DESSINS D'ENFANTS AND SUPERPOTENTIALS

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ABSTRACT. Grothendieck's theory of dessin d'enfants, which are graphs on surfaces, are combinatorial objects used to study branched covers between Riemann surfaces and the absolute Galois group of the field of rational numbers. The superpotential algebra is an algebraic object attached to bipartite graphs on surfaces, and arises from mathematical physics. Our first goal in this paper is to discuss the relationship between superpotential algebras and dessin d'enfants, in particular the construction of conjectural Galois invariants. A tool is the use of Belyi-extending maps, as defined by Ellenberg and elaborated upon by Wood. The second goal of the paper is to completely understand the superpotential algebra of trees; we give an algorithm to compute their superpotential algebras, as well as providing a catalog for low degrees.

1. Introduction

Although dessins d'enfants have been studied as early as the nineteenth century, it was rediscovered by Grothendieck in the twentieth century and discussed in his ambitious research outline [7]. This research outline indicates a way to understand the absolute Galois group of the rationals via its action on dessins d'enfants, which are certain graphs on surfaces associated to branched covers of Riemann surfaces. A way to do this is to understand properties that are Galois invariant on dessins d'enfants, and since then numerous interesting objects associated to dessins d'enfants have been found to be Galois invariants (see [5, 6, 10, 11] for some examples). The purpose of the present paper is to study superpotential algebras, an algebraic object from the theory of Calabi-Yau algebras, in the framework of the theory of dessins d'enfants. Briefly speaking, the superpotential algebra of a dessin d'enfant is defined via the vertices and edges of its dual graph; the precise definition is given in Section 3.

On the algebraic side of things (Section 4), we provide a method to compute the superpotential algebras of bipartite trees. The method we provide not only tells us that the superpotential algebras of trees must be isomorphic to $\mathbb{Z}[x]/(x^l - 1)$ or $\mathbb{Z}[x, x^{-1}]$, but also tells us how how each edge of the tree, which are used to define the superpotential algebra, corresponds to a power of the indeterminate x. This characterization of superpotential algebra of trees uses various reduction methods to relate trees of different degrees, and the most simple example of such a reduction can be found without proof in [4, Section 2.1].

As for the geometric side of things (Sections 5 and 6), we will discuss superpotential algebras via the language of Galois invariants. On the level of trees, one can define the superpotential rank using l as in the previous paragraph, and we conjecture that the superpotential rank is a Galois invariant of trees. This conjecture can be proved in a special case where our trees satisfies certain conditions (Proposition 5.6). We do not have such a conjecture for general dessins d'enfants. However, one can define a different algebra $B_{\mathcal{D}}$ on a dessin d'enfant $\mathcal{D}$ via its vertices and edges and show that it agrees with the superpotential algebra if we use the clean dual $\mathcal{D}_c^\vee$ instead (the clean dual is the bipartite version of the dual of a graph). Furthermore, this algebra is a Galois invariant on a class of dessins d'enfants that includes trees. The tools to do this are *Belyi-extending maps*, which are rational functions $\mathbb{P}^1 \longrightarrow \mathbb{P}^1$ defined over $\mathbb{Q}$, ramified only over $\{0, 1, \infty\}$, and maps $\{0, 1, \infty\}$ into $\{0, 1, \infty\}$. These maps have the nice property that they not only create new dessin d'enfants from old, but also create new Galois invariants from old. In particular, we can use Belyi-extending maps to show that the clean dual is a Galois invariant, which provides us the gateway to understand superpotential algebras using a different algebra.

2. Dessins d'enfants

In this section we review the basics of dessins d'enfants based on [10]. A *dessin*, or *dessin d'enfant*, is defined to be a bipartite graph (with finitely many black and white vertices) embedded on a Riemann surface, which we will assume throughout to be compact and connected. Suppose a vertex v on a dessin has

n edges. Then the theory of Riemann surfaces tells us that locally at v the angle between two adjacent edges must be $2\pi/n$. Dessins d'enfants are connected to algebraic geometry, specifically algebraic curves, via the following remarkable theorem due to Belyi.

Theorem 2.1 (Belyi, [2]). *A compact Riemann surface S is defined over $\overline{\mathbb{Q}}$ if and only if there is a holomorphic map $f : S \longrightarrow \mathbb{P}^1$ with at most three branched points.*

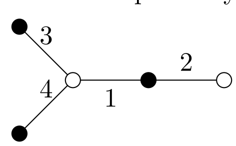
Such a pair (S, f) as in Belyi's theorem is called a *Belyi pair*, and f is called a *Belyi map*. We can assume the branched points of f are $\{0, 1, \infty\}$ up to a Möbius transformation on $\mathbb{P}^1$. By drawing a straight line from 0 to 1, we get a dessin on S by looking at the graph $f^{-1}([0, 1])$. To get the bipartite coloring, the convention is to let the preimages of 0 be colored black, and the preimages of 1 be colored white.

There is a correspondence between Belyi pairs and dessins. To do this we need to define equivalences between dessins and Belyi pairs. Two dessins $\mathcal{D}$ and $\mathcal{D}'$ embedded on a compact Riemann surface S are *equivalent* if there exists an orientation-preserving isomorphism on S that restricts to isomorphisms on $\mathcal{D}$ and $\mathcal{D}'$ as graphs. Two Belyi pairs (S, f) and (S, g) are *equivalent* if there exists an isomorphism $i : S \longrightarrow S$ such that $g = i \circ f$.

Theorem 2.2 ([10, Section 1.5]). *There is a bijection between the set of equivalence classes of dessins and the set of equivalence classes of Belyi pairs.*

This theorem tells us that every dessin determines a unique Belyi pair up to equivalence. Due to this, for the rest of this section we denote a dessin by $(\mathcal{D}, f)$, where f is the Belyi map corresponding to the dessin.

A dessin determines which compact Riemann surface it has to be embedded on, as we now explain. To every dessin with n edges we can associate to it its *constellation*, which is a pair of elements σ_0, σ_1 of the symmetric group $\mathfrak{S}_n$. To get this, first label its edges arbitrarily with $1, \ldots, n$ such that if we walk along an edge from the black to the white vertex we see the labeling at the left side. Next, for every vertex write down the edges adjacent to it counterclockwise to form a cycle. Then σ_0 and σ_1 are the products of all cycles associated to black and white vertices respectively. An example is given for the dessin above.



$$\begin{cases} \sigma_0 = (1\ \ 2) \\ \sigma_1 = (1\ \ 3\ \ 4) \end{cases}$$

Note that it is possible for a dessin to determine more than one constellations by simply changing the numbering of the edges. This corresponds to conjugating the constellations by an element of the symmetric group S_n. In fact, we have the following correspondence between dessins and constellations:

Theorem 2.3 ([10]). *Every dessin determines a unique constellation up to conjugation.*

To see this, we let $\sigma_\infty = \sigma_1^{-1}\sigma_0^{-1}$, where we multiply permutations from left to right. This σ_∞ can be interpreted as follows: add a vertex at the center of each face of a dessin (including unbounded faces), read off the edge labelings enclosed in the face counterclockwise to form a cycle, and multiply all these cycles together. Using this new data, it should be clear why Theorem 2.3 is true. We also take this chance to introduce a terminology: the *monodromy group* associated to the constellation $\{\sigma_0, \sigma_1, \sigma_\infty\}$ is the subgroup generated by them. As we are most interested in trees, it is easily seen that every tree has σ_∞ equal to some n-cycle.

Another remarkable fact about dessins is that, just by knowing its graph, we can tell which compact Riemann surface it has to embed on. This computation is due to the next theorem. Let $\#\sigma$ denote the number of cycles of a permutation σ.

Theorem 2.4 (Riemann-Hurwitz). *Consider a dessin with n edges and constellations $\sigma_0, \sigma_1, \sigma_\infty$. Then the genus g of the compact Riemann surface it embeds into is*

$$g = 1 + \frac{n - (\#\sigma_0 + \#\sigma_1 + \#\sigma_\infty)}{2}.$$

The classification of closed orientable 2-manifolds tells us that compact Riemann surfaces are classified by their genus up to homeomorphism (their complex analytic structures may not be preserved, the canonical example being elliptic curves). For example, genus zero Riemann surfaces are homeomorphic to the Riemann sphere $\mathbb{P}^1$, and in fact isomorphic by the uniformization theorem. As for branched covers from a Riemann surface S to $\mathbb{P}^1$, we can relate it to dessins d'enfants via Riemann's existence theorem.

Theorem 2.5 (Riemann existence)**.** *Let $P = \{p_1, \ldots, p_k\} \in \mathbb{P}^1$, and suppose $\widehat{f} : S \setminus f^{-1}(P) \longrightarrow \mathbb{P}^1 \setminus P$ is a covering map. Then, up to equivalence, there exists a unique branched cover $f : S \longrightarrow \mathbb{P}^1$ that agrees with $\widehat{f}$ in $S \setminus f^{-1}(P)$.*

The Riemann existence theorem tells us that, up to equivalence, the complex structure of the Riemann surface is determined by the dessin. Also, the data obtained from the Riemann-Hurwitz formula is sufficient to tell us the surface the dessin lives in. For example, a simple computation tells us every tree is embedded on the Riemann sphere.

3. The superpotential algebra of a quiver

In this section we define the notion of a superpotential algebra as given in [4]. By a *quiver*, we mean a directed finite graph. To each dessin $\mathcal{D}$, we consider the dual graph $\mathcal{D}^\vee$ of $\mathcal{D}$. This is the graph such that we associate a vertex to every face of $\mathcal{D}$, and connect two vertices if the corresponding faces in $\mathcal{D}$ share a boundary. Turn $\mathcal{D}^\vee$ into a quiver by assigning the arrow orientation to each edge of $\mathcal{D}^\vee$ such that the faces dual to white and black faces are oriented counterclockwise and clockwise respectively. With this we can construct the superpotential algebra of $\mathcal{D}$, following [4]. Let $\mathcal{D}_0$ and $\mathcal{D}_1$ be the set of vertices and edges of $\mathcal{D}^\vee$ respectively, and let $h, t : \mathcal{D}_1 \longrightarrow \mathcal{D}_0$ be the *head* and *tail* functions that assign to each $e \in \mathcal{D}_1$ the vertices corresponding to its head $h(e)$ and tail $t(e)$. The *path algebra* of $\mathcal{D}$ is defined to be

$$\mathbb{Z}[\mathcal{D}] := \mathbb{Z}\langle \mathcal{D}_0, \mathcal{D}_1 \rangle \Big/ \begin{matrix} e_i^2 = e_i \text{ and } e_i e_j = 0 \text{ for } e_i, e_j \in \mathcal{D}_0 \text{ and } i \neq j \\ v = t(v) v h(v) \text{ for } v \in \mathcal{D}_1 \end{matrix}.$$

Note that we can represent each face as a monomial in $\mathbb{Z}[\mathcal{D}]$ up to cyclic permutations by writing the edges of its boundary according to the orientation it is given. For each monomial α in $\mathbb{Z}[\mathcal{D}]$ representing a face, and for each each $v \in \mathcal{D}_1$, we can define the partial derivative $\partial_v \alpha$ by cyclically permuting α until its first element is v, then deleting v from α. If α does not contain a v, we set $\partial_v \alpha$ to be zero. If we also let $\mathcal{D}_2$ be the set of faces of $\mathcal{D}^\vee$, we can define its *superpotential* to be

$$W_{\mathcal{D}_2} := \sum_{f \in \mathcal{D}_2} (-1)^f \partial f,$$

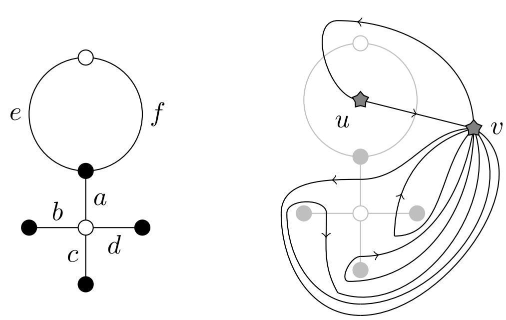
where $(-1)^f$ is either 1 or -1, depending on whether it corresponds to a white or black vertex of $\mathcal{D}$. Its corresponding two-sided ideal is called the *superpotential ideal* and defined as

$$I_{W_{\mathcal{D}_2}} := \langle \partial_v W : v \in \mathcal{D}_1 \rangle.$$

The *superpotential algebra* of $\mathcal{D}$ is then defined to be

$$A_{\mathcal{D}} := \mathbb{Z}[\mathcal{D}]/I_{W_{\mathcal{D}_2}}.$$

Example 3.1. Consider the female symbol $\mathcal{F}$ as the genus zero dessin below, with its dual drawn out.



Also consider the labeling of the vertices and edges of the dual graph $\mathcal{F}^\vee$ as above, where we have labeled the edges of $\mathcal{F}^\vee$ on the edges corresponding to $\mathcal{F}$ for clarity. Then, by definition

$$W_{\mathcal{F}} = abcd + ef - aef - b - c - d,$$
$$I_{W_{\mathcal{F}}} = \begin{pmatrix} bcd - ef, cda - v, dab - v \\ abc - v, f - fa, e - ae \end{pmatrix}.$$

Observe that in $A_{\mathcal{F}}$ we will have the relations $b = c = d$, and $ab = ba$, and $b = efa$. Hence, a computation tells us that

$$A_{\mathcal{F}} \cong \mathbb{Z}[a, e, f, u, v] \Big/ (efef - v, f - fa, e - ae, S)$$
$$\cong \mathbb{Z}[e, f, u, v] \Big/ (efef - v, S),$$

where S is the set of conditions imposed on the path algebra of $\mathcal{F}$.

The example above provides us a geometrical way to construct the superpotential algebra via the dual of the dessin in the genus zero case. For each directed edge e on the dual dessin, with startpoint and endpoint $h(e)$ and $t(e)$, we obtain two paths P_1, P_2 that starts at $t(e)$ and ends at $h(e)$ by tracing out counterclockwise the boundaries of the two regions with e on the boundary. The paths P_1 and P_2 corresponds to two terms in the path algebra, and we set them equal in the construction of the superpotential algebra.

To prepare for the next section, let us observe an easy fact on general associative algebras.

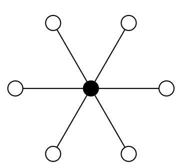
Lemma 3.2. *Let A be an associative algebra, and let $v, c_1, \ldots, c_n \in A$. Suppose $c_{i+1} \cdots c_n v c_1 \cdots c_{i-1} = 1$ for all $i = 1, \ldots, n$. Then $c_i = c$ for all i, $cv = vc$, and $c^{n-1}v = 1$.*

Proof. By assumption $c_2 \cdots c_n v = 1 = c_3 \cdots c_n v c_1$. Therefore $c_1 = c_2$ as left and right inverses are equal if they exist. Iterating gives the result. ☐

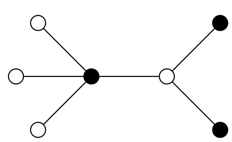
4. The superpotential algebra for trees

For the case of trees, our path algebra has only one nontrivial idempotent as generator. Hence we can suppress its notation. We now show that the superpotential algebra of a tree is always commutative. Actually we will prove something much stronger (see Theorem 4.7).

We start by computing the superpotential algebras for star and double star graphs. Denote Star_n to be the bipartite star graph with n leaves, and denote $\mathrm{DStar}_{n,m}$ to be $\mathrm{Star}_n \sqcup \mathrm{Star}_m$, where the internal vertices of Star_n and Star_m are oppositely colored, and we draw one additional edge connecting the two internal vertices. Note that in principle there are two ways to color the star and double star graphs, but for the purposes of computing its superpotential algebra this does not matter.



Star_6



$\mathrm{DStar}_{3,2}$

Lemma 4.1. *Consider the bipartite star graph* Star_n *with $n \geq 1$ edges labeled $\alpha_1, \ldots, \alpha_n$ respectively.*

(a) $A_{\mathrm{Star}_n} \cong \mathbb{Z}[x]/(x^{n-1} - 1)$.

(b) Consider $\mathrm{DStar}_{n,m}$ *as defined above. Then*

$$A_{\mathrm{DStar}_{n,m}} \cong \begin{cases} \mathbb{Z}[x, x^{-1}] & \text{if } n = m, \\ \mathbb{Z}[x]/(x^{m-n} - 1) & \text{if } n < m. \end{cases}$$

(c) The tree with a unique vertex and no edges has superpotential algebra $\mathbb{Z}$.

Proof. (a) We can assume the internal vertex is colored white and the edges are labeled counterclockwise. Then $W_{\mathrm{Star}_n} = \alpha_1 \cdots \alpha_n - \sum_{i=1}^n \alpha_i$, so

$$I_{W_{\mathrm{Star}_n}} = (\alpha_{i+1} \cdots \alpha_n \alpha_1 \cdots \alpha_{i-1} - 1)_i.$$

Lemma 3.2 then gives us the result immediately.

(b) We can assume the internal vertex of Star_n and Star_m are colored white and black respectively, and the $\alpha_1, \dots, \alpha_n$ and $\beta_1, \dots, \beta_m$ are arranged counterclockwise and clockwise respectively. Let v be the edge connecting Star_n and Star_m to form $\mathrm{DStar}_{n,m}$, such that the leaf-edge counterclockwise to v about the white vertex is α_1, and the leaf-edge clockwise to v about the black vertex is β_1. Then a computation tells us that $W_G = v\alpha_1 \cdots \alpha_n - v\beta_1 \cdots \beta_m - \sum_{i=1}^n \alpha_i + \sum_{j=1}^m \beta_j$, so

$$I_{W_G} = \begin{pmatrix} \alpha_1 \cdots \alpha_n - \beta_1 \cdots \beta_m, \\ \alpha_{i+1} \cdots \alpha_n v \alpha_1 \cdots \alpha_{i-1} - 1, \\ \beta_{j+1} \cdots \beta_m v \beta_1 \cdots \beta_{j-1} - 1 \end{pmatrix}_{i,j},$$

where the subscripts i, j indicate that our relations span over all i and j. Lemma 3.2 tells us that

- $\alpha_i = \alpha$ and $\beta_j = \beta$ for all i, j, implying $\alpha^n = \beta^m$,
- $\alpha v = v\alpha$ and $\beta v = v\beta$,
- $\alpha^{n-1} v = \beta^{m-1} v = 1$.

Notice that $\alpha = \alpha^n v = \beta^m v = \beta$. Hence $A_G \cong \mathbb{Z}[a, v]/(a^{n-1}v - 1)$ if $n = m$. But observe that

$$\mathbb{Z}[a, v]/(a^{n-1}v - 1) \cong \mathbb{Z}[a, a^{-(n-1)}] \cong \mathbb{Z}[a, a^{-1}],$$

as claimed. If $n \neq m$ however, then the equality $\alpha^n = \beta^m = \alpha^m$ gives us more information. Assume $n < m$ without loss of generality. We get $\alpha^{n(m-1)} = \alpha^{m(m-1)}$, implying

$$1 = v^n \alpha^{n(m-1)} = v^n \alpha^{m(m-1)} = \alpha^{(m-n)(m-1)}.$$

Hence $v^{m-n} = v^{m-n}\alpha^{(m-n)(m-1)} = 1$, implying $v = \alpha^{(m-n-1)(m-1)} = \alpha^{(m-n-1)(n-1)}$. Furthermore we have $1 = \alpha^{(m-n)(m-1)} = \alpha^{m^2-m-nm+m} = \alpha^{m(m-n)}$, so that $\alpha^{m-n} = \alpha^{m(m-n)} = 1$. This discussion tells us that

$$A_G \cong \mathbb{Z}[a]/(a^n - a^m, a^{m-n} - 1) = \mathbb{Z}[a]/(a^{m-n} - 1),$$

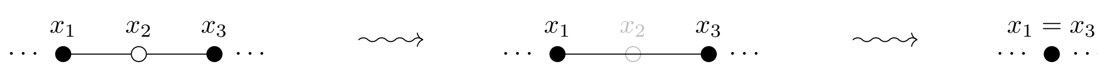
as desired.

(c) This is clear. □

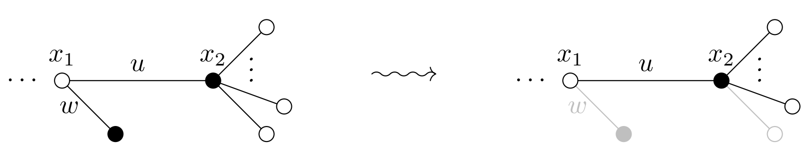
The following lemma gives us the key reduction methods to compute superpotential algebras for trees.

Lemma 4.2. *Let $\mathcal{T}$ be a tree.*

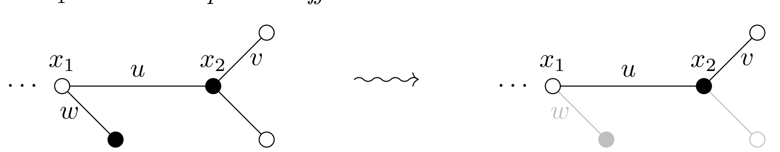
(a) Suppose there exists a consecutive sequence of vertices $x_1x_2x_3$ on $\mathcal{T}$, with $x_1 \neq x_3$, such that x_2 has degree two. Let $\mathcal{T}'$ be the resulting graph after we delete x_2 and contract vertices x_1 and x_3 along uw, where $u = x_1x_2$ and $w = x_2x_3$. Then $A_{\mathcal{T}} \cong A_{\mathcal{T}'}$.



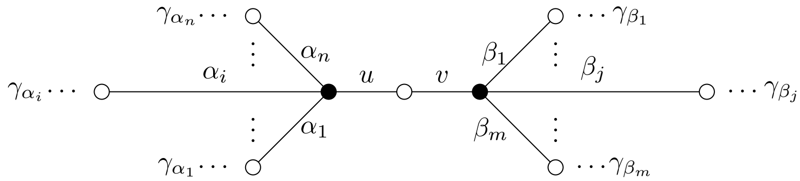
(b) Let $u = x_1x_2$ be an edge of $\mathcal{T}$ such that x_2 is adjacent to $m \geq 3$ leaves and x_1 (and nothing else). Suppose there is a leaf w adjacent to x_1 such that it is clockwise of u about x_1. Then $A_{\mathcal{T}} \cong A_{\mathcal{T}''}$, where $\mathcal{T}''$ is the tree obtained by pruning off w and one of the m leaves adjacent to x_2.



(c) Let $u = x_1x_2$ be an edge of $\mathcal{T}$ such that x_2 is adjacent to two leaves and x_1 (and nothing else). Suppose there is a leaf w adjacent to x_1 such that it is clockwise of u about x_1. Then $A_{\mathcal{T}} \cong A_{\mathcal{T}'''}[v^{-1}]$, where $\mathcal{T}'''$ is the tree obtained by pruning off w and one of the m leaves adjacent to x_2, and v is the other leaf adjacent to x_1 that is not pruned off.



Proof. Let us start by proving part (a). Without loss of generality we let x_1 and x_3 both be black vertices. Let $u = \alpha_0, \alpha_1, \ldots, \alpha_n$ be a clockwise orientation of edges adjacent to x_1, and similarly let $v = \beta_0, \beta_1, \ldots, \beta_m$ be a clockwise orientation of edges adjacent to x_3. Furthermore, let γ_{α_i} (resp. γ_{β_j}) be the edges adjacent to α_i (resp. β_j) at the other vertex that is not x_1 (resp. x_3) in arbitrary order. Since $\mathcal{T}$ is assumed to be a tree, this means that these γ_x's intersect trivially pairwise. A local picture about uv is shown below.



A computation of its superpotential tells us that

$$W_{\mathcal{T}} = vu + \sum_{i=1}^{n} \alpha_i \gamma_{\alpha_i} + \sum_{j=1}^{m} \beta_j \gamma_{\beta_j} - u\alpha - v\beta + \cdots,$$

where we write $\alpha = \alpha_1 \cdots \alpha_n$ and $\beta = \beta_1 \cdots \beta_m$. Therefore

$$I_{W_{\mathcal{T}}} = \begin{pmatrix} v - \alpha, u - \beta, \\ \alpha_{i+1} \cdots \alpha_n u \alpha_1 \cdots \alpha_{i-1} - \gamma_{\alpha_i}, \\ \beta_{j+1} \cdots \beta_m v \beta_1 \cdots \beta_{j-1} - \gamma_{\beta_j}, \\ \cdots \end{pmatrix}_{i,j}.$$

In the above, the subscripts i, j means that the relations written out spans over all i and j just as before. We have also used ellipses above to denote relations that will not play a role in our argument. That is, the ellipses indicate relations that defines the superpotential ideal of both $I_{W_{\mathcal{T}}}$ and $I_{W_{\mathcal{T}'}}$ (described below). We will keep this notation throughout the course of the proofs of the lemmas in this section.

Now consider $\mathcal{T}'$, which is the result of contracting edges uv and identify x_1 with x_3. Then,

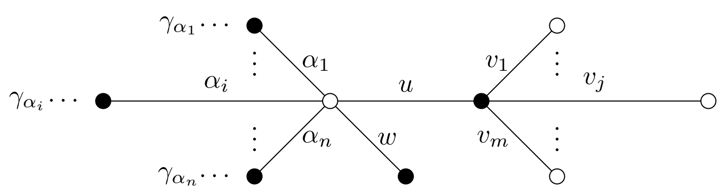
$$W_{\mathcal{T}'} = \sum_{i=1}^{n} \alpha_i \gamma_{\alpha_i} + \sum_{j=1}^{m} \beta_j \gamma_{\beta_j} - \alpha\beta + \cdots,$$

so that

$$I_{W_{\mathcal{T}'}} = \begin{pmatrix} \alpha_{i+1} \cdots \alpha_n \beta \alpha_1 \cdots \alpha_{i-1} - \gamma_{\alpha_i}, \\ \beta_{j+1} \cdots \beta_m \alpha \beta_1 \cdots \beta_{j-1} - \gamma_{\beta_j}, \\ \cdots \end{pmatrix}_{i,j}.$$

As in the previous part we can effectively delete the variables u and v from $A_{\mathcal{T}}$ since $u = \alpha$ and $v = \beta$. This is isomorphic to $A_{\mathcal{T}'}$ as all the other unwritten out relations in $I_{W_{\mathcal{T}}}$ do not depend on u and v, hence the corresponding relations remain unchanged in $I_{W_{\mathcal{T}'}}$.

We prove parts (b) and (c) simultaneously. Without loss of generality, let x_2 be a black vertex. Let $u = v_0, v_1, \ldots, v_m$ be the clockwise orientation of edges adjacent to x_2, and let $u = \alpha_0, \alpha_1, \ldots, \alpha_n, w$ be the counterclockwise orientation of edges adjacent to x_1. Also let γ_i be the edges adjacent to each α_i at the other vertex that is not x_1 in arbitrary order. As before the γ_i's intersect trivially pairwise, and a local picture about u is shown below.



A computation of its superpotential tells us that

$$W_{\mathcal{T}} = \sum_{i=1}^{m} v_i + u\alpha w - uv' - w - \sum_{j=1^n} \alpha_j \gamma_j + \cdots,$$

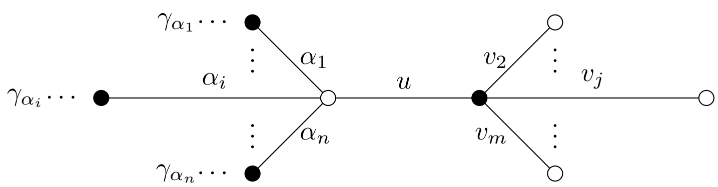
where $\alpha = \alpha_1 \cdots \alpha_n$ and $v' = v_1 \cdots v_m$. Therefore

$$I_{W_{\mathcal{T}}} = \begin{pmatrix} \alpha w - v',\ u\alpha - 1, \\ 1 - v_{i+1} \cdots v_m u v_1 \cdots v_{i-1}, \\ \alpha_{j+1} \cdots \alpha_n w u \alpha_1 \cdots \alpha_{j-1} - \gamma_j, \\ \cdots \end{pmatrix}_{i,j}.$$

Lemma 3.2 then implies $v_i = v$ for all $i = 1, \ldots, m$, $uv = vu$, and $v^{m-1}u = 1$. Together with $\alpha w = v' = v^m$ and $u\alpha = 1$, this means $v = uv^m = u\alpha w = w$. Also, $\alpha - v^{m-1} = (\alpha v - v^m)v^{m-2}u + (\alpha - v^{m-1})(1 - v^{m-1}u)$ and $\alpha v - v^m = (\alpha - v^{m-1})v$. Using this, we observe that

$$\begin{aligned} A_{\mathcal{T}} &= \mathbb{Z}\langle u, v, \alpha_1, \ldots, \alpha_n, \cdots \rangle \Big/ \begin{pmatrix} \alpha v - v^m,\ u\alpha - 1, \\ uv - vu,\ 1 - v^{m-1}u, \\ \alpha_{j+1} \cdots \alpha_n v u \alpha_1 \cdots \alpha_{j-1} - \gamma_j, \\ \cdots \end{pmatrix}_j \\ &= \mathbb{Z}\langle u, v, \alpha_1, \ldots, \alpha_n, \cdots \rangle \Big/ \begin{pmatrix} \alpha - v^{m-1},\ u\alpha - 1, \\ uv - vu, 1 - v^{m-1}u, \\ \alpha_{j+1} \cdots \alpha_n v u \alpha_1 \cdots \alpha_{j-1} - \gamma_j, \\ \cdots \end{pmatrix}_j \\ &= \mathbb{Z}\langle u, v, \alpha_1, \ldots, \alpha_n, \cdots \rangle \Big/ \begin{pmatrix} \alpha - v^{m-1},\ uv - vu,\ 1 - v^{m-1}u, \\ \alpha_{j+1} \cdots \alpha_n v u \alpha_1 \cdots \alpha_{j-1} - \gamma_j, \\ \cdots \end{pmatrix}_j. \end{aligned}$$

If we were to draw $\mathcal{T}'''$, a local picture about u is shown below.



In this case a computation of its superpotential tells us that

$$W_{\mathcal{T}'''} = \sum_{i=2}^{m} v_i + u\alpha - uv_2 \cdots v_m - \sum_{j=1^n} \alpha_j \gamma_j + \cdots,$$

where $\alpha = \alpha_1 \cdots \alpha_n$. Therefore

$$I_{W_{\mathcal{T}'''}} = \begin{pmatrix} \alpha - v_2 \cdots v_m, \\ 1 - v_{i+1} \cdots v_m u v_2 \cdots v_{i-1}, \\ \alpha_{j+1} \cdots \alpha_n u \alpha_1 \cdots \alpha_{j-1} - \gamma_j, \\ \cdots \end{pmatrix}_{i,j}.$$

The same argument as before gives us $A_{\mathcal{T}'''} = \mathbb{Z}\langle u, v, \alpha_1, \ldots, \alpha_n, \cdots \rangle / K$, where

$$K = \begin{pmatrix} \alpha - v^{m-1},\ uv - vu,\ 1 - v^{m-2}u, \\ \alpha_{j+1} \cdots \alpha_n u \alpha_1 \cdots \alpha_{j-1} - \gamma_j, \\ \cdots \end{pmatrix}_j.$$

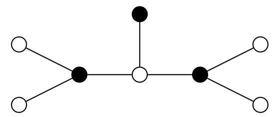
If $m \geq 3$, define a map $\varphi : \mathbb{Z}\langle u, v, \alpha_1, \ldots, \alpha_n, \cdots \rangle \longrightarrow A_{\mathcal{T}}$ by sending $u \mapsto uv$, and sending all other generators to itself. This map is surjective since $\varphi(v^{m-3}u^2) = v^{m-1}u^2 = u$, and $\ker \varphi = K$. Hence $A_{\mathcal{T}'''} \cong A_{\mathcal{T}}$ as well by the first isomorphism theorem. On the other hand, if $m = 2$, an inspection tells us that

$$\begin{aligned} A_{\mathcal{T}} &= \mathbb{Z}\langle u, v, \alpha_1, \ldots, \alpha_n, \cdots \rangle \Big/ \begin{pmatrix} \alpha - v,\ uv - vu,\ 1 - vu, \\ \alpha_{j+1} \cdots \alpha_n \alpha_1 \cdots \alpha_{j-1} - \gamma_j, \\ \cdots \end{pmatrix}_j, \\ A_{\mathcal{T}'''} &= \mathbb{Z}\langle v, \alpha_1, \ldots, \alpha_n, \cdots \rangle \Big/ \begin{pmatrix} \alpha - v, \\ \alpha_{j+1} \cdots \alpha_n \alpha_1 \cdots \alpha_{j-1} - \gamma_j, \\ \cdots \end{pmatrix}_j, \end{aligned}$$

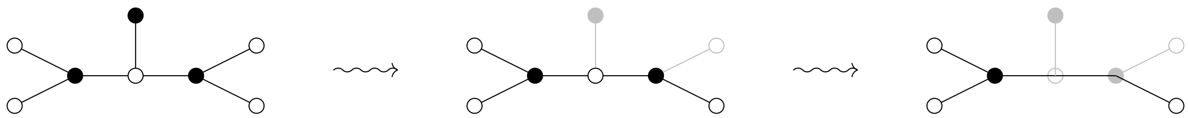
and so $A_{\mathcal{T}} \cong A_{\mathcal{T}'''}[v^{-1}]$. □

Notice that the reduction methods of Lemma 4.2 agrees with the computations of the superpotential algebra of the double star in Lemma 4.1. In fact, the computation for the double star trees $\mathrm{DStar}_{n,m}$ follows immediately from that of the star trees Star_k.

Example 4.3. Consider the following bipartite tree.



Its superpotential algebra is isomorphic to $\mathbb{Z}[x, x^{-1}]/(x^2 - 1) \cong \mathbb{Z}[x]/(x^2 - 1)$ via the following diagrammatic reduction and applying the previous two lemma.

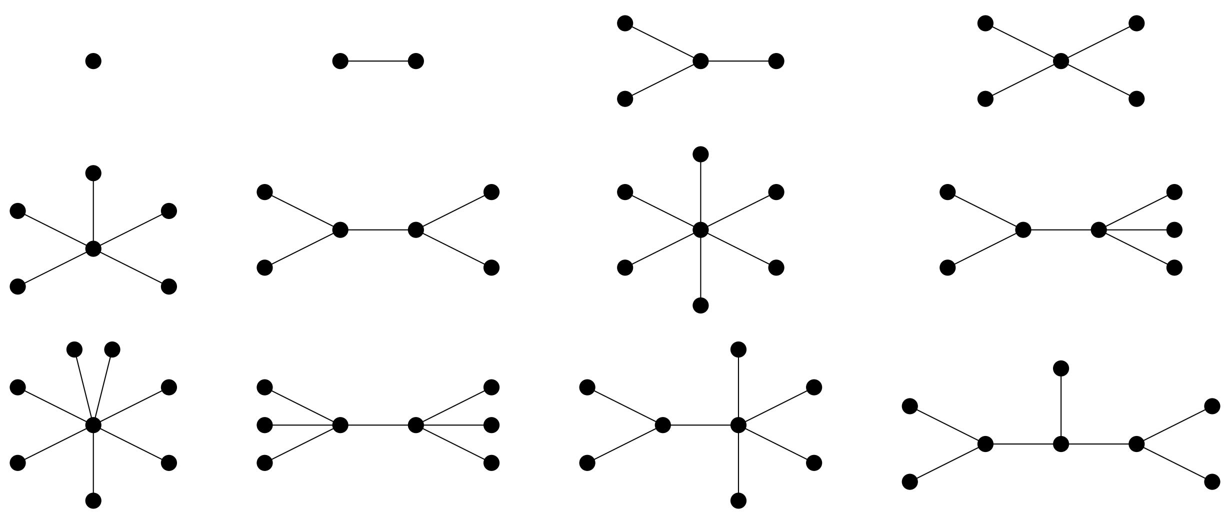


The same answer will be achieved if we directly use the definition of the superpotential algebra in Section 3.

Due to Lemma 4.2, for the purposes of computing the superpotential algebra of tree it suffices to consider a stronger version of homeomorphically irreducible trees (which are trees with non-leaf vertices having degrees at least three). Say that a tree $\mathcal{T}$ is *irreducible* if none of the reduction steps of Lemma 4.2 applies to $\mathcal{T}$.

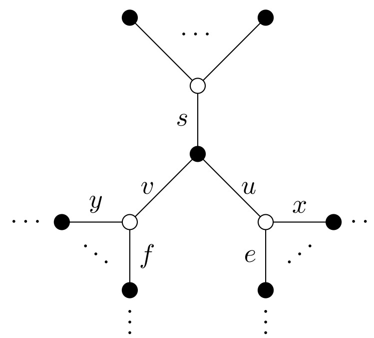
Definition 4.4. A non-leaf edge e of an irreducible tree $\mathcal{T}$ is a *stem* of $\mathcal{T}$ if $\mathcal{T} \setminus \{e\}$ is the disjoint union of a tree and a star graph. An edge e' of $\mathcal{T}$ is an *internal edge* if, for any stem $e \in \mathcal{T}$, the edge e' is not on the star graph component of $\mathcal{T} \setminus \{e\}$.

Note that a stem is an internal edge by definition. For any (not necessarily bipartite) irreducible tree $\mathcal{T}$, we define its *internal order* $o(\mathcal{T})$ to be the number of internal edges of $\mathcal{T}$. This index is necessarily a non-negative integer by definition of an irreducible tree. If $o(\mathcal{T}) = 0$, then it is a star graph, and if $o(\mathcal{T}) = 1$, then it is isomorphic to the graph $\mathrm{DStar}_{n,m}$. Observe that $o(\mathcal{T}) \neq 2$ due to Lemma 4.2, and by the same lemma trees with internal order three must have the three internal edges connected to a vertex of degree exactly three. Below is the list of all possible irreducible trees with internal order at most seven, with none of the leaves drawn out.



Lemma 4.5. *In an irreducible tree $\mathcal{T}$ with internal order $o(\mathcal{T}) \geq 3$, there exists two stems s_1 and s_2 that share a vertex.*

Proof. The cases for $3 \leq o(\mathcal{T}) \leq 7$ is clear by the enumeration above. As for $o(\mathcal{T}) \geq 8$, we prove by contradiction. Suppose not. Then the edges clockwise and counterclockwise to every stem must not be leaves or stems. If there is a stem s that is connected to a vertex of degree at least four, then we can delete s, together with all its leaves, to get an irreducible tree of one less internal order with no stems sharing a vertex. This is a contradiction by induction on $o(\mathcal{T})$, since the claimed property holds for $o(\mathcal{T}) = 7$. Thus we can assume s is connected to a vertex of degree exactly three. A local picture around s in this case is shown below, with all leaves omitted.



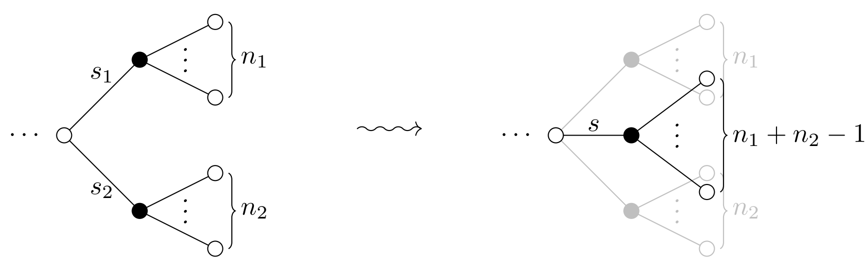
We now have three cases to consider, depending on whether e, f, x, y are stems or not.

Case 1: Suppose one of x, y is not a stem, and one of e, f is not a stem. Then we can delete s together with all its leaves, followed by contracting u and v, to get an irreducible tree of three less internal order. This is a contradiction by induction.

Case 2: Suppose one of x, y is not a stem, and both e, f are stems. Say x is not a stem. Delete s together with all its leaves. Then delete e and sufficiently many stems to the right of it until we have a non-stem (possible since x is not a stem. Then, as before, get a contradiction by contracting u and v.

Case 3: Suppose x, y are stems. Then, after deleting s together with all its leaves, the procedure to achieve a contradiction is exactly the same as the two cases above. □

Lemma 4.6. *Let $\mathcal{T}$ be a tree with two stems s_1 and s_2 that shares a vertex. Suppose s_1 and s_2 are adjacent to n_1 and n_2 number of leaves respectively, where $n_1, n_2 \geq 2$. Let $\mathcal{T}'$ be the resulting graph obtained from $\mathcal{T}$ by deleting s_1, s_2 and all the leaves adjacent to it, and then drawing a stem s with $n_1 + n_2 - 1$ leaves as shown below.*



Then all the leaves adjacent to s in $\mathcal{T}'$ have the same image w in $A_{\mathcal{T}'}$, and $A_{\mathcal{T}}$ and $A_{\mathcal{T}'}$ are related to each other by

$$A_{\mathcal{T}} = A_{\mathcal{T}'}\langle s_1, s_2\rangle \Big/ \begin{pmatrix} s - s_1 s_2, \\ s_1 w - w s_1, s_2 w - w s_2, \\ s_1 w^{n_1 - 1} - 1, s_2 w^{n_2 - 1} - 1 \end{pmatrix}.$$

In particular, $A_{\mathcal{T}}$ is a quotient of $A_{\mathcal{T}'}$.

Proof. To see $A_{\mathcal{T}}$ related to $A_{\mathcal{T}'}$ as described above, let $\epsilon_1, \ldots, \epsilon_l$ be the edges adjacent to the vertex shared by s_1 and s_2 labeled in counterclockwise order (where we assumed this vertex is white). By the same argument as the previous lemmas, the variables corresponding to the leaves of s_1 and s_2 are all equal in the superpotential algebra, and we set them equal to w. Then

$$I_{W_{\mathcal{T}}} = \begin{pmatrix} s_1 w^{n_1 - 1} - 1,\ s_1 w - w s_1,\ s_2 w^{n_2 - 1} - 1,\ s_2 w - w s_2, \\ s_2 \epsilon_1 \cdots \epsilon_l - w^{n_1}, \\ \epsilon_1 \cdots \epsilon_l s_1 - w^{n_2}, \\ \epsilon_{i+1} \cdots \epsilon_l s_1 s_2 \epsilon_1 \cdots \epsilon_{i-1} + \gamma_i, \\ \cdots \end{pmatrix}_i,$$

where γ_i correspond to the collection of edges adjacent to ϵ_i. Also,

$$I_{W_{\mathcal{T}'}} = \begin{pmatrix} s w^{n_1 + n_2 - 2} - 1, \\ \epsilon_1 \cdots \epsilon_l - w^{n_1 + n_2 - 1}, \\ \epsilon_{i+1} \cdots \epsilon_l s \epsilon_1 \cdots \epsilon_{i-1} + \gamma_i, \\ \cdots \end{pmatrix}_i.$$

We see at once that $A_{\mathcal{T}}$ is the quotient of $A_{\mathcal{T}'}$, subject to the additional relations $s - s_1 s_2$, $s_1 w^{n_1-1} - 1$, $s_1 w - w s_1$, $s_2 w^{n_2-1} - 1$, and $s_2 w - w s_2$. □

We can now prove the following theorem. The theorem not only characterizes the superpotential algebras for trees, but also tells us how to compute them.

Theorem 4.7. *The superpotential algebra $A_{\mathcal{T}}$ for a tree $\mathcal{T}$ is isomorphic to $\mathbb{Z}[x, x^{-1}]$ or $\mathbb{Z}[x]/(x^l - 1)$ for some non-negative integer l. Furthermore,*

- *every edge z is identified with $x^{m(z)}$ in $A_{\mathcal{T}}$ for some integer $m(z)$,*
- *leaves are identified with x in $A_{\mathcal{T}}$,*
- *every stem s is identified with x^{n_s-1} or x^{1-n_s} if s is connected to n_s leaves.*

Proof. It suffices to assume $\mathcal{T}$ is an irreducible tree. We would like to do induction on the internal order $o(\mathcal{T})$, under the hypothesis that, in $A_{\mathcal{T}}$,

- leaves of $\mathcal{T}$ correspond to some indeterminate x,
- every non-leaf edge e of $\mathcal{T}$ is subject to a relation $e = x^k$ or $ex^k = 1$ for some positive integer k.

Notice these two hypothesis holds in case $o(\mathcal{T}) \leq 3$. For the inductive step, Lemma 4.5 tells us that in an irreducible tree of internal order at least three there exists two stems s_1 and s_2 that shares a vertex, and Lemma 4.6 tells us that $A_{\mathcal{T}}$ is a certain quotient of $A_{\mathcal{T}'}$ that satisfies the two conditions in the hypothesis above. This prove the theorem. □

Notice that, in principle, the reduction steps of lemma 4.2 might yield $\mathbb{Z}[x, x^{-1}]/(x^l - 1)$, but this algebra is isomorphic to $\mathbb{Z}[x]/(x^l - 1)$. Also, due to the above theorem, we can define the following notion.

Definition 4.8. For a tree $\mathcal{T}$, define its *superpotential rank* $l(\mathcal{T})$ to be

$$l(\mathcal{T}) = \begin{cases} l & \text{if } A_{\mathcal{T}} \cong \mathbb{Z}[x]/(x^l - 1), \\ \infty & \text{if } A_{\mathcal{T}} \cong \mathbb{Z}[x] \text{ or if } A_{\mathcal{T}} \cong \mathbb{Z}[x, x^{-1}]. \end{cases}$$

The superpotential rank is an indication of the number of leaves used to define the superpotential of a tree after reducing it to a star or double star using the various reduction lemmas in this section. Another interpretation of the superpotential rank is given in the next section. We provide a catalog of the computations of superpotential algebras for low-degree trees in Section 7.

5. Galois invariants

A main problem in the theory of dessin d'enfants is to study the action under the absolute Galois group $\Gamma := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Such an action exists by Belyi's theorem. In genus zero, the uniformization theorem implies that all compact Riemann surfaces of genus zero is isomorphic to $\mathbb{P}^1$, and since Belyi maps $\mathbb{P}^1 \longrightarrow \mathbb{P}^1$ are given by rational functions in $\overline{\mathbb{Q}}(x)$, the action of Γ is given by acting on the coefficients of its Belyi map. There is a remarkable result of this Galois action.

Theorem 5.1 ([11])**.** *The Γ-action is faithful on the set of all dessins.*

A natural question is to ask if there are any properties or assignments to dessins that are invariant under the Galois action (up to equivalence of course). The following theorem summarizes some well-known invariants of dessins under the action of Γ.

Theorem 5.2 ([6])**.** *The following properties of a dessin d'enfant remains invariant under the Galois action:*

(a) the number of edges, white vertices, black vertices, and faces of $\mathcal{D}$,
(b) the degree of white vertices, black vertices, and faces of $\mathcal{D}$,
(c) the constellation associated to $\mathcal{D}$ (up to conjugation),
(d) the monodromy group associated to the constellation of $\mathcal{D}$,
(e) the genus of $\mathcal{D}$.

Note that, in the above theorem, the degree of a face F is defined to be half the count of edges on the boundary of F as we walk along its boundary counterclockwise (this is generally not equal to half the number of edges on the boundary of F). With this, we can state an easy observation.

Proposition 5.3. *The internal order is a Galois invariant on trees.*

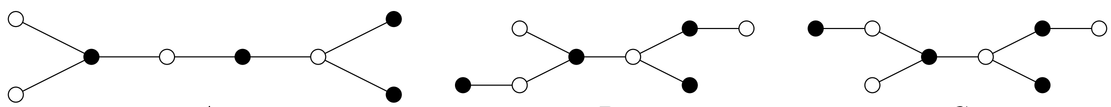
Proof. This follows from Theorem 5.2, since trees in the same Galois orbit have equal number of vertices of a certain degree, in particular of degree one. □

Motivated by the computations of the previous section, we are led to the following conjecture.

Conjecture 5.4. The superpotential rank is a Galois invariant on trees.

This conjecture is supported by the catalog of superpotential algebras for Galois conjugates in Section 7. One might ask for the entire superpotential algebra to be a Galois invariant on trees up to $\mathbb{Z}$-algebra isomorphism. Unfortunately this is not the case by the example below.

Example 5.5. Consider the three trees below.



A B C

It is easily seen, using the reduction methods described in Lemma 4.2 and the computations of Lemma 4.1, that the superpotential algebra of tree A is isomorphic to $\mathbb{Z}[x, x^{-1}]$, while trees B and C have superpotential algebra isomorphic to $\mathbb{Z}[x]$. However, tree C is in a different Galois orbit from the first two trees by the classification of [3]. This can be easily deduced between trees B and C as the order of central rotational symmetry is a Galois invariant (see [10, section 2.4]). The fact that trees A and C are not in the same Galois orbit cannot be told apart using the well-known Galois invariants of Theorem 5.2, not even by their monodromy groups! We can deduce this by passing over to the monodromy groups of their associated clean dessins, which have nonequal cardinalities $2 \cdot 7!$ and $(7!)^2$ (see section 6 for the definition of a clean dessin).

The validity of Conjecture 5.4 up to the catalog of trees available in the literature is somewhat surprising. The difficulty in the conjecture lies in the fact that applying the reduction lemmas of the previous section to two Galois conjugate trees do not proceed in the same fashion, and so it is not possible to keep track of their associated irreducible trees. However, if the conjecture is true, then it tells us that after reducing two Galois conjugate trees the resulting star or double star are actually related in terms of the number of edges.

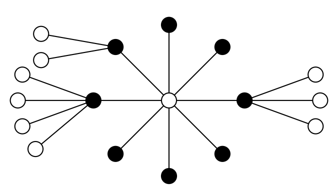
We can prove one case of the conjecture. Let us introduce the following notation. For a tree of degree n, we will write $\langle b_1, \ldots, b_n; w_1, \ldots, w_m \rangle$, where $b_1, \ldots, b_n$ enumerate the degrees of the black vertices, and $w_1, \ldots, w_m$ enumerate the degrees of the white vertices.

Proposition 5.6. *Suppose we have two Galois conjugate trees $\mathcal{T}_1, \mathcal{T}_2$ of datum $\langle b_1, \ldots, b_n; w_1, \ldots, w_m \rangle$ satisfying the following two conditions:*

- *none of $b_1, \ldots, b_n$ equals 2,*
- *exactly one of $w_1, \ldots, w_m$ is greater than 2.*

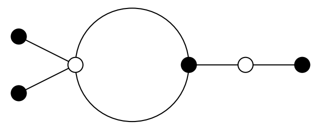
Then the superpotential indices of $\mathcal{T}_1$ and $\mathcal{T}_2$ are equal. The claim is still true if we exchange the roles of $b_1, \ldots, b_n$ and $w_1, \ldots, w_m$.

Proof. Suppose $w_1 \geq 3 > w_2 \geq \cdots \geq w_m \geq 1$. Apply part (a) of Lemma 4.2 to reduce $\mathcal{T}_1, \mathcal{T}_2$ to irreducible trees $\mathcal{T}_1', \mathcal{T}_2'$. The irreducible tree we obtained will have all degree one black vertices adjacent to the white vertex of degree w_1. An illustration of such a tree we will obtain is drawn below.



Now apply parts (b) or (c) of Lemma 4.2 to $\mathcal{T}_1', \mathcal{T}_2'$ in order to eliminate all the degree one vertex adjacent to the central white vertex above. Then do the collapsing adjacent stem reduction as given in the proof of Theorem 4.7 and applying part (a) of Lemma 4.2. We get two star graphs with the same number of vertices, proving the proposition. □

One can hope to get a statement of Conjecture 5.4 for non-tree dessins of genus zero, or in the general case. The examples in the literature suggests, as in the case of trees, that Galois-invariant superpotential algebras are essentially isomorphic. For example, the female symbol introduced in Example 3.1 is in a Galois orbit of order two, with the following graph as its only Galois conjugate, and a computation reveals that their superpotential algebras are isomorphic.

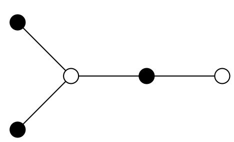


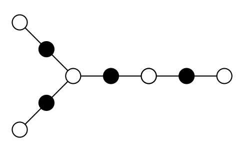
We will not attempt to give such a generalized conjecture here because of the following two difficulties. Firstly, it is computationally very difficult to classify dessin d'enfants in general; for example, after much effort existing results have only classified clean dessins for degrees at most eight (see [1]), which means that we essentially have data up to dessins of degrees at most four, especially from the point of view of understanding their superpotential algebras. Secondly, superpotential algebras behave very differently when there is more than one face in the dessin, so it is hard to predict how they behave in general with the little examples and classification we have.

6. Belyi-extending maps

In this section, we discuss various aspects of the conjecture above using the idea of Belyi-extending maps. In particular, we show how the dual graphs of two Galois conjugate maps are essentially conjugate to each other, and how we can reduce the conjecture to conditions that are always satisfied by trees.

The original theory considered by Grothendieck only dealt with *clean dessins*, i.e. those dessins in which all white vertices have degree 2. The corresponding Belyi map of a clean dessin is called a *clean Belyi map*. This is not less general, because we can turn every dessin $(\mathcal{D}, f)$ into a clean dessin $(\mathcal{D}_c, f_c)$, and furthermore relate its Belyi map to the corresponding clean Belyi map. The transformation from $\mathcal{D}$ to $\mathcal{D}_c$ is as follow: make all vertices in the original dessin white, and to each edge add a black vertex in between corresponding to a preimage of $1/2$. A pictorial example on a tree is given below (left is the original dessin, right is its clean dessin).





Notice $\mathcal{D}$ and $\mathcal{D}_c$ determine the same complex structure on the surface it is embedded on. We now describe the relationship between f and f_c. Notice the critical points of $f_c - 1$ are the union of the solutions of $f = 0$ and $f = 1$, and the ramification indices of each face doubles due to twice more edges on each face boundary. Therefore $f_c = 1 + \alpha f(f-1)$ for some constant α. To determine α, evaluating f_c at any black vertex of $\mathcal{D}_c$ equal 0, while evaluating it there with respect to f yields the value $1/2$. Hence

$$0 = 1 + \alpha \cdot \frac{1}{2} \cdot \left(\frac{1}{2} - 1 \right),$$

implying $\alpha = 4$. We summarize this discussion in the following proposition.

Proposition 6.1. *Let $(\mathcal{D}, f)$ be a dessin, and consider its corresponding clean dessin $\mathcal{D}_c$ described as above. Then f_c relates to f by $f_c = 1 + 4f(f-1)$.*

More generally, we can define a *Belyi-extending map* (due to Ellenberg [5]). This is a rational function $\alpha : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ satisfying the following three conditions:

- α has critical values a subset of $\{0, 1, \infty\}$,
- α is defined over the rationals,
- $\alpha(\{0, 1, \infty\}) \subset \{0, 1, \infty\}$.

The point of Belyi-extending maps is to study how the dessin of $\alpha \circ f$ relates to the dessin of f. In particular, it is a useful technique to produce, and deduce, Galois invariant properties of dessins via post-composing with a suitable Belyi-extending map.

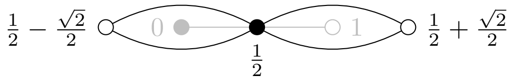
Example 6.2. Let us now explain via example how to construct the dessin after post-composition with a Belyi-extending map, using

$$\alpha(x) = \frac{(x-1/2)^4}{x(x-1)}.$$

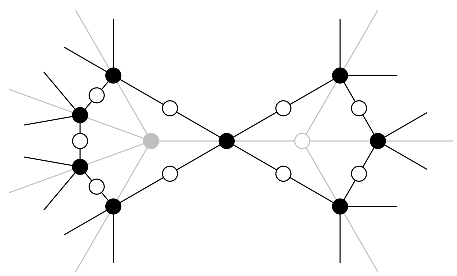
A computation tells us that the critical points of this map are:

- the two solutions $\frac{1}{2} \pm \frac{\sqrt{2}}{2}$ to the polynomial $2x^2 - 2x - \frac{1}{2}$, which are of order two and maps to 1 under α,
- the point $\frac{1}{2}$, which is of order four and maps to 0 under α,
- the points $0, 1$, which are of order one and maps to ∞ under α,
- the point ∞, which is of order two and maps to ∞ under α

Hence α is a Belyi-extending map, and the dessin corresponding to α is as below.



Hence the dessin corresponding to $\alpha \circ f$ is a clean dessin with all black vertices of degree four. Locally it looks like the drawing below.



The above example gives a combinatorial model of how to understand the dessin associated to a Belyi-extending map. More complicated examples, which uses similar ideas as the example given above, can be found in [6] and [12].

Let us now observe that Galois invariants are stable under composition with Belyi-extending maps in the sense described below.

Proposition 6.3. *Suppose P is a property for Belyi maps. For a Belyi-extending map α, define P_α by $P_\alpha(f) = P(\alpha \circ f)$ for any Belyi map $f : S \longrightarrow \mathbb{P}^1$.*

(a) If P is a Galois invariant, then so is P_α.

(b) If P_α is a Galois invariant on $\Gamma \cdot \{\alpha \circ f\}$, then it is a Galois invariant on $\Gamma \cdot \{f\}$.

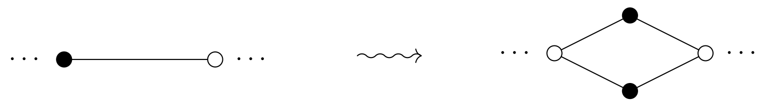
Proof. This is immediate from the observation that $\sigma(\alpha \circ f) = \alpha \circ \sigma(f)$ for any $\sigma \in \Gamma$. □

Hence Belyi-extending maps serves at least two purposes. Firstly, by the proposition above they create new Galois invariants from old by the procedure described in the proposition above. In more detail, for a dessin $\mathcal{D}$ corresponding to the Belyi map f, by post-composing with a Belyi-extending map α we create a new dessin $\mathcal{D}_\alpha$ by looking at the preimage $(\alpha \circ f)^{-1}([0,1])$, and the property P_α is not only a Galois invariant for $\mathcal{D}_\alpha$, but for $\mathcal{D}$ as well. Secondly, if a property P is not a Galois invariant, then it is possible that P_α is a Galois invariant after some choice of α. A trivial example is to choose $\alpha(x) = 1 + 4x(x-1)$ and P to be the superpotential algebra associated to a tree. Then the reduction methods of Lemma 4.2 tells us that P_α is always trivial, hence an uninteresting Galois invariant of trees.

We now list two applications of Belyi-extending maps. One is to reinterpret superpotential algebras via clean duals, and the other is the usage of Belyi-extending maps to a problem related to the theory of dessin d'enfants, called Hurwitz's problem on branched covers of compact Riemann surfaces.

Clean dual maps of dessins. Let us define the *clean dual* $\mathcal{D}_c^\vee$ of a dessin $\mathcal{D}$ by coloring the vertices of its dual graph $\mathcal{D}^\vee$ white, and then adding a black vertex to the midpoint of each edge. On the surface, this definition is clearly motivated by our discussion of clean dessins, and in fact is a clean dessin itself. However, since the dual of a bipartite connected graph is in general not bipartite (though it is Eulerian, i.e. has a path that transverses each edge exactly once), we have made an obvious alteration to carry over the notion of a dual that makes sense in the world of dessin d'enfants.

Example 6.4 (The clean double dual)**.** The clean double dual $(\mathcal{D}_c^\vee)_c^\vee$ of a dessin $\mathcal{D}$ is a thickened version of the original dessin, in the following sense. It replaces every vertex of $\mathcal{D}$ by a white vertex, and to every edge of $\mathcal{D}$ a four-cycle as drawn below.



Note that the clean double dual is still a clean dessin.

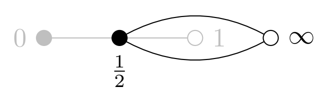
A deeper reason why we defined this notion is because we would like to discuss the Galois-invariant property of dual dessins. In particular, we would like to prove a complementary result to Lando and Zvonkin [10], which states that the property of a map to be self-dual is a Galois invariant.

Theorem 6.5. *Let $\mathcal{D}$ be a dessin, and let $\mathcal{D}_c^\vee$ be its clean dual. Then $\sigma(\mathcal{D})_c^\vee = \sigma(\mathcal{D}_c^\vee)$ for any $\sigma \in \Gamma$. In other words, the clean dual of the conjugate of a dessin is the conjugate of its clean dual.*

Proof. Let $f : S \longrightarrow \mathbb{P}^1$ be the Belyi map for $\mathcal{D}$. We need to find a Belyi-extending map $\alpha : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ such that the dessin for $\alpha \circ f$ is $\mathcal{D}_c^\vee$. Consider the function

$$\alpha(x) = \frac{(x - 1/2)^2}{x(x-1)}.$$

A computation tells us that the critical points are $\{0, 1/2, 1, \infty\}$, with $f(1/2) = 0$ and $f(\infty) = 1$ and $f(0) = f(1) = \infty$. Furthermore, the points $0, 1$ are critical points of order one, while the points $1/2$ and ∞ are of order two. Hence α is a Belyi-extending map, and the dessin corresponding to α is given as below.



Hence the dessin of $\alpha \circ f$ is $\mathcal{D}_c^\vee$, with:

- all the original vertices of $\mathcal{D}$ mapping to ∞ under $\alpha \circ f$,
- all of the black vertices of $\mathcal{D}_c^\vee$ corresponding to the midpoint of the edges of $\mathcal{D}$,
- all of the white vertices of $\mathcal{D}_c^\vee$ corresponding to the faces of $\mathcal{D}$.

This proves the claim. □

The theorem above enables us to calculate an algebra related to the superpotential algebra for genus zero dessins. For a Belyi map f with genus zero dessin $\mathcal{D}$, consider the following algebra $B_{\mathcal{D}}$ constructed via the following steps.

(i) Assign, to each vertex of $\mathcal{D}$, an idempotent v_i, and enumerate the variables as $v_1, \ldots, v_m$.
(ii) Assign, to each edge of $\mathcal{D}$, a variable x_i, and enumerate the variables as $x_1, \ldots, x_n$.
(iii) Let S_1 be the set containing the defining relations for a path algebra, but replacing instances of x_i in each relation by x_i^2.
(iv) Let S_2 be the set containing relations $x_j = x_i$ if the edges corresponding to x_i and x_j are adjacent to the same black vertex.
(v) Let S_3 be the set containing relations described as follow: if b is a black vertex of degree two such that $\mathcal{D} \setminus \{b\}$ is not a disconnected graph, then include into S_3 the relation

$$x_{k_1} \cdots x_{k_{2r}} - x_{l_1} \cdots x_{l_{2s}},$$

where $x_{k_1} \cdots x_{k_{2r}}$ and $x_{l_1} \cdots x_{l_{2s}}$ trace out in counterclockwise direction the boundaries of the two faces with b on the boundary, excluding the two edges adjacent to b.
(vi) Let $P(f)$ be the algebra $\mathbb{Z}[x_1^2, \ldots, x_n^2, v_1, \ldots, v_m]$ subject to the relations $S_1 \cup S_2 \cup S_3$.

Note that the terms used in the fifth step must have an even number of variables since every cycle in a bipartite graph is of even length. Thus, under the relation $S_1 \cup S_2 \cup S_3$, a term $x_{k_1} \cdots x_{k_{2r}}$ used in the fifth step are of the form $x_{k'_1}^2 \cdots x_{k'_r}^2$. This tells us that the algebra constructed in step six makes sense.

Two comments about this algebra $B_{\mathcal{D}}$ are in order. Firstly, the set S_1 is stable under the Galois action, up to relabeling of the edges of the dessin, by the elementary invariants of Theorem 5.2. If a dessin has every degree two black vertex as a bridge, or have no degree two black vertices, then the set S_2 is empty. In particular, $B_{\mathcal{D}}$ is a Galois invariant for trees. Secondly, note that the $B_{\mathcal{D}_c^\vee}$ is the superpotential algebra $A_{\mathcal{D}}$ after one replaces all the variables x_i^2 by x_i. We summarize this discussion as a corollary.

Corollary 6.6. *For a dessin $\mathcal{D}$, let $B_{\mathcal{D}}$ be the associated algebra defined by the above six steps.*
(a) The algebra $B_{\mathcal{D}}$ is a Galois invariant for trees and dessins with no degree two black vertices.
(b) The algebra $B_{\mathcal{D}_c^\vee}$ is the superpotential algebra $A_{\mathcal{D}}$.

Hurwitz's problem on branched covers. Let Σ_g be a compact Riemann surface of genus g, so $\Sigma_0 \cong \mathbb{P}^1$. The theory of Riemann surfaces tells us that every branched cover $f : \Sigma_g \longrightarrow \Sigma_0$ of degree d, with branched points $y_1, \ldots, y_n \in \Sigma_0$, can be associated to a collection of partitions $A_1, \ldots, A_n$ of d, where each A_i is associated to y_i via the rule

$$A_i := [k_z]_{z \in f^{-1}(y)}, \text{ where } k_z \text{ is the local degree of } f \text{ at point } z.$$

This collection of partitions must satisfy the Riemann-Hurwitz formula

$$\sum_{i=1}^{n} (d - \#A_i) = 2d - 2 + 2g.$$

Hurwitz's problem on branched covers of Riemann surfaces asks for the converse of this, i.e. for a collection of partitions $A_1, \ldots, A_n$ satisfying the Riemann-Hurwitz formula for some nonnegative integer g, does it correspond to a branched cover $f : \Sigma_g \longrightarrow \Sigma_0$?

Hurwitz's problem is unsolved in general, and there are vast amounts of literature attacking it in special cases via an extension of the different interpretations of branched covers described in Section 2, but in the general case of more than three branched points (see [10] for an overview of such interpretations). An obvious observation is that the order of the collection of partitions do not matter for Hurwitz's problem (i.e. they define isomorphic covers). The most important case for Hurwitz's problem, however, is when we have exactly three branched points, since almost all the interesting cases reduces to this seemingly trivial case (but in fact is the hardest).

Our aim here is not to discuss Hurwitz's problem in detail. Rather, we see how Belyi-extending maps, which arises from the study of dessin d'enfants, would be useful in Hurwitz's problem. For, if we have a collection of partitions A_1, A_2, A_3 that corresponds to a branched cover $f : \Sigma_g \longrightarrow \Sigma_0$, then by choosing a nontrivial Belyi-extending map $\alpha : \Sigma_0 \longrightarrow \Sigma_0$, we would be able to get another collection of partitions A_1', A_2', A_3' realizing the branched cover $\alpha \circ f : \Sigma_g \longrightarrow \Sigma_0$. However, as with the techniques in this subject, it does not cover all the cases that we would want. In particular, the collection of Belyi-extending maps does not give us all possible collections of partitions that corresponds to a branched cover.

Example 6.7. Our four examples of Belyi-extending maps generates more realizable partitions from one as follows. If A_1, A_2, A_3 corresponds to some branched cover, where $A_i = [a_{i,1}, \ldots, a_{i,n_i}]$, then so does the following collection of partitions:

- $A_1' = A_1 \cup A_2, A_2' = [2, \ldots, 2], A_3' = 2A_3$ via the clean dessin,
- $A_1' = [4, \ldots, 4], A_2' = [2, \ldots, 2], A_3' = A_1 \cup A_2 \cup 2A_3$ via Example 6.2,
- $A_1' = 2A_3, A_2' = [2, \ldots, 2], A_3' = A_1 \cup A_2$ via the clean dual,
- $A_1' = 2A_1 \cup 2A_2, A_2' = [2, \ldots, 2], A_3' = [2a_{3,1}, \ldots, 2a_{3,n_3}, 2, \ldots, 2]$ via the clean double dual.

7. Computations of superpotential algebras for trees of low degrees

In this section, we provide the calculation of the superpotential algebras for trees of degrees at most ten, following the classifications done by Bétréma-Zvonkin [3] and Kochetkov [8, 9]. In fact, we will only do the calculation for trees of degrees between six and ten, since those for degrees at most five are easy to enumerate, and furthermore the Galois orbits are singletons. Recall the following notation: for a tree of degree n, we will write $\langle b_1, \ldots, b_n; w_1, \ldots, w_m \rangle$, where $b_1, \ldots, b_n$ enumerate the degrees of the black vertices, and $w_1, \ldots, w_m$ enumerate the degrees of the white vertices. Our catalog is provided below. We do not draw out the pictures of the relevant trees, but follow the numberings of the relevant papers for easy comparison.

The case of degree six trees. The complete drawing of trees in this case can be found in [3].

(1) Trees of $\langle 6; 1, 1, 1, 1, 1, 1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^5 - 1)$.
(2) Trees of $\langle 5, 1; 2, 1, 1, 1, 1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^3 - 1)$.
(3) Trees of $\langle 4, 2; 2, 1, 1, 1, 1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^3 - 1)$.
(4) Trees of $\langle 3, 3; 2, 1, 1, 1, 1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^3 - 1)$.
(5) Trees of $\langle 4, 1, 1; 3, 1, 1, 1\rangle$ all have superpotential algebra $\mathbb{Z}$.
(6) Trees of $\langle 4, 1, 1; 2, 2, 1, 1\rangle$ all have superpotential algebra $\mathbb{Z}$.
(7) Trees of $\langle 2, 2, 2; 3, 1, 1, 1\rangle$ all have superpotential algebra $\mathbb{Z}$.
(8) Trees of $\langle 2, 2, 2; 2, 2, 1, 1\rangle$ all have superpotential algebra $\mathbb{Z}$.
(9) Trees of $\langle 3, 2, 1; 3, 1, 1, 1\rangle$ all have superpotential algebra $\mathbb{Z}$.
(10) Trees of $\langle 3, 2, 1; 2, 2, 1, 1\rangle$ all have superpotential algebra $\mathbb{Z}$.

The case of degree seven trees. The complete drawing of trees in this case can be found in [3].

(1) Trees of $\langle 7; 1, 1, 1, 1, 1, 1, 1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^6 - 1)$.
(2) Trees of $\langle 6, 1; 2, 1, 1, 1, 1, 1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^4 - 1)$.
(3) Trees of $\langle 5, 2; 2, 1, 1, 1, 1, 1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^4 - 1)$.
(4) Trees of $\langle 4, 3; 2, 1, 1, 1, 1, 1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^4 - 1)$.
(5) Trees of $\langle 5, 1, 1; 3, 1, 1, 1, 1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^2 - 1)$.
(6) Trees of $\langle 5, 1, 1; 2, 2, 1, 1, 1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^2 - 1)$.
(7) Trees of $\langle 4, 2, 1; 3, 1, 1, 1, 1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^2 - 1)$.
(8) Trees of $\langle 4, 2, 1; 2, 2, 1, 1, 1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^2 - 1)$.
(9) Trees of $\langle 3, 3, 1; 3, 1, 1, 1, 1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^2 - 1)$.
(10) Trees of $\langle 3, 3, 1; 2, 2, 1, 1, 1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^2 - 1)$.
(11) Trees of $\langle 3, 2, 2; 3, 1, 1, 1, 1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^2 - 1)$.
(12) Trees of $\langle 3, 2, 2; 2, 2, 1, 1, 1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^2 - 1)$.
(13) Trees of $\langle 4, 1, 1, 1; 4, 1, 1, 1\rangle$ all have superpotential algebra $\mathbb{Z}[x, x^{-1}]$.
(14) Trees of $\langle 4, 1, 1, 1; 3, 2, 1, 1\rangle$ all have superpotential algebra $\mathbb{Z}[x, x^{-1}]$.
(15) Trees of $\langle 4, 1, 1, 1; 2, 2, 2, 1\rangle$ all have superpotential algebra $\mathbb{Z}[x]$.
(16) Trees of $\langle 3, 2, 1, 1; 3, 2, 1, 1\rangle$ have superpotential algebra $\mathbb{Z}[x]$ or $\mathbb{Z}[x, x^{-1}]$.
(17) Trees of $\langle 3, 2, 1, 1; 2, 2, 2, 1\rangle$ all have superpotential algebra $\mathbb{Z}[x]$.
(18) Trees of $\langle 2, 2, 2, 1; 2, 2, 2, 1\rangle$ all have superpotential algebra $\mathbb{Z}[x]$.

The case of degree eight trees. The complete drawing of trees in this case can be found in [3].

(1) Trees of $\langle 8; 1, 1, 1, 1, 1, 1, 1, 1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^7 - 1)$.
(2) Trees of $\langle 7, 1; 2, 1, 1, 1, 1, 1, 1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^5 - 1)$.
(3) Trees of $\langle 6, 2; 2, 1, 1, 1, 1, 1, 1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^5 - 1)$.
(4) Trees of $\langle 5, 3; 2, 1, 1, 1, 1, 1, 1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^5 - 1)$.
(5) Trees of $\langle 4, 4; 2, 1, 1, 1, 1, 1, 1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^5 - 1)$.
(6) Trees of $\langle 6, 1, 1; 3, 1, 1, 1, 1, 1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^3 - 1)$.
(7) Trees of $\langle 6, 1, 1; 2, 2, 1, 1, 1, 1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^3 - 1)$.
(8) Trees of $\langle 5, 2, 1; 3, 1, 1, 1, 1, 1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^3 - 1)$.
(9) Trees of $\langle 4, 3, 1; 3, 1, 1, 1, 1, 1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^3 - 1)$.
(10) Trees of $\langle 5, 2, 1; 2, 2, 1, 1, 1, 1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^3 - 1)$.
(11) Trees of $\langle 4, 3, 1; 2, 2, 1, 1, 1, 1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^3 - 1)$.
(12) Trees of $\langle 4, 2, 2; 3, 1, 1, 1, 1, 1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^3 - 1)$.
(13) Trees of $\langle 4, 2, 2; 2, 2, 1, 1, 1, 1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^3 - 1)$.
(14) Trees of $\langle 3, 3, 2; 3, 1, 1, 1, 1, 1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^3 - 1)$.
(15) Trees of $\langle 3, 3, 2; 2, 2, 1, 1, 1, 1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^3 - 1)$.
(16) Trees of $\langle 5, 1, 1, 1; 4, 1, 1, 1, 1\rangle$ all have superpotential algebra $\mathbb{Z}$.
(17) Trees of $\langle 5, 1, 1, 1; 3, 2, 1, 1, 1\rangle$ all have superpotential algebra $\mathbb{Z}$.
(18) Trees of $\langle 5, 1, 1, 1; 2, 2, 2, 1, 1\rangle$ all have superpotential algebra $\mathbb{Z}$.
(19) Trees of $\langle 4, 2, 1, 1; 4, 1, 1, 1, 1\rangle$ all have superpotential algebra $\mathbb{Z}$.
(20) Trees of $\langle 4, 2, 1, 1; 3, 2, 1, 1, 1\rangle$ all have superpotential algebra $\mathbb{Z}$.
(21) Trees of $\langle 4, 2, 1, 1; 2, 2, 2, 1, 1\rangle$ all have superpotential algebra $\mathbb{Z}$.

(22) Trees of $\langle 3,3,1,1;4,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}$.
(23) Trees of $\langle 3,3,1,1;3,2,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}$.
(24) Trees of $\langle 3,3,1,1;2,2,2,1,1\rangle$ all have superpotential algebra $\mathbb{Z}$.
(25) Trees of $\langle 3,2,2,1;4,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}$.
(26) Trees of $\langle 3,2,2,1;3,2,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}$.
(27) Trees of $\langle 3,2,2,1;2,2,2,1,1\rangle$ all have superpotential algebra $\mathbb{Z}$.
(28) Trees of $\langle 2,2,2,2;4,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}$.
(29) Trees of $\langle 2,2,2,2;3,2,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}$.
(30) Trees of $\langle 2,2,2,2;2,2,2,1,1\rangle$ all have superpotential algebra $\mathbb{Z}$.

The case of degree nine trees. The complete drawing of trees in this case can be found in [8].

(1) Trees of $\langle 9;1,1,1,1,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^8-1)$.
(2) Trees of $\langle 8,1;2,1,1,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^6-1)$.
(3) Trees of $\langle 7,2;2,1,1,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^6-1)$.
(4) Trees of $\langle 7,1,1;3,1,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^4-1)$.
(5) Trees of $\langle 7,1,1;2,2,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^4-1)$.
(6) Trees of $\langle 6,3;2,1,1,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^6-1)$.
(7) Trees of $\langle 6,2,1;3,1,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^4-1)$.
(8) Trees of $\langle 6,2,1;2,2,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^4-1)$.
(9) Trees of $\langle 6,1,1,1;4,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^2-1)$.
(10) Trees of $\langle 6,1,1,1;3,2,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^4-1)$.
(11) Trees of $\langle 6,1,1,1;2,2,2,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}$.
(12) Trees of $\langle 5,4;2,1,1,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^6-1)$.
(13) Trees of $\langle 5,3,1;3,1,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^4-1)$.
(14) Trees of $\langle 5,3,1;2,2,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^4-1)$.
(15) Trees of $\langle 5,2,2;3,1,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^4-1)$.
(16) Trees of $\langle 5,2,2;2,2,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^4-1)$.
(17) Trees of $\langle 5,2,1,1;4,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^2-1)$.
(18) Trees of $\langle 5,2,1,1;3,2,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^2-1)$.
(19) Trees of $\langle 5,2,1,1;2,2,2,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^2-1)$.
(20) Trees of $\langle 5,1,1,1,1;5,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x,x^{-1}]$.
(21) Trees of $\langle 5,1,1,1,1;4,2,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x,x^{-1}]$.
(22) Trees of $\langle 5,1,1,1,1;3,3,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x,x^{-1}]$.
(23) Trees of $\langle 5,1,1,1,1;3,2,2,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x,x^{-1}]$.
(24) Trees of $\langle 5,1,1,1,1;2,2,2,2,1\rangle$ all have superpotential algebra $\mathbb{Z}$.
(25) Trees of $\langle 4,4,1;3,1,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^4-1)$.
(26) Trees of $\langle 4,4,1;2,2,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^4-1)$.
(27) Trees of $\langle 4,3,2;3,1,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^4-1)$.
(28) Trees of $\langle 4,3,2;2,2,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^4-1)$.
(29) Trees of $\langle 4,3,1,1;4,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^2-1)$.
(30) Trees of $\langle 4,3,1,1;3,2,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^2-1)$.
(31) Trees of $\langle 4,3,1,1;2,2,2,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^2-1)$.
(32) Trees of $\langle 4,2,2,1;4,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^2-1)$.
(33) Trees of $\langle 4,2,2,1;3,2,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^2-1)$.
(34) Trees of $\langle 4,2,2,1;2,2,2,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^2-1)$.
(35) Trees of $\langle 4,2,1,1,1;4,2,1,1,1\rangle$ have superpotential algebra $\mathbb{Z}[x]$ or $\mathbb{Z}[x,x^{-1}]$.
(36) Trees of $\langle 4,2,1,1,1;3,3,1,1,1\rangle$ have superpotential algebra $\mathbb{Z}[x]$ or $\mathbb{Z}[x,x^{-1}]$.
(37) Trees of $\langle 4,2,1,1,1;3,2,2,1,1\rangle$ have superpotential algebra $\mathbb{Z}[x]$ or $\mathbb{Z}[x,x^{-1}]$.
(38) Trees of $\langle 4,2,1,1,1;2,2,2,2,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]$.
(39) Trees of $\langle 4,1,1,1,1,1;3,3,2,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^2-1)$.
(40) Trees of $\langle 4,1,1,1,1,1;3,2,2,2\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^2-1)$.
(41) Trees of $\langle 3,3,3;3,1,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^4-1)$.
(42) Trees of $\langle 3,3,3;2,2,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^4-1)$.
(43) Trees of $\langle 3,3,2,1;3,2,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^2-1)$.

(44) Trees of $\langle 3,3,2,1;2,2,2,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^2-1)$.
(45) Trees of $\langle 3,3,1,1,1;3,3,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x,x^{-1}]$.
(46) Trees of $\langle 3,3,1,1,1;3,2,2,1,1\rangle$ have superpotential algebra $\mathbb{Z}[x]$ or $\mathbb{Z}[x,x^{-1}]$.
(47) Trees of $\langle 3,3,1,1,1;2,2,2,2,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]$.
(48) Trees of $\langle 3,2,2,2;3,2,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^2-1)$.
(49) Trees of $\langle 3,2,2,2;2,2,2,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^2-1)$.
(50) Trees of $\langle 3,2,2,1,1;3,2,2,1,1\rangle$ have superpotential algebra $\mathbb{Z}[x]$ or $\mathbb{Z}[x,x^{-1}]$.
(51) Trees of $\langle 3,2,2,1,1;2,2,2,2,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]$.
(52) Trees of $\langle 2,2,2,2,1;2,2,2,2,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]$.

The case of degree ten trees. Partial drawing of trees in this case can be found in [9].

(1) Trees of $\langle 10;1,1,1,1,1,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^9-1)$.
(2) Trees of $\langle 9,1;2,1,1,1,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^7-1)$.
(3) Trees of $\langle 8,2;2,2,1,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^5-1)$.
(4) Trees of $\langle 8,1,1;3,1,1,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^5-1)$.
(5) Trees of $\langle 8,1,1;2,2,1,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^5-1)$.
(6) Trees of $\langle 7,3;2,1,1,1,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^7-1)$.
(7) Trees of $\langle 7,2,1;3,1,1,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^5-1)$.
(8) Trees of $\langle 7,2,1;2,2,1,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^5-1)$.
(9) Trees of $\langle 7,1,1,1;4,1,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^3-1)$.
(10) Trees of $\langle 7,1,1,1;3,2,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^3-1)$.
(11) Trees of $\langle 7,1,1,1;2,2,2,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^3-1)$.
(12) Trees of $\langle 6,4;2,1,1,1,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^7-1)$.
(13) Trees of $\langle 6,3,1;3,1,1,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^7-1)$.
(14) Trees of $\langle 6,3,1;2,2,1,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^5-1)$.
(15) Trees of $\langle 6,2,2;3,1,1,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^5-1)$.
(16) Trees of $\langle 6,2,2;2,2,1,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^5-1)$.
(17) Trees of $\langle 6,2,1,1;4,1,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^3-1)$.
(18) Trees of $\langle 6,2,1,1;3,2,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^3-1)$.
(19) Trees of $\langle 6,2,1,1;2,2,2,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^3-1)$.
(20) Trees of $\langle 6,1,1,1,1;5,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}$.
(21) Trees of $\langle 6,1,1,1,1;4,2,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}$.
(22) Trees of $\langle 6,1,1,1,1;3,3,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}$.
(23) Trees of $\langle 6,1,1,1,1;3,2,2,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}$.
(24) Trees of $\langle 6,1,1,1,1;2,2,2,2,1,1\rangle$ all have superpotential algebra $\mathbb{Z}$.
(25) Trees of $\langle 5,5;2,1,1,1,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^7-1)$.
(26) Trees of $\langle 5,4,1;3,1,1,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^5-1)$.
(27) Trees of $\langle 5,4,1;2,2,1,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^3-1)$.
(28) Trees of $\langle 5,3,2;3,1,1,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^3-1)$.
(29) Trees of $\langle 5,3,2;2,2,1,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^3-1)$.
(30) Trees of $\langle 5,3,1,1;4,1,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^3-1)$.
(31) Trees of $\langle 5,3,1,1;3,2,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^3-1)$.
(32) Trees of $\langle 5,3,1,1;2,2,2,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^3-1)$.
(33) Trees of $\langle 5,2,2,1;4,1,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^3-1)$.
(34) Trees of $\langle 5,2,2,1;3,2,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^3-1)$.
(35) Trees of $\langle 5,2,2,1;2,2,2,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^3-1)$.
(36) Trees of $\langle 5,2,1,1,1;5,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}$.
(37) Trees of $\langle 5,2,1,1,1;4,2,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}$.
(38) Trees of $\langle 5,2,1,1,1;3,3,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}$.
(39) Trees of $\langle 5,2,1,1,1;3,2,2,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}$.
(40) Trees of $\langle 5,2,1,1,1;2,2,2,2,1,1\rangle$ all have superpotential algebra $\mathbb{Z}$.
(41) Trees of $\langle 5,1,1,1,1,1;4,3,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}$.
(42) Trees of $\langle 5,1,1,1,1,1;4,2,2,1,1\rangle$ all have superpotential algebra $\mathbb{Z}$.
(43) Trees of $\langle 5,1,1,1,1,1;3,3,2,1,1\rangle$ all have superpotential algebra $\mathbb{Z}$.

(44) Trees of $\langle 5,1,1,1,1,1;3,2,2,2,1\rangle$ all have superpotential algebra $\mathbb{Z}$.
(45) Trees of $\langle 5,1,1,1,1,1;2,2,2,2,2\rangle$ all have superpotential algebra $\mathbb{Z}$.
(46) Trees of $\langle 4,4,2;3,1,1,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^5-1)$.
(47) Trees of $\langle 4,4,2;2,2,1,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^5-1)$.
(48) Trees of $\langle 4,4,1,1;4,1,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^3-1)$.
(49) Trees of $\langle 4,4,1,1;3,2,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^3-1)$.
(50) Trees of $\langle 4,4,1,1;2,2,2,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^3-1)$.
(51) Trees of $\langle 4,3,3;3,1,1,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^5-1)$.
(52) Trees of $\langle 4,3,3;2,2,1,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^5-1)$.
(53) Trees of $\langle 4,3,2,1;4,1,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^3-1)$.
(54) Trees of $\langle 4,3,2,1;3,2,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^3-1)$.
(55) Trees of $\langle 4,3,2,1;2,2,2,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^3-1)$.
(56) Trees of $\langle 4,3,1,1,1;4,2,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}$.
(57) Trees of $\langle 4,3,1,1,1;3,3,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}$.
(58) Trees of $\langle 4,3,1,1,1;3,2,2,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}$.
(59) Trees of $\langle 4,3,1,1,1;2,2,2,2,1,1\rangle$ all have superpotential algebra $\mathbb{Z}$.
(60) Trees of $\langle 4,2,2,2;4,1,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^3-1)$.
(61) Trees of $\langle 4,2,2,2;3,2,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^3-1)$.
(62) Trees of $\langle 4,2,2,2;2,2,2,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^3-1)$.
(63) Trees of $\langle 4,2,2,1,1;4,2,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}$.
(64) Trees of $\langle 4,2,2,1,1;3,3,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}$.
(65) Trees of $\langle 4,2,2,1,1;3,2,2,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}$.
(66) Trees of $\langle 4,2,2,1,1;2,2,2,2,1,1\rangle$ all have superpotential algebra $\mathbb{Z}$.
(67) Trees of $\langle 4,2,1,1,1,1;3,3,2,1,1\rangle$ all have superpotential algebra $\mathbb{Z}$.
(68) Trees of $\langle 4,2,1,1,1,1;3,2,2,2,1\rangle$ all have superpotential algebra $\mathbb{Z}$.
(69) Trees of $\langle 4,2,1,1,1,1;2,2,2,2,2\rangle$ all have superpotential algebra $\mathbb{Z}$.
(70) Trees of $\langle 4,1,1,1,1,1,1;3,3,3,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^3-1)$.
(71) Trees of $\langle 4,1,1,1,1,1,1;3,3,2,2\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^3-1)$.
(72) Trees of $\langle 3,3,3,1;3,2,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^3-1)$.
(73) Trees of $\langle 3,3,3,1;2,2,2,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^3-1)$.
(74) Trees of $\langle 3,3,2,2;3,2,1,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^3-1)$.
(75) Trees of $\langle 3,3,2,2;2,2,2,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^3-1)$.
(76) Trees of $\langle 3,3,2,1,1;3,3,1,1,1,1\rangle$ all have superpotential algebra $\mathbb{Z}$.
(77) Trees of $\langle 3,3,2,1,1;3,2,2,1,1,1\rangle$ have superpotential algebra $\mathbb{Z}[x]$ or $\mathbb{Z}[x,x^{-1}]$.
(78) Trees of $\langle 3,3,2,1,1;2,2,2,2,1,1\rangle$ all have superpotential algebra $\mathbb{Z}$.
(79) Trees of $\langle 3,3,1,1,1,1;3,2,2,2,1\rangle$ all have superpotential algebra $\mathbb{Z}$.
(80) Trees of $\langle 3,3,1,1,1,1;2,2,2,2,2\rangle$ all have superpotential algebra $\mathbb{Z}[x]/(x^9-1)$.
(81) Trees of $\langle 3,2,2,2,1;3,2,2,1,1,1\rangle$ have superpotential algebra $\mathbb{Z}[x]$ or $\mathbb{Z}[x,x^{-1}]$.
(82) Trees of $\langle 3,2,2,2,1;2,2,2,2,1,1\rangle$ all have superpotential algebra $\mathbb{Z}$.
(83) Trees of $\langle 3,2,2,1,1,1;2,2,2,2,2\rangle$ all have superpotential algebra $\mathbb{Z}$.
(84) Trees of $\langle 2,2,2,2,2;2,2,2,2,1,1\rangle$ all have superpotential algebra $\mathbb{Z}$.

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