ is of length $2k$. The number of longest playing sequences is $2 \cdot 3^{k-2}$ for $k \geq 2$.*

Proof. By the same argument as in **Proposition 4.5.1**, the number of operations R_j we can play is at most the number of parts in τ_k, which is $2k$. The maximum is achieved by playing $[12^{2k-1}1]$.
We prove the second statement by induction. The base case $k=2$ is obvious and is shown in **Figure 14**.
From the groups of playing sequences of $\mathcal{G}_k$ as in 16, those longest playing sequences include

- twice the number of longest playing sequences in $\mathcal{G}_{k-1}$ given by group 2 and group 5,
- the sum of all numbers of longest playing sequences from $\mathcal{G}_0$ to $\mathcal{G}_{k-2}$ given by groups 3 and 4,
- the sum of all numbers of longest playing sequences from $\mathcal{G}_1$ to $\mathcal{G}_{k-2}$ given by group 7.

However, the portion involving $\mathcal{G}_1$ in the sequences $[12\ldots1^{2s}]$, $s \geq 1$ has no [2] branch coming out of it, so we don't count it in the longest sequences. Hence, the total number of longest playing sequences is

$$\begin{aligned}&2(2\cdot 3^{k-3}) + (\sum_{j=2}^{k-2} 2\cdot 3^{j-2} + 1 - 1) + (\sum_{j=2}^{k-2} 2\cdot 3^{j-2} + 1)\\ =&4\cdot 3^{k-3} + 2(3^{k-3}-1) + 2\\ =&2\cdot 3^{k-2}\end{aligned}$$

□

Proposition 4.5.4. *The number of partitions in $\mathcal{O}_{(BW)^k}$ with $4k-2$ parts is 3^{k-2} for $k \geq 2$.*

Proof. This number is equal to the number of partitions with largest part $4k-2$. We prove the asserted formula by induction. First, the base case $k=2$ is shown in **Figure 14**.

Since we only have $2k$ plays, the largest part size $4k-2$ is obtained if and only if we play only R_j with $j \geq 2$ after the compulsory initial R_1. Thus, from the recurrence structure (16), we see that those partitions only come from either the sequences leading to partitions with $4(k-1)-2$ parts in $\mathcal{G}_{k-1}$ obtained by playing $[12^2\ldots]$ (group 5), or the longest playing sequences in $\mathcal{G}_{k-1}$ obtained by playing group 1 - that is $[1], [12], [123[\geq 2]]$. Hence the total number of partitions of length $4k-2$ is

$$3^{k-3} + 2\cdot 3^{k-3} = 3^{k-2}.$$

□

This concludes the proof of all parts of Proposition 3.3.1.

5 Analysis of $\mathcal{O}_{P^k}$ for $P = BWW$ and BBW

5.1 On the reverse BS game graph for $(BWW)^k$

Proposition 2.2.4 implies that $n = |\lambda| = \binom{3k}{2} + k = \frac{9k^2-k}{2}$ for any partitions λ in the orbit $\mathcal{O}_{(BWW)^k}$. Similarly to the last section, we show some initial observations about $\mathcal{O}^{op}_{(BWW)^k}$ and their difference labelings in **Figure 17**.

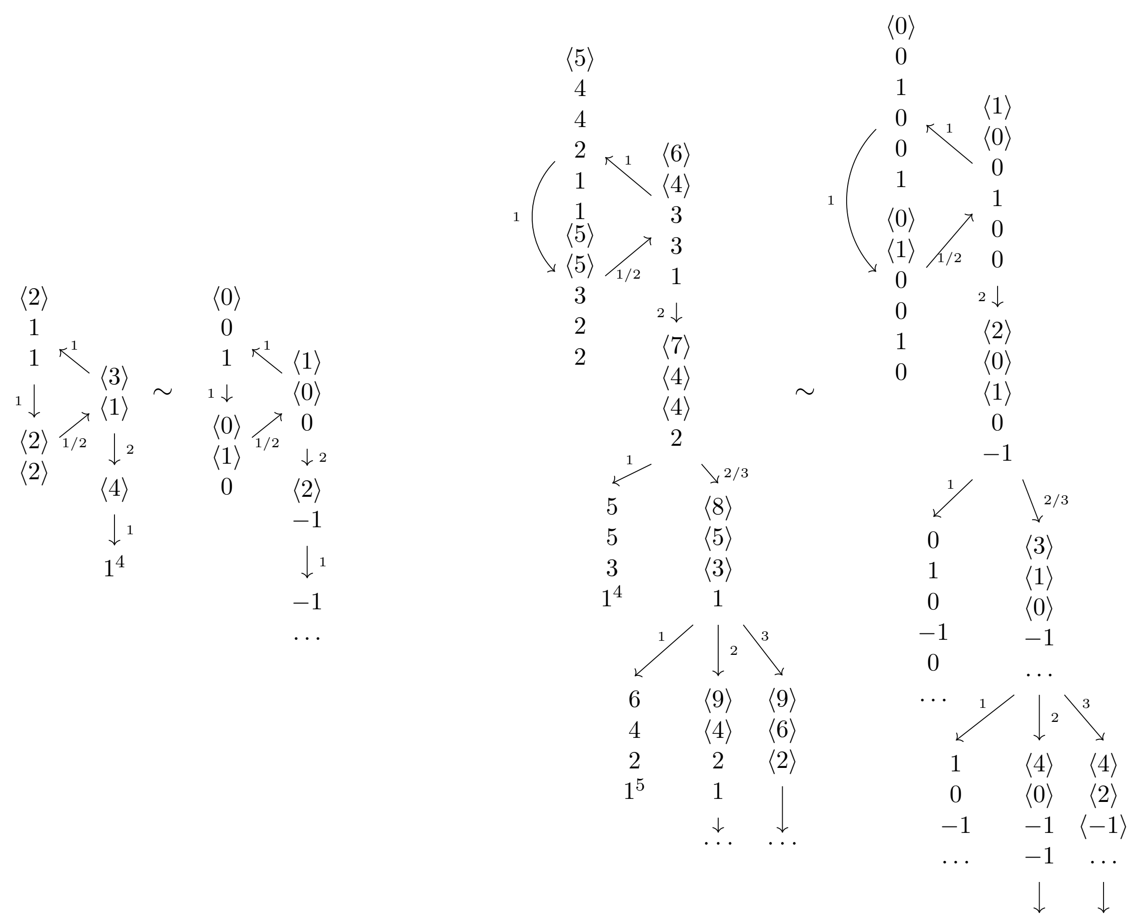


Figure 17: The reversed game graphs $\mathcal{O}^{op}_{(BWW)^k}$ and their difference labellings for $k = 1, 2$ up to level 3.

The recurrent 3-cycle $\mathcal{C}_{(BWW)^k}$ consists of three partitions $(3k, 3k-2, 3k-3, \ldots, 3, 1)$, $(3k-1, 3k-1, 3k-3, \ldots, 2, 2)$ and $(3k-1, 3k-2, 3k-2, \ldots, 2, 1, 1)$ whose difference labelings are $((100)^k)^T$, $((010)^k)^T$ and $((001)^k)^T$, respectively. We will use these elements as roots of the trees in the *BWW quasi-infinite forest*. First, we prove an important property for $(BWW)^k$ orbits:

Proposition 5.1.1. *Given a playing sequence σ from some recurrent partition $\lambda \in C_{(BWW)^k}$ in an orbit $\mathcal{O}^{op}_{(BWW)^k}$, then $\sigma_{j+1} \le \sigma_j + 1$ for any element of σ. Moreover, there exists an index t such that*

$$\sigma_1 \le \sigma_2 \le \ldots \le \sigma_t > \sigma_{t+1} > \ldots . \tag{17}$$

Proof. The only element in $C_{(BWW)^k}$ that is a root of a tree outside of the cycle is

$$\lambda = \begin{matrix} \langle 3k \rangle \\ \langle 3k-2 \rangle \\ 3k-3 \\ \vdots \\ 3 \\ 3 \\ 1 \end{matrix}$$

We will prove the Proposition 5.1.1 for playing sequences starting with λ, since any playing sequence starting from an arbitrary element in the orbit is contained in a playing sequence starting at λ. Let $\sigma[:j] = (\sigma_1, \ldots, \sigma_j)$ be the indices of σ up to index j. We will prove both the assertions by induction on j, that is, $\sigma[:j]$ has the properties in the proposition for any j.

- Base case: $j = 1$. If $\sigma_1 = 1$, then $R_1(\lambda)$ gets back to the cycle, so $\sigma_2 = 1$. If $\sigma_1 = 2$, then

  $$R_2(\lambda) = \begin{matrix} \langle 3k+1 \rangle \\ \langle 3k-2 \rangle \\ \langle 3k-2 \rangle \\ 3k-4 \\ \vdots \\ 4 \\ 4 \\ 2 \end{matrix}$$

  so $\sigma_2 \le 3 = \sigma_1 + 1$.

- Inductive step: Assume $\sigma[:j]$ has the form as in (17) and $\sigma_{h+1} \le \sigma_h$ for any $h < j$. If $\sigma_j = \max \sigma[:j] = m$, that means any parts below σ_j have not been played so far. Let $\psi = R_{\sigma[:(j-1)]}(\lambda)$. Note that ψ_s for any $s \ge m$ are parts of λ. Hence, there are 3 cases for $\begin{matrix} \psi_m \\ \psi_{m+1} \\ \psi_{m+2} \\ \psi_{m+3} \end{matrix}$, which are

  $$\begin{matrix} t+1 \\ t-1 \\ t-2 \\ t-2 \end{matrix} \qquad\qquad \begin{matrix} t \\ t \\ t-2 \\ t-3 \end{matrix} \qquad\qquad \begin{matrix} t \\ t-1 \\ t-1 \\ t-3 \end{matrix}$$

  for some t.

  Hence, $\sigma_{j+1} \le m + 1$ due to **Claim 1**, that is, the next playable part must differ from ψ_m by at most 2.

  Moreover in this case, $\sigma[:(j+1)]$ trivially has the form in (17). Another point is that $\sigma_{j+1} = m + 1$ happens only in the second case above. In that case, if we let $\omega = R_{\sigma[:(j+1)]}(\lambda)$ then $\omega_m - \omega_{m+1} = 3$.

  Now if σ_j is not the maximum in $\sigma[:j]$ then there is $s < j$ such that $\sigma_1 \le \ldots \le \sigma_s > \ldots > \sigma_j$ and $\sigma_s > \sigma_j$. The above argument shows that whenever we strictly increase σ_h, there is a gap of 3 between the parts of indices $\sigma_h - 1$ and σ_h, which cannot be decreased by playing a part of index larger or equal than σ_h. Thus $R_{\sigma[:s]}$ has a difference of at least 3 between consecutive parts up to σ_s. Recalling **Claim 1** and let $\psi = R_{\sigma[:(h-1)]}$ for some $h > s$, we see that any $R_{\sigma_h}(\psi)_i$ with $i \geq \sigma_h$ are not playable any time in the future because $\psi_{\sigma_h - 1} - \psi_{\sigma_h} \geq 3$. Thus $\sigma_{j+1} < \sigma_j$.

The proposition is proved.

□

5.2 The limit for $\mathcal{O}_{(BWW)^k}$ by level sizes and proof of Theorem 3.2.2

Similarly to the BW case, we consider the *BWW quasi-infinite forest* as in **Figure18**.

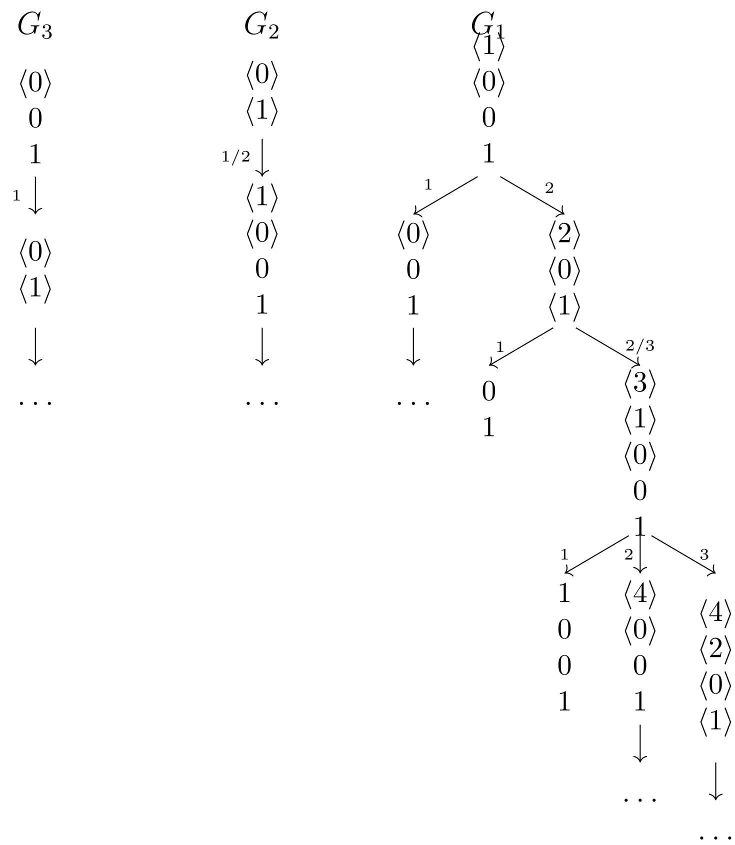


Figure 18: BWW quasi-infinite game graph.

Rule for $\xrightarrow{i}$ *in the BWW quasi-infinite forest:*

1. Delete i^{th} bracket. If the i^{th} bracket is $\langle 0 \rangle$ followed immediately by a $\langle 1 \rangle$, playing either vertex i or vertex $i+1$ has the same result, so we label the play by $\xrightarrow{i/i+1}$.
2. Increase all entries in rows above it by 1 each, and bracket them.
3. Bracket the new i^{th} number (if there is one) if it differs at most 1 from the old one. If there are two consecutive entries $\begin{matrix} 0 \\ 1 \end{matrix}$ and 0 is bracketed, so is 1.
4. If $\begin{matrix} \langle 0 \rangle \\ \langle 1 \rangle \end{matrix}$ are the lowest parts that are bracketed, and if part $\langle 0 \rangle$ among those two is played, append $\begin{matrix} \langle 0 \rangle \\ 0 \\ 1 \end{matrix}$ at the end.

The forest $\mathcal{F}_{BWW}$ consists of three trees G_1, G_2 and G_3 rooted at the three elements in $\mathcal{C}_{(BWW)^k}$:

$$\gamma_1 = \begin{matrix} \langle 1 \rangle \\ \langle 0 \rangle \\ 0 \\ 1 \end{matrix}, \gamma_2 = \begin{matrix} \langle 0 \rangle \\ \langle 1 \rangle \end{matrix} \text{ and } \gamma_3 = \begin{matrix} \langle 0 \rangle \\ 0 \\ 1 \end{matrix},$$

respectively. Let g_i be the generating function by level sizes of G_i for $i = 1, 2, 3$ and let $g = g_1 + g_2 + g_3$ be the generating function for the BWW quasi-infinite forest by level sizes.

Proposition 5.2.1. *In the tree G_1 of the BWW quasi-infinite forest, for any playing sequence that starts with* 2*, the top part is always playable. Moreover, the playing sequences* $[2[\geq 2]1]$ *lead to the leaves.*

Proof. By rule 2, the top part is always bracketed unless it is played. However, the first move $R_2(\gamma_1) = \begin{matrix}\langle 2\rangle\\ \langle 0\rangle\\ \langle 1\rangle\end{matrix}$ separates the top part from the second part by 2, and the difference between them is not decreasing due to rule 2 unless we play R_1. Now if we play R_1, the second and any lower parts cannot be bracketed due to rule 3, thus we reach a leaf. □

Similarly to the BW case, when we play R_1 from any roots, we get an entire copy of the tree rooted from the next node. In addition, in G_1, the playing sequences $[2[\geq 2]]$ form a subtree that is isomorphic to G_2 by neglecting the top part. By the **Proposition 5.2.1**, playing R_1 after those playing sequences is always valid and leads to the leaves. Hence we have the table of contribution of the playing sequences in the tree G_1 as below:

Sequence	**Contribution**
$[1\ldots]$	xg_3
$[2[\geq 2]]$	xg_2
$[2[\geq 2]1]$	x^2g_2

Table 4: Growth function for tree G_1.

Therefore, we get

$$\begin{aligned}g_3(x) &= 1 + xg_2(x)\\ g_2(x) &= 1 + xg_1(x)\\ g_1(x) &= 1 + xg_3(x) + (x + x^2)g_2(x)\end{aligned}$$

Putting them all together gives

$$\begin{aligned}g_1(x) &= 1 + (x + x^2)(1 + xg_1(x)) + x(1 + x + x^2g_1(x))\\ &= 1 + 2x + 2x^2 + (2x^3 + x^2)g_1(x)\end{aligned}$$

Thus one has

$$g_1(x) = -\frac{2x^2 + 2x + 1}{2x^3 + x^2 - 1}.$$

Therefore,

$$g(x) = 2 + x + (1 + x + x^2)g_1(x) = \frac{x^3 - 3x^2 - 4x - 3}{2x^3 + x^2 - 1}.$$

Now, we desire the generating function for the level sizes of $\mathcal{O}^{op}_{(BWW)^k}$ in the limit as $k \to \infty$. Let the generating function for heights of the elements of the BWW quasi-infinite forest to be H_{BWW}. As constructed, our quasi-infinite forest has an entire copy of itself after playing R_1 from each root. Moreover, H_{BWW} is the generating function by level sizes for the remaining part after disregarding that copy, including the whole trees G_2, G_3 from level 1 and the left branch of G_1. Hence

$$g(x) = xg(x) + H_{BWW}(x)$$

That leads to

$$\begin{aligned}H_{BWW}(x) = (1 - x)g(x) &= \frac{(1 - x)(x^3 - 3x^2 - 4x - 3)}{2x^3 + x^2 - 1}\\ &= 3 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 17x^7 + 25x^8 + 39x^9 + 59x^{10} + \ldots.\end{aligned}$$

In this case, we also have the same result as in **Theorem 4.3.1**, that is the infinite forest coincides with the finite game graph for $(BWW)^k$ at least up to level k. Thus we conclude $H_{BWW}(x)$ as the limit of the generating functions of $(BWW)^k$ game graphs by level sizes.

Next, the limit by level sizes of $\mathcal{O}^{op}_{(BBW)^k}$ can be computed easily by the same technique. The set of playing sequences in the *BBW quasi-infinite forest* turns out to be exactly the same as in the BWW case. That is because the **Proposition 5.2.1** holds true for the BBW quasi-infinite forest. **Figure 19** shows the analogous *BBW quasi-infinite forest*. Therefore, the limit $H_{BBW}(x)$ for generating functions of $\mathcal{O}^{op}_{(BBW)^k}$ by level sizes is the same as $H_{BWW}(x)$ shown above, completing the proof of **Theorem 3.2.2**.

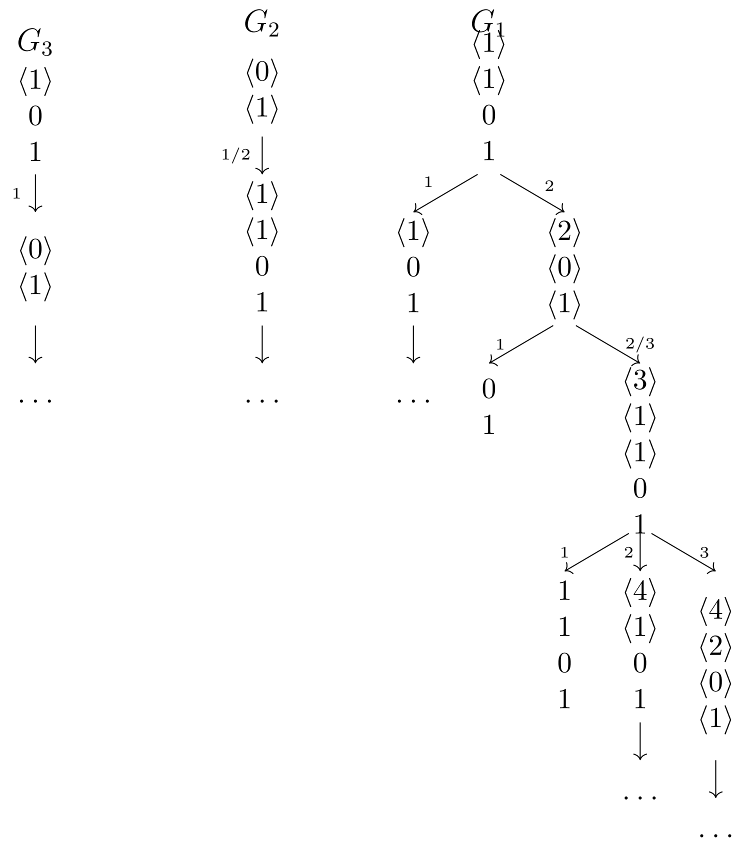


Figure 19: BBW quasi-infinite forest.

5.3 Proof of Theorem 3.1.2

5.3.1 Notation

Throughout this section, we will use the same symbol for the tree and its size. The top parts $+n$ means that there are n playable parts at the top but by playing them, we disable any parts below them (including itself) from being playable, according to the second rule in **Section 5.2**. Those extra parts will occur during the recurrences and can be neglected so that we can recognize the isomorphism between a part of a tree and another tree. We don't count playing those top extras in the cardinality of the trees. As before, the playable parts will be put in brackets $\langle . \rangle$.
For a sequence σ, we use $\sigma[j:] = (\sigma_j, \sigma_{j+1}, \ldots)$.

5.3.2 Three types of trees

Those three types of trees will occur in the orbit $\mathcal{O}_{(BWW)^k}$. Recall the staircase is of the form $\Delta_k = (k, k-1, \ldots, 1, 0)$. When describing the trees, we will use the *disrupted staircase* partition, defined as

$$\Delta_{j+k}^m = (m + 3(j+k), m + 3(j+k) - 1, \ldots, m + 3k + 1, m + 3k - 1, \ldots, m)$$

which is the staircase starting with m, having $3(j+k)$ entries and there is a separation of 1 in between. More specifically, using the notation $\lambda + m = (\lambda_1 + m, \ldots, \lambda_n + m)$ for a partition λ of length n and allowing the last part λ_n being 0, we can define the *disrupted staircase* as

$$\Delta_{j+k}^m = \begin{matrix} \Delta_{3j-1} + (3k+1) + m \\ \Delta_{3k-1} + m \end{matrix}$$

That is, the lowest part of the higher partition is 2 more than the highest part of the lower partition. This disrupted staircase will be used as a base to add the "blocks" on. In particular, $\Delta_{3j-1} = \Delta_{j+0}^0$.

Type 1: T_k^j has j blocks $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ followed by a separation and k blocks $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. If $k = 0$, the base is Δ_{j+0}^0, and T_0^j is exactly one of the elements in the recurrent set of $\mathcal{O}^{op}_{(BWW)^j}$. Otherwise the base is Δ^1_{j+k} and has one extra playable part on top. We also denote the root of T^j_k by τ^j_k. For example:

$$\tau^3_0 = \begin{array}{c} \langle 8\rangle \\ \langle 8\rangle \\ 6 \\ 5 \\ 5 \\ 3 \\ 2 \\ 2 \end{array}; \quad \tau^2_1 = \begin{array}{c} +1 \\ \langle 10\rangle \\ \langle 10\rangle \\ 8 \\ 7 \\ 7 \\ 5 \\ 3 \\ 2 \\ 2 \end{array}; \quad \tau^0_3 = \begin{array}{c} +1 \\ \langle 9\rangle \\ 8 \\ 8 \\ 6 \\ 5 \\ 5 \\ 3 \\ 2 \\ 2 \end{array}; \quad \tau^j_0 = \begin{array}{c} \langle 3j-1\rangle \\ \langle 3j-1\rangle \\ 3j-3 \\ \vdots \\ 2 \\ 2 \end{array}; \quad \tau^j_k = \begin{array}{c} +1 \\ \langle 3(j+k)+1\rangle \\ \langle 3(j+k)+1\rangle \\ \vdots \\ 3k+4 \\ 3k+4 \\ 3k+2 \\ 3k \\ \vdots \\ 3 \\ 2 \\ 2 \end{array}$$

Type 2: A^j_k has two extra playable parts on top, j blocks $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ followed by a separation and k blocks $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. If $k = 0$, the base is Δ^3_{j+0}. Otherwise the base is Δ^2_{j+k}. The root here is denoted α^j_k. For example:

$$\alpha^3_0 = \begin{array}{c} +2 \\ \langle 11\rangle \\ \langle 11\rangle \\ 9 \\ 8 \\ 8 \\ 6 \\ 5 \\ 5 \\ 3 \end{array}; \quad \alpha^2_1 = \begin{array}{c} +2 \\ \langle 11\rangle \\ \langle 11\rangle \\ 9 \\ 8 \\ 8 \\ 6 \\ 4 \\ 3 \\ 3 \end{array}; \quad \alpha^0_3 = \begin{array}{c} +2 \\ \langle 10\rangle \\ 9 \\ 9 \\ 7 \\ 6 \\ 6 \\ 4 \\ 3 \\ 3 \end{array}; \quad \alpha^j_0 = \begin{array}{c} +2 \\ \langle 3j+2\rangle \\ \langle 3j+2\rangle \\ 3j \\ \vdots \\ 5 \\ 5 \\ 3 \end{array}; \quad \alpha^j_k = \begin{array}{c} +2 \\ \langle 3(j+k)+2\rangle \\ \langle 3(j+k)+2\rangle \\ \vdots \\ 3k+5 \\ 3k+5 \\ 3k+3 \\ 3k+1 \\ \vdots \\ 4 \\ 3 \\ 3 \end{array}$$

The purpose of this tree type is to disable any added parts (at the bottom) from being playable in the future.

Type 3: B^j_k has two extra playable parts on top, j blocks $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ followed by a separation and k blocks $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and one extra playable part at the bottom. If $k = 0$, the base is $(\Delta^4_{j+0}, 2)$. Otherwise the base is $(\Delta^3_{j+k}, 2)$. The root for B^j_k is denoted β^j_k. For example:

$$\beta^3_0 = \begin{array}{c} +2 \\ \langle 12\rangle \\ \langle 12\rangle \\ 10 \\ 9 \\ 9 \\ 7 \\ 6 \\ 6 \\ 4 \\ 2 \end{array}; \quad \beta^2_1 = \begin{array}{c} +2 \\ \langle 12\rangle \\ \langle 12\rangle \\ 10 \\ 9 \\ 9 \\ 7 \\ 5 \\ 4 \\ 4 \\ 2 \end{array}; \quad \beta^0_3 = \begin{array}{c} +2 \\ \langle 11\rangle \\ 10 \\ 10 \\ 8 \\ 7 \\ 7 \\ 5 \\ 4 \\ 4 \\ 2 \end{array}; \quad \beta^j_0 = \begin{array}{c} +2 \\ \langle 3j+3\rangle \\ \langle 3j+3\rangle \\ 3j+1 \\ \vdots \\ 6 \\ 6 \\ 4 \\ 2 \end{array}; \quad \beta^j_k = \begin{array}{c} +2 \\ \langle 3(j+k)+3\rangle \\ \langle 3(j+k)+3\rangle \\ \vdots \\ 3k+6 \\ 3k+6 \\ 3k+4 \\ 3k+2 \\ \vdots \\ 5 \\ 4 \\ 4 \\ 2 \end{array}$$

5.3.3 Recurrent Relationship between the tree types

Some examples are given in the figures below. The rules for coloring are:

- Blue nodes are new nodes that appear in the current tree.
- Black nodes are from an embedding of another tree, specified in the figures, into the current tree.
- Red nodes are directly obtained from the blue nodes by playing R_1 on them.
- Green nodes are the effects of the extra playable part at the bottom of B_k^j.

Figure 20: Tree A_0^0

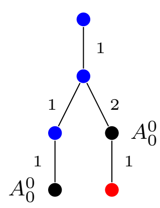


Figure 21: Tree A_0^1

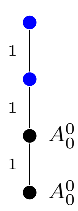


Figure 22: Tree A_1^0

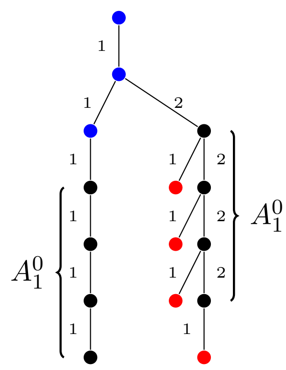


Figure 23: Tree A_1^1

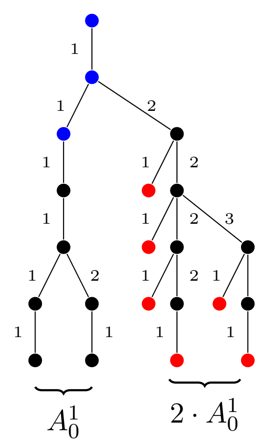


Figure 24: Tree A_0^2

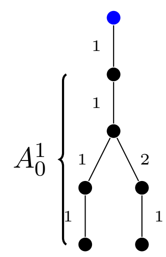


Figure 25: Tree B_1^0

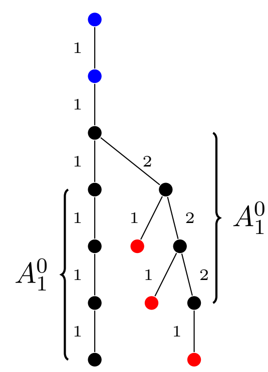


Figure 26: Tree $T_2^0 = A_2^0$

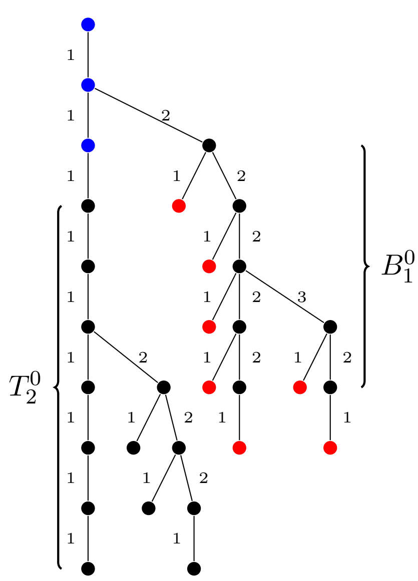


Figure 27: Tree T_1^1

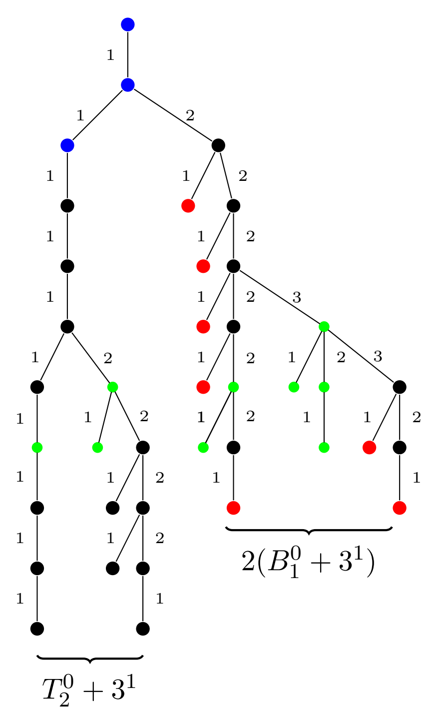


Figure 28: Tree B_1^1

Here are the intertwined recurrence relations between the three types of trees defined in the previous subsection. The proofs are also given.

(a) For the trees A_k^j:

$$A_k^j = \begin{cases} 1 & \text{if } j = k = 0 \\ 1 + 3A_{k-1}^0 & \text{if } j = 0, k > 0 \\ 3 + 3A_k^{j-1} & \text{if } j > 0, k > 0 \end{cases} \tag{18}$$

From the recurrences above, we get

$$A_k^j = \frac{3^{j+k+1} - 3}{2} + 3^j$$

$$A_k^0 = \frac{3^{k+1} - 1}{2} \quad , \quad A_0^j = \frac{3^{j+1} - 3}{2} + 3^j$$

Proof. We examine the trees in three cases:

- If $j = k = 0$, we have no blocks, so $A_0^0 = \begin{array}{c} +2 \\ \emptyset \end{array} = 1$.
- If $j = 0, k > 0$, we start with $3k + 2$ parts and play

$$A_k^0 = \begin{array}{c} +2 \\ \langle 3k+1 \rangle \\ 3k \\ 3k \\ \vdots \\ 4 \\ 3 \\ 3 \end{array} \xrightarrow{1} \begin{array}{c} +2 \\ \langle 3k+1 \rangle \\ \langle 3k+1 \rangle \\ 3k-1 \\ \vdots \\ 5 \\ 4 \\ 4 \end{array} \xrightarrow{1} \begin{array}{c} +2 \\ \langle 3k+2 \rangle \\ \langle 3k \rangle \\ 3k-1 \\ 3k-1 \\ \vdots \\ 6 \\ 5 \\ 5 \\ \mathbf{1} \end{array} \xrightarrow{1} \begin{array}{c} +2 \\ \langle 3k+1 \rangle \\ 3k \\ 3k \\ \vdots \\ 7 \\ 6 \\ 6 \\ \mathbf{2} \\ \mathbf{1} \\ \mathbf{1} \end{array} \longrightarrow \cdots$$

$$\Big\downarrow{\scriptstyle 2} \quad \begin{array}{c} +3 \\ \langle 3k \rangle \\ \langle 3k \rangle \\ 3k-2 \\ \vdots \\ 7 \\ 6 \\ 6 \\ \mathbf{2} \end{array} \longrightarrow \cdots$$

Figure 29

The bold parts are never playable according to rule 3 in **Section 5.2** because the first bold part differs from its immediate part above by 4, so we can neglect them. By playing $R_{[1^3]}$, we get a partition with $k - 1$ blocks of $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Let $\mathcal{R} := \{R_\sigma(\alpha_k^0) : \sigma = [1^3 \ldots]\}$. Specifically,

$$R_{[1^3]} = \begin{array}{c} \alpha_{k-1}^0 + 3 \\ \mathbf{2} \\ \mathbf{1} \\ \mathbf{1} \end{array} .$$

We define the map $\varphi : \mathcal{R} \longrightarrow A_{k-1}^0$ to be

$$\varphi(R_\sigma(\alpha_k^0)) = R_{\sigma[4:]}(\alpha_{k-1}^0)$$

where $R_\sigma(\alpha_k^0) \in \mathcal{R}$. We easily see that φ is a graph isomorphism with the operation R because: (1) the bijection is recognized by corresponding a playing sequence σ' in A_{k-1}^0 with $\sigma = [1^3\sigma']$ in $\mathcal{R}$, and (2) $\varphi(R_{[1^3]}) = \alpha_{k-1}^0$.
Similarly, playing $[1^2]$ and $[1^2 2[\geq 2]]$ we have a part that is isomorphic to A_{k-1}^0 (by neglecting the top part), since

$$R_{[1^2]} = \begin{array}{c} \langle 3k+2 \rangle \\ \alpha_{k-1}^0 + 2 \\ \mathbf{1} \end{array} \qquad \text{and} \qquad R_{[1^2 2]} = \begin{array}{c} R_1(\alpha_{k-1}^0) + 2 \\ \mathbf{2} \end{array} .$$

Each of them, except the root, has an extra playable top part, which means playing sequences $[1^2 2[\geq 2]1]$ give another A_{k-1}^0. Thus we have:

$$A_k^0 = 3 + A_{k-1}^0 + 2(A_{k-1}^0 - 1) = 1 + 3A_{k-1}^0.$$

That leads to

$$\left(A_k^0 + \frac{1}{2}\right) = 3\left(A_{k-1}^0 + \frac{1}{2}\right) = \ldots = 3^k\left(A_0^0 + \frac{1}{2}\right) = \frac{3^{k+1}}{2}.$$

Hence

$$A_k^0 = \frac{3^{k+1} - 1}{2}.$$

- When $j > 0$, we start with $3(j+k)+2$ parts and play

$$\begin{array}{cccccccccc}
A_k^j = & \begin{array}{c} +2\\ \langle 3(j+k)+2\rangle \\ \langle 3(j+k)+2\rangle \\ 3(j+k) \\ \vdots \\ 4\\ 3\\ 3\end{array} & \xrightarrow{1} & \begin{array}{c} +2\\ \langle 3(j+k)+3\rangle \\ \langle 3(j+k)+1\rangle \\ \vdots \\ 5\\ 4\\ 4\\ \mathbf{1}\end{array} & \xrightarrow{1} & \begin{array}{c} +2\\ \langle 3(j+k)+2\rangle \\ 3(j+k)+1 \\ \vdots \\ 6\\ 5\\ 5\\ \mathbf{2}\\ \mathbf{1}\\ \mathbf{1}\end{array} & \xrightarrow{1} & \begin{array}{c} +2\\ \langle 3(j+k)+2\rangle \\ \langle 3(j+k)+2\rangle \\ 3(j+k) \\ \vdots \\ 7\\ 6\\ 6\\ \mathbf{3}\\ \mathbf{2}\\ \mathbf{2}\end{array} & \to \cdots \\
 & & & 2\downarrow & & & & & \\
 & & & \begin{array}{c} +3\\ \langle 3(j+k)+1\rangle \\ \langle 3(j+k)+1\rangle \\ 3(j+k)-1 \\ \vdots \\ 6\\ 4\\ 4\\ \mathbf{2}\end{array} & \to \cdots & & & &
\end{array}$$

Figure 30

The bold parts are never playable, so we can neglect them, and we proceed as in the previous case. By playing $[1^3]$ we get the root of A_k^{j-1}. Similarly, the playing sequences $[12[\geq 2]]$ form a subtree that is isomorphic to A_k^{j-1} (by neglecting one top entry). Each of them has an extra playable at the top. Thus we have:

$$A_k^0 = 3 + A_{k-1}^0 + 2A_{k-1}^0 = 3 + 3A_{k-1}^0.$$

That leads to

$$A_k^j = \frac{3^{j+k+1} - 3}{2} + 3^j.$$

□

We also prove a lemma for the trees A_k^j that will be useful in the analysis of the trees B_k^j.

Definition 5.3.1. *An* ending branch *is obtained by playing sequence* $[1^{\alpha_1} 2^{\alpha_2} \ldots k^{\alpha_k}(k-1)\ldots 1]$ *where* $[k]$ *is played as many times as possible.*

Example 5.3.2. ***Figure 21, 22, 23, 24*** *show that*

- A_0^1 *has* 2 *ending branches:* $[1^3$ *and* $[121]$.
- A_1^0 *has* 1 *ending branch* $[1^3]$.
- A_1^1 *has* 2 *ending branches:* $[1^6]$ *and* $[12^4 1]$.
- A_0^2 *has* 4 *ending branches:* $[1^6], [1^4 21], [12^4 1]$ *and* $[12^2 321]$.
- T_1^1 *has* 4 *ending branches:* $[1^9], [1^5 2^3 1], [12^5 1]$ *and* $[12^3 321]$.
- B_1^1 *has* 4 *ending branches:* $[1^{10}], [1^5 2^4 1], [12^6 1]$ *and* $[12^3 3^2 21]$.

Lemma 5.3.3. *The tree A_k^j has 2^{j+k-1} equal-length ending branches of length $3(j+k)+2$. Moreover, each ending branch lead to a partition with the largest number of parts in A_k^j.*

Proof. From the argument for the recurrence (18) of A_k^j, we see that A_k^j has twice as many ending branches as A_k^{j-1} if $j > 0$, and twice as many ending branches as A_{k-1}^j if $j = 0$. Since A_1^0 has only one ending branch, the first part of the lemma follows. Moreover, if $j > 0$, although it requires one fewer play to get to the second subtree that is isomorphic to A_k^{j-1} than it does to reach the first subtree ([12] in comparison to $[1^3]$), we can always play exactly one more R_1 for each ending branches in the second subtree. Thus, the two branches have the same height. A similar argument applies for $j = 0$.

In the tree A_k^j, the added new parts are not playable at any time, so each ending branch actually plays all and only the $3(j+k)+2$ original parts. This gives the length $3(j+k)+2$ for each of them.

To see the last statement regarding the largest number of parts in partitions in A_k^j, we refer to the recurrence (18) again, with the trivial base case of A_0^1 and A_1^0. If $j > 0$, the first branch of A_k^j resulted from playing $[1^3\ldots]$ leads to the leaves of A_k^{j-1}, while the second branch resulted from playing $[12[\geq 2]]$ leads to the leaves of A_k^{j-1} and one extra play of [1] at each leaf. While $[1^3]$ adds 3 new parts at the end, [12] only adds 1 new part, as we see from the Figure 30 above. However, playing [2] add 1 to the top part, so it is $3(j+k)+4)$ and is 2 more than the original top $(3(j+k)+2)$. This part will only be played at the very end of the ending branch and make up the shortage of 2 parts in comparison with the other branch. Playing $[1^3]$ does not change the value of top part. Hence, the ending branches of the form $[12\ldots 1]$ lead to partitions of the same number of parts as the leaves of A_k^{j-1}. A similar argument applies to $j = 0$. $\square$

(b) For the trees T_k^j:

$$T_k^j = \begin{cases} A_k^0 & \text{if } j = 0 \\ 3 + T_{k+1}^{j-1} + 2B_k^{j-1} & \text{if } j > 0,\ k > 0 \\ 9 + 2T_1^{j-2} + 4B_0^{j-2} & \text{if } k = 0 \end{cases} \tag{19}$$

From the recurrences above, we get

$$T_k^0 = \frac{3^{k+1}-1}{2}$$

$$\begin{aligned} T_0^j &= 9 + 6(j-2) + 2T_{j-1}^0 + 4\sum_{t=0}^{j-2} B_t^{j-2-t} \\ &= 6j + 3^j - 4 + 4\sum_{t=0}^{j-2} B_t^{j-2-t}. \end{aligned}$$

Proof. • When $j = 0$, any added parts (bold in the figure) at the bottom will never be playable:

$$T_k^0 = \begin{array}{c} +1 \\ \langle 3k+2\rangle \\ 3k \\ 3k \\ \vdots \\ 3 \\ 2 \\ 2 \end{array} \xrightarrow{1} \begin{array}{c} +1 \\ \langle 3k+1\rangle \\ \langle 3k+1\rangle \\ 3k-1 \\ \vdots \\ 4 \\ 3 \\ 3 \end{array} \xrightarrow{1} \begin{array}{c} +1 \\ \langle 3k+2\rangle \\ \langle 3k\rangle \\ 3k-1 \\ 3k-1 \\ \vdots \\ 5 \\ 4 \\ 4 \\ \mathbf{1} \end{array} \longrightarrow \cdots$$

Figure 31

So T_k^0 is exactly the tree A_k^0.

- When $j > 0, k > 0$, we start with $3(j+k)+1$ parts:

$$
T_k^j = \begin{array}{c} +1 \\ \langle 3(j+k)+1 \rangle \\ \langle 3(j+k)+1 \rangle \\ 3(j+k)-1 \\ \vdots \\ 3k+4 \\ 3k+4 \\ 3k+2 \\ 3k \\ \vdots \\ 3 \\ 2 \\ 2 \end{array}
\xrightarrow{1}
\begin{array}{c} +1 \\ \langle 3(j+k)+2 \rangle \\ \langle 3(j+k) \rangle \\ \vdots \\ 4 \\ 3 \\ 3 \\ \mathit{1} \end{array}
\xrightarrow{1}
\begin{array}{c} +1 \\ \langle 3(j+k)+1 \rangle \\ 3(j+k) \\ 3(j+k) \\ \vdots \\ 5 \\ 4 \\ 4 \\ \mathit{2} \\ \mathit{1} \\ \mathit{1} \end{array}
\xrightarrow{1}
\begin{array}{c} +1 \\ \langle 3(j+k)+1 \rangle \\ \langle 3(j+k)+1 \rangle \\ \vdots \\ 6 \\ 5 \\ 5 \\ \mathit{3} \\ \mathit{2} \\ \mathit{2} \end{array}
\to \cdots
$$

$$
\begin{array}{c} +1 \\ \langle 3(j+k)+2 \rangle \\ \langle 3(j+k) \rangle \\ \vdots \\ 4 \\ 3 \\ 3 \\ \mathit{1} \end{array}
\quad 2\downarrow \quad
\begin{array}{c} +2 \\ \langle 3(j+k) \rangle \\ \langle 3(j+k) \rangle \\ 3(j+k)-2 \\ \vdots \\ 5 \\ 4 \\ 4 \\ \mathit{2} \end{array}
\to \cdots
$$

Figure 32: Caption

The italicized parts are playable in the future because the first italicized part differs from its neighborly immediately above by only 2. By playing $[1^3]$, we add one block $\begin{pmatrix} 0 \\ 0+\Delta^0_{2+0} \\ 1 \end{pmatrix}$ at the bottom, and also use all of the top block $\begin{array}{c} \langle 0 \rangle \\ \langle 1 \rangle \\ 0 \end{array}$, so we reach a part that is isomorphic to T_{k+1}^{j-1}. By playing [12], we use up the top block $\begin{array}{c} \langle 0 \rangle \\ \langle 1 \rangle \\ 0 \end{array}$ and the playing sequences $[12[\geq 2]]$ form a part that is isomorphic to B_k^{j-1} (by neglecting one top part). Each of them has one extra playable part at the top. Hence

$$T_k^j = 3 + T_{k+1}^{j-1} + 2B_k^{j-1}.$$

- When $j > 0, k = 0$, we start with $3j+2$ parts:

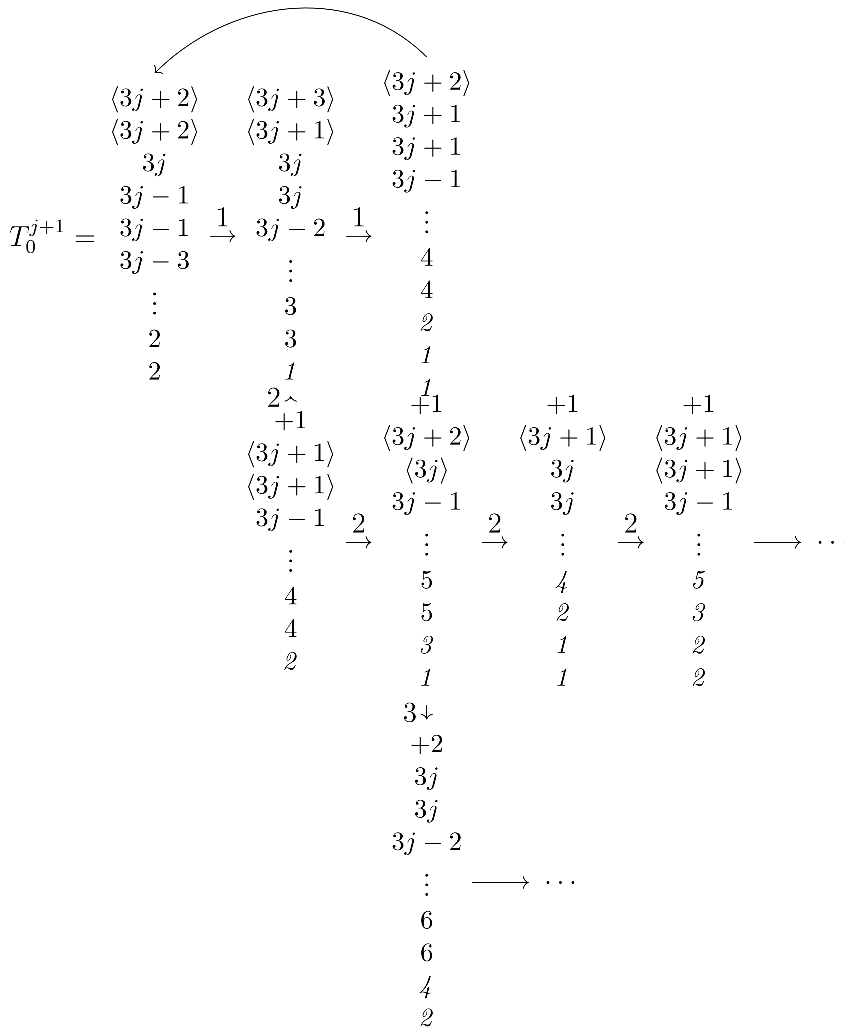


Figure 33

Playing $[1^3]$ gets we back to the root. Playing $[12^4]$ gets to the tree T_1^{j-1} because we used up 2 top blocks $\begin{pmatrix}0\\1\\0\end{pmatrix}$, plus one extra playable top part. Playing [123] gets to the tree B_0^{j-1} and two extra playables at top. If we play the second to top part, we can only play [1] next (by definition of extra top playables) and terminate, so each node in this part (which is in the tree B_0^{j-1}), we count as 4. Hence:

$$T_0^{j+1} = 9 + 2T_1^{j-1} + 4B_0^{j-1}.$$

□

(c) For the trees B_k^j:

$$B_k^j = \begin{cases} 1 + A_0^k & \text{if } j = 0 \\ 3 + (T_{k+1}^{j-1} + 3^{j+k-1}) + 2(B_k^{j-1} + 3^{j+k-1}) & \text{if } j > 0 \end{cases} \tag{20}$$

From the recurrences above, we get

$$B_k^j = \begin{cases} \dfrac{3^{k+1}-1}{2} + 3^k & \text{if } j = 0 \\ 3j + 3^{j+k} + T_{j+k}^0 + 2\sum_{t=0}^{j-1} B_{k+t}^{j-1-t} & \text{if } j > 0 \end{cases}$$

so for $j > 0$:

$$\begin{aligned} 2B_k^j &= 6j + 2\cdot 3^{j+k} + 3^{j+k+1} - 1 + 4\sum_{t=0}^{j-1} B_{k+t}^{j-1-t} \\ &= 6j + 5\cdot 3^{j+k} + 4\sum_{t=0}^{j-1} B_{k+t}^{j-1-t}. \end{aligned}$$

Proof.

- When $j = 0$, we start with $3k + 3$ parts:

$$B_k^0 = \begin{array}{c} +2 \\ \langle 3k+2 \rangle \\ 3k \\ 3k \\ \vdots \\ 5 \\ 4 \\ 4 \\ 2 \end{array} \xrightarrow{1} \begin{array}{c} +2 \\ \langle 3k+1 \rangle \\ \langle 3k+1 \rangle \\ 3k-1 \\ \vdots \\ 5 \\ 5 \\ 3 \end{array} = A_0^k$$

Figure 34

From the diagram, we have

$$B_k^0 = 1 + A_0^k.$$

- When $j > 0$, we start with $3(j + k) + 3$ parts:

$$B_k^j = \begin{array}{c} +2 \\ \langle 3(j+k)+3 \rangle \\ \langle 3(j+k)+3 \rangle \\ 3(j+k)+1 \\ \vdots \\ 3k+6 \\ 3k+6 \\ 3k+4 \\ 3k+2 \\ \vdots \\ 5 \\ 4 \\ 4 \\ \mathbf{2} \end{array} \xrightarrow{1} \begin{array}{c} +2 \\ \langle 3(j+k)+4 \rangle \\ \langle 3(j+k)+2 \rangle \\ 3(j+k)+1 \\ \vdots \\ 5 \\ 5 \\ \mathbf{3} \\ \mathit{1} \\ 2 \searrow \\ +3 \\ \langle 3(j+k)+2 \rangle \\ \langle 3(j+k)+2 \rangle \\ 3(j+k) \\ \vdots \\ 7 \\ 6 \\ 6 \\ \mathbf{4} \\ \mathit{2} \end{array} \begin{array}{c} \xrightarrow{1} \begin{array}{c} +2 \\ \langle 3(j+k)+3 \rangle \\ 3(j+k)+2 \\ 3(j+k)+2 \\ \vdots \\ 6 \\ 6 \\ \mathbf{4} \\ \mathit{2} \\ \mathit{1} \\ \mathit{1} \end{array} \xrightarrow{1} \begin{array}{c} +2 \\ \langle 3(j+k)+3 \rangle \\ \langle 3(j+k)+3 \rangle \\ \vdots \\ 8 \\ 7 \\ 7 \\ \mathbf{5} \\ \mathit{3} \\ \mathit{2} \\ \mathit{2} \end{array} \to \cdots \\ \\ \to \cdots \end{array}$$

Figure 35: Caption

Playing $[1^3]$ we get to a subtree that is nearly isomorphic to the tree T_{k+1}^{j-1} except that there are two additional plays at the end of each ending branch of the part above the bold part, caused by the bold parts. Specifically, from Figure 35 above, we see that

$$R_{[1^3]}(\beta_k^j) = \begin{array}{c} \tau_k^{j-1} + 5 \\ \mathbf{5} \\ 3 \\ 2 \\ 2 \end{array} = \begin{array}{c} \alpha_k^{j-1} + 4 \\ \mathbf{5} \\ 3 \\ 2 \\ 2 \end{array} \tag{21}$$

Let $\boldsymbol{x}$ denotes the bold last part of β_k^j from the beginning. Let $\mathcal{R} := \{R_\sigma(\beta_k^j) : \sigma = [1^3], \boldsymbol{x} \text{ is not played}\}$. We define a map $\varphi : \mathcal{R} \longrightarrow T_{k+1}^{j-1}$ by

$$\varphi(\lambda) = \varphi\begin{pmatrix} \lambda^1 \\ \boldsymbol{x} \\ \lambda^2 \end{pmatrix} = \begin{cases} \begin{pmatrix} \lambda^1 - 2 \\ \lambda^2 \end{pmatrix} & \text{if } \lambda^1 \neq \emptyset \\ \lambda^2 & \text{otherwise} \end{cases}.$$

Since the difference between two consecutive parts is preserved unless one of them is played, we have that the last part of λ^1 is exactly 4 greater than the first part of λ^2 unless it is played. Together with $\varphi(R_{[1^3]}(\beta_k^j)) = \tau_{k+1}^{j-1}$, φ is a graph isomorphism.

Now we consider the set $\mathcal{S} = B_k^j - \mathcal{R}$, which contains partitions resulting from playing the bold part. The bold part can only be played after one of the two parts immediately above it is played, i.e. before the ending branches of the subtree formed by playing the parts above the bold part decrease. That subtree is easily recognized as A_k^{j-1} from (21).

In particular, if the ending branch is obtained by playing $[1^{\alpha_1}2^{\alpha_2}\ldots s^{\alpha_s}(s-1)\ldots 1]$ then we can play the bold part at either vertex s (if the immediate part above it was played in the previous turn) or vertex $s+1$ (if the part two above it was played). The corresponding playing sequences are $[1^{\alpha_1}2^{\alpha_2}\ldots s^{\alpha_s+1}(s-1)\ldots 1]$ and $[1^{\alpha_1}2^{\alpha_2}\ldots s^{\alpha_s}(s+1)s\ldots 1]$. Similar pattern happens when playing $[12]$. We get to the tree B_k^{j-1} plus one extra playable at top, except there are two additional "maximal" plays at each ending branch of A_k^{j-1}.

If the ending branch is $[1^{\alpha_1}2^{\alpha_2}\ldots s^{\alpha_s}(s-1)\ldots 1]$ and we have one extra play R_s before the sequence declines, then there are 2^{s-1} elements added to the tree. This is because there are 2^{s-1} strictly decreasing sequences σ that can be made from $s-1, s-2, \ldots, 1$, with which $[1^{\alpha_1}2^{\alpha_2}\ldots s^{\alpha_s+1}\sigma]$ is a legal playing sequence in B_k^j. From the argument for the recurrence (18) of A_k^j, we see that the total number of extra elements for A_k^j is 3 times the number of extra elements for A_k^{j-1} if $j > 0$, and 3 times the number of extra elements for A_{k-1}^j if $j = 0$. Since A_1^0 has only one ending branch (by playing $[1^3]$) - thus $3 = 3^1$ extra elements by playing $[1^4]$ or $[1^2 2]$ and $[1^2 21]$, we conclude that the total number of extra elements for A_k^j is 3^{j+k}.

So the total number of additional elements after A_k^{j-1} in the tree B_k^{j-1} is 3^{j+k-1}. Therefore, we obtain the recurrence

$$B_k^j = 3 + (T_{k+1}^{j-1} + 3^{j+k-1}) + 2(B_k^{j-1} + 3^{j+k-1}) = 3 + T_{k+1}^{j-1} + 2B_k^{j-1} + 3^{j+k}.$$

□

5.3.4 The computation

We want to compute B_k^n first:

$$\begin{aligned}
2B_k^n &= 6n + 5\cdot 3^{n+k} - 1 + 4\sum_{t=0}^{n-1} B_{k+t}^{n-1-t} \\
&= 6n + 5\cdot 3^{n+k} - 1 + 2\sum_{t=0}^{n-2}\left(6(n-1-t) + 5\cdot 3^{n+k-1} - 1 + 4\sum_{s=0}^{n-2-t} B_{k+t+s}^{n-2-t-s}\right) + 4B_{k+n-1}^0 \\
&= 6n + 5\cdot 3^{n+k} - 1 + 2\sum_{t=0}^{n-2}\left(6(n-1-t) + 5\cdot 3^{n+k-1} - 1 + 4\sum_{s=t}^{n-2} B_{k+s}^{n-2-s}\right) + 4B_{k+n-1}^0 \\
&= \left[6n + 6\cdot 2\cdot\binom{n}{2}\right] + \left[(5\cdot 3^{n+k} - 1) + (5\cdot 3^{n+k-1} - 1)\cdot 2\cdot\binom{n-1}{1}\right] + \\
&\quad + 4\left[B_{k+n-1}^0 + 2\cdot\binom{n-1}{1}B_{k+n-2}^0\right] + 4\cdot 2\sum_{t=0}^{n-3}(t+1)\cdot B_{k+t}^{n-2-t} \\
&= \ldots \\
&= 6\cdot\sum_{t=0}^{n-1}\binom{n}{t+1}2^t + \sum_{t=0}^{n-1}\binom{n-1}{t}2^t\cdot(5\cdot 3^{n+k-t} - 1) + 4\sum_{t=0}^{n-1}\binom{n-1}{t}2^t\cdot B_{k+n-1-t}^0
\end{aligned}$$

We will show for any $0 \le m \le n-1$ that

$$\begin{aligned}
2B_k^n = 6\cdot\sum_{t=0}^{m}\binom{n}{t+1}2^t + \sum_{t=0}^{m}\binom{n-1}{t}2^t\cdot(5\cdot 3^{n+k-t} - 1) + 4\sum_{t=0}^{m}\binom{n-1}{t}2^t\cdot B_{k+n-1-t}^0 + \\
+2^{m+1}\cdot 2\sum_{t=0}^{n-2-m}\binom{t+m}{m}\cdot B_{k+t}^{n-1-m-t}.
\end{aligned} \tag{22}$$

Base case: The first and the fourth equation showed the base cases for $m = 1$.

Induction: Assume that (22) is true for m, we want to show it for $m+1$:

$$
\begin{aligned}
2\sum_{t=0}^{n-2-m}&\binom{t+m}{m}\cdot B_{k+t}^{n-1-m-t}\\
&=\sum_{t=0}^{n-2-m}\binom{t+m}{m}\left(6(n-1-m-t)+5\cdot 3^{n+k-1-m}-1+4\sum_{s=t}^{n-2-m}B_{k+s}^{n-2-m-s}\right)\\
&=6\sum_{t=m}^{n-2}\binom{t}{m}(n-1-t)+(5\cdot 3^{n+k-1-m}-1)\sum_{t=m}^{n-2}\binom{t}{m}+\\
&\quad+4\cdot B_{n+k-2-m}^{0}\sum_{t=m}^{n-2}\binom{t}{m}+2\cdot 2\sum_{t=0}^{n-3-m}\left(\sum_{s=0}^{t}\binom{s+m}{m}\right)\cdot B_{k+t}^{n-2-m-t}
\end{aligned}
$$

We use some familiar identities:

$$
\begin{aligned}
\sum_{t=m}^{n-2}\binom{t}{m}&=\binom{n-1}{m+1}\\
\sum_{t=m}^{n-2}\binom{t}{m}(n-1-t)&=\binom{n}{m+2}\\
\sum_{s=0}^{t}\binom{s+m}{m}&=\binom{t+m+1}{m+1}
\end{aligned}
$$

to get

$$
\begin{aligned}
2\sum_{t=0}^{n-2-m}\binom{t+m}{m}\cdot B_{k+t}^{n-1-m-t}=6&\binom{n}{m+2}+(5\cdot 3^{n+k-1-m}-1)\binom{n-1}{m+1}+\\
&+4\cdot B_{n+k-2-m}^{0}\binom{n-1}{m+1}+\\
&+2\cdot 2\sum_{t=0}^{n-2-(m+1)}\binom{t+m+1}{m+1}\cdot B_{k+t}^{n-1-(m+1)-t}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
2B_k^n=6\cdot\sum_{t=0}^{m+1}\binom{n}{t+1}2^t+\sum_{t=0}^{m+1}\binom{n-1}{t}2^t\cdot\left(5\cdot 3^{n+k-t}-1\right)+4\sum_{t=0}^{m+1}\binom{n-1}{t}2^t\cdot B_{k+n-1-t}^{0}+\\
+2^{m+2}\cdot 2\sum_{t=0}^{n-2-(m+1)}\binom{t+m+1}{m+1}\cdot B_{k+t}^{n-1-(m+1)-t}.
\end{aligned}
$$

We have proved (22). Let $m = n-1$ we have

$$
2B_k^n=6\cdot\sum_{t=0}^{n-1}\binom{n}{t+1}2^t+\sum_{t=0}^{n-1}\binom{n-1}{t}2^t\cdot\left(5\cdot 3^{n+k-t}-1\right)+4\sum_{t=0}^{n-1}\binom{n-1}{t}2^t\cdot B_{k+n-1-t}^{0}
$$

That gives

$$\begin{aligned}
2B_k^n &= 3 \cdot \sum_{t=0}^{n} \binom{n}{t+1} 2^t - 3 + 15 \cdot 3^k \cdot \sum_{t=0}^{n-1} \binom{n-1}{t} 2^t \cdot 3^{n-1-t} - \sum_{t=0}^{n-1} \binom{n-1}{t} 2^t + \\
&\quad + 2 \left(\sum_{t=0}^{n-1} \binom{n-1}{t} 2^t \cdot (3^{k+n-t} - 1 + 2 \cdot 3^{k+n-1-t}) \right) \\
&= 3 \cdot 3^n - 3 + 5 \cdot 3^{k+1} \cdot 5^{n-1} - 3^{n-1} + 2 \cdot 3^{k+1} \cdot 5^{n-1} - 2 \cdot 3^{n-1} + 4 \cdot 3^k \cdot 5^{n-1} \\
&= 2 \cdot 3^n + 3^k \cdot 5^{n+1} - 3.
\end{aligned}$$

Plugging in the recurrence formula for T_0^n, we get:

$$\begin{aligned}
T_0^n &= 6n + 3^n - 4 + 2 \sum_{t=0}^{n-2} \left(2 \cdot 3^{n-2-t} + 3^t \cdot 5^{n-1-t} - 3\right) \\
&= 6n + 3^n - 4 + 2 \cdot (3^{n-1} - 1) - 6(n-1) + 2 \sum_{t=0}^{n-1} 3^t \cdot 5^{n-1-t} - 2 \cdot 3^{n-1} \\
&= 3^n + 5^n - 3^n \\
&= 5^n.
\end{aligned}$$

The tree T_0^n is the one we want with n blocks WBW or BWW.

5.4 More properties of partitions in $\mathcal{O}_{(BWW)^k}$ and proof of Proposition 3.3.2

Proposition 5.4.1. *The largest part size of a partition in $\mathcal{O}_{(BWW)^k}$ is $9k-5$.*

Proof. Start with the root of τ_0^k, and let $\lambda = R_1(\tau_0^k)$ so $\lambda_0 = 3k$. By playing [2], we play the last part in the first $\begin{matrix}\langle 0\rangle \\ \langle 1\rangle \\ 0\end{matrix}$ block and create a separation of 3 between the top part and the rest, as in the proof of **Proposition 5.1.1**. Now if we play any parts not of index 1, the top part gets 1 added to it. Moreover, playing $[2^3]$ creates a playable block $\begin{pmatrix}0\\0\\1\end{pmatrix}$ at the bottom as shown in **Figure 33**. However, after playing all the original k blocks of $\begin{pmatrix}0\\1\\0\end{pmatrix}$ and reaching the first added playable block $\begin{pmatrix}0\\0\\1\end{pmatrix}$, no new parts are added during the two plays of the last 0 in the former block and the first 0 in the latter block. Thus new parts added after this moment are not playable at any time, because the first added part differs from the part immediately above it by at least 3. Hence, the longest playing sequence must be $[12^{3s}1]$, which takes care of $k-1$ original blocks and $k-1$ added blocks while playing R_2 consecutively. That means we can play at most $3 \cdot 2(k-1)$ consecutive moves R_2. Therefore, the largest part we can get is

$$3k + 1 + 3 \cdot 2(k-1) = 9k - 5.$$ □

Corollary 5.4.2. *Partitions in $\mathcal{O}_{(BWW)^k}$ have at most $9k-5$ parts.*

Proof. Easily deduced by performing one reverse BS operation on elements with the largest part size. □

Proposition 5.4.3. *The number of partitions with $9k-5$ parts in $\mathcal{O}_{(BWW)^k}$ is 2^{k-2}.*

Proof. This is the same as the number of partitions with largest part $9k-5$. Those result from the playing sequences $[12^{1+3(k-1)}]$ that lead to the tree T_{k-1}^0, having 2^{k-2} equal-length ending branches. Each one of the ending branches contributes one partition with largest part $9k-5$, as proven in **Lemma 5.3.3**. □

Part III
A few final remarks

6 Which recurrent partitions are roots of nontrivial trees?

Looking at the BS game graph for n that is not a triangular number, e.g. $n = 8$ in Figure 3, one sees that there are partitions in the recurrent sets of the orbits that are not roots of any trees outside of the cycle. In other words, those elements do not have more than one image under R - the reversed Bulgarian Solitaire operation. This proposition below will count and characterize such partitions[1]:

Proposition 6.0.1. *Let N be a necklace with $|N| = m$ beads of which r are black. Then in the recurrent set $\mathcal{C}_N$ for the reverse Bulgarian Solitaire orbit $\mathcal{O}_N^{\mathrm{op}}$, there will be exactly $2\binom{m-2}{r-1}$ recurrent elements with no images outside of the cycle $\mathcal{C}_N$.*

Proof. First, recall that for a partition λ in the reverse BS system, λ_j is playable if and only if $\lambda_j \geq l(\lambda) - 1$. In particular, if λ_j is playable, then so are all of $\lambda_1, \lambda_2, \ldots, \lambda_j$. The recurrent partitions λ in $\mathcal{C}_N$ therefore must all have λ_1 playable, and have no images outside of the cycle $\mathcal{C}_N$ if and only if λ_2 is not playable.

Elements in cycles in the BS system of $\mathcal{P}(n)$ with size $n = \frac{m(m-1)}{2} + r$ are of the form $\lambda = \Delta_{m-1} + \sigma = ((m-1) + \sigma_{m-1}, \ldots, 1 + \sigma_1, \sigma_0)$ where σ is a binary sequence with sum r. Now λ has exactly one image $R_1(\lambda)$ if and only if one of the two cases below happen:

(i) If $\sigma_0 = 0$, then $l(\lambda) = m-1$, so $\lambda_1 = m-1+\sigma_{m-1} \geq m-2$ and $\lambda_2 = m-2+\sigma_{m-2} \leq m-3$, that is
$$\sigma_{m-1} \geq -1 \text{ and } \sigma_{m-2} \leq -1 \text{ which cannot happen .}$$

(ii) If $\sigma_0 = 1$ then $l(\lambda) = m$, so $\lambda_1 = m-1+\sigma_{m-1} \geq m-1$ and $\lambda_2 = m-2+\sigma_{m-2} \leq m-2$, that is
$$\sigma_{m-1} \geq 0 \text{ and } \sigma_{m-2} \leq 0.$$

Hence, the only possibility for elements in a cycle that have exactly one image R is when

$$\sigma = (\sigma_{m-1}, 0, \sigma_{m-3}, \sigma_{m-4}, \ldots, \sigma_2, \sigma_1, 1).$$

The number of such σ's is $\binom{m-2}{r-1}$, since we can freely choose $\sigma_{m-1}, \sigma_{m-3}, \ldots, \sigma_1$ in $\{0, 1\}$. ☐

7 Relation between finite orbit recurrence and limit generating function

Note that Theorem 3.1.1 gives a suggestive alternate proof of the form of the limiting level size generating function $H_{BW}(x)$ in Theorem 3.2.1, as follows. Since Theorem 3.1.1 asserts that for $k \geq 2$ one has

$$\mathcal{D}_{(BW)^k}(x) = x(3x+1)\mathcal{D}_{(BW)^{k-1}}(x) - x^3\mathcal{D}_{(BW)^{k-2}}(x) + (x-1)^2(3x+2)$$

and since $H_{BW}(x) = \lim_{k\to\infty} \mathcal{D}_{(BW)^k}(x)$, one must have

$$H_{BW}(x) = (3x^2 + x)H_{BW}(x) - x^3 H_{BW}(x) + (x-1)^2(3x+2).$$

Solving thi for $H_{BW}(x)$ gives Theorem 3.2.1.

We similarly expect a recurrence for the level size generating functions of $\mathcal{O}((BWW)^k)$ of the form

$$\mathcal{D}_{(BWW)^k}(x) = p(x)\mathcal{D}_{(BWW)^{k-1}}(x) + q(x)$$

for some polynomials $p(x), q(x)$ satisfying

[1]This proposition is irrelevant for our earlier results, but crucial to the proof of **Theorem 3.2.4**, omitted in this thesis.

- $p(1) = 5, q(1) = 0$ to agree with $\left|\mathcal{O}_{(BWW)^k}\right| = 5^k$ from Theorem 3.1.2, and
- $H_{BWW}(x) = \lim_{k\to\infty} \mathcal{D}_{(BWW)^k}(x) = \frac{q(x)}{1-p(x)}$

However, the rational function expression given in (6)

$$H_{BWW}(x) = (1-x)\frac{x^3 - 3x^2 - 4x - 3}{2x^3 + x^2 - 1}$$

has denominator which when evaluated at $x = 1$ gives 2 rather than ± 4 as we would have expected from $1 - p(x)$. This suggests that $q(x)$ and $1 - p(x)$ share a common factor whose evaluation at $x = 1$ is ± 2.

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