Bulletin of the London Mathematical Society Advance Access published January 22, 2015

BOOK REVIEW

Combinatorics of minuscule representations
(Cambridge Tracts in Mathematics 199)

By R. M. Green: 320 pp., £50.00 (US$80.00), ISBN 9781107026247
(Cambridge University Press, Cambridge, 2013).

doi:10.1112/blms/bdu112

To be a *minuscule* representation of a complex simple Lie algebra $\mathfrak{g}$ is to be 'as small as possible': not only irreducible and finite-dimensional, but with all weights in the same orbit under the action of the Weyl group W. In particular, the *highest weight* λ has one-dimensional weight space, so all of its weight spaces are one-dimensional. It turns out to be fruitful to partially order the set of weights $W\lambda$, defining the *weight poset* in which $\mu \leqslant \nu$ when $\nu - \mu$ is a non-negative sum of positive roots. As explained in R. M. Green's book under review here, poset structures are the key to a minuscule representation's many special properties, making it simpler than a typical $\mathfrak{g}$-irreducible.

Minuscule representations exist only in types A, B, C, D, E_6, E_7. In type A, they are the exterior powers $\wedge^k\mathbb{C}^n$ of the natural representation $\mathbb{C}^n$ for the Lie algebra $\mathfrak{g} := \mathrm{sl}_n(\mathbb{C})$ of trace zero matrices in $\mathbb{C}^{n\times n}$. Here the Weyl group W of type A_{n-1} can be taken to be the symmetric group S_n permuting the standard basis $e_1, \ldots, e_n$. The monomial wedges $\{e_{i_1} \wedge \cdots \wedge e_{i_k}\}_{1\leqslant i_1<\cdots<i_k\leqslant n}$ give a basis of weight vectors for $\wedge^k\mathbb{C}^n$, depicted here for $\wedge^2\mathbb{C}^5$, ordered as in the weight poset:

$$
\begin{tikzcd}[column sep=small, row sep=small]
e_1 \wedge e_2 \arrow[dr, dash] & & & \\
& e_1 \wedge e_3 \arrow[dl, dash] \arrow[dr, dash] & & \\
e_2 \wedge e_3 \arrow[dr, dash] & & e_1 \wedge e_4 \arrow[dl, dash] \arrow[dr, dash] & \\
& e_2 \wedge e_4 \arrow[dl, dash] \arrow[dr, dash] & & e_1 \wedge e_5 \arrow[dl, dash] \\
e_3 \wedge e_4 \arrow[dr, dash] & & e_2 \wedge e_5 \arrow[dl, dash] & \\
& e_3 \wedge e_5 \arrow[dl, dash] & & \\
e_4 \wedge e_5 & & &
\end{tikzcd}
\tag{1}
$$

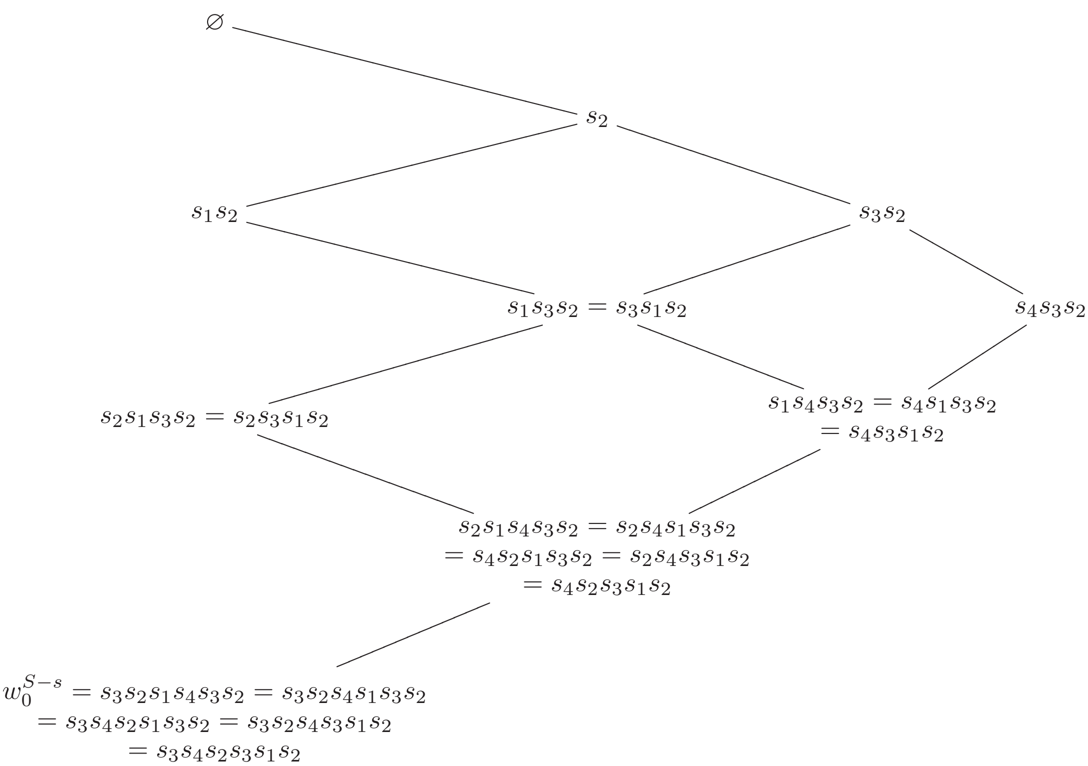
It is useful to reinterpret the transitive W-action on $W\lambda$ as the action on cosets W/W_{S-s}, where W_{S-s} is the *maximal parabolic subgroup* of W fixing λ; a minuscule highest weight λ is always one of the *fundamental weights*, fixed by all but one simple reflection s among the Coxeter generators S for W. Stembridge [8] showed that the set W^{S-s} of *minimum-length coset representatives* for W/W_{S-s} enjoys two extra properties in the minuscule setting.

(i) Each w in W^{S-s} is *fully commutative*: its minimum-length factorizations in S (*reduced* S-*words*) can be connected by sequences of commutation moves $s_i s_j = s_j s_i$ valid in W, with no need for longer braid relations from W.

(ii) The elements w in W^{S-s} are characterized as those with a reduced word which is a *suffix* of a reduced word for the unique element w_0^{S-s} in W^{S-s} that carries the highest weight to the lowest weight.

From this, one can show that the weight poset on $W\lambda$ is (the opposite of) either the (*strong*) *Bruhat order* or (*left*) *weak Bruhat order* restricted from W to W^{S-s}.

For example, when $\mathfrak{g} = \mathrm{sl}_2(\mathbb{C})$ as above, one can take the adjacent transpositions $s_i = (i, i+1)$ as the simple reflections $S = \{s_1, \ldots, s_{n-1}\}$. When $n = 5$, the minuscule representation $\wedge^2\mathbb{C}^5$ has its highest weight vector v_λ with weight λ fixed by $\{s_1, s_3, s_4\} \subset S$, that is, by $S - s$ for $s = s_2$. The elements of W^{S-s} are the permutations $w = (w_1, w_2, w_3, w_4, w_5)$ having $w_1 < w_2$ and $w_3 < w_4 < w_5$. Below is the same weight poset as in (1), depicted as the opposite of the Bruhat order on W^{S-s}, with w in W^{S-s} indicated by its set of reduced S-words $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$.



Its bottom element is the unique permutation $w_0^{S-s} = (4, 5, 1, 2, 3)$ in W^{S-s} carrying the highest weight $e_1 \wedge e_2$ to the lowest weight $e_4 \wedge e_5$, and every w in W^{S-s} has an S-reduced word that is a suffix for one of the S-reduced words of w_0^{S-s}.

This weight poset $W\lambda$, or Bruhat order on W^{S-s}, is far better behaved than your average poset. It is a *lattice*, meaning that pairs of elements have a well-defined *join* (least upper bound) and *meet* (greatest lower bound). Better yet, it is *distributive*: the meet and join operations distribute over each other. A theorem of Birkhoff (see, for example, [**6**, Theorem 3.4.1]) then implies that it is determined in a simple way by its subposet E of *meet-irreducibles* (the poset elements covered by only one other element): the distributive lattice is isomorphic to the opposite of the inclusion poset $J(E)$ on the collection of *order filters* (subsets closed under going upward) in E. Furthermore, the above-mentioned fully commutative and suffix properties let one identify E with the *minuscule heap* associated to λ and w_0^{S-s}, which projects vertically to the Dynkin diagram Γ, as shown here:

$$
\begin{tikzcd}[column sep=small]
& s_2 \arrow[dl, dash] \arrow[dr, dash] & & & \\
s_1 \arrow[dr, dash] & & s_3 \arrow[dl, dash] \arrow[dr, dash] & & \\
& s_2 \arrow[dr, dash] & & s_4 \arrow[dl, dash] & \text{heap poset } E \\
& & s_3 & &
\end{tikzcd}
$$

$$
\Big\downarrow \epsilon \tag{2}
$$

$$
s_1 - s_2 - s_3 - s_4 \quad \text{Dynkin diagram } \Gamma
$$

The linear extensions of the heap poset E are the S-reduced words for w_0^{S-s}. More generally, the isomorphism alluded to above sends an order filter of the heap poset to the element w in W^{S-s} factored by its set of linear extensions. For example, it maps this order filter as follows:

$$
\begin{tikzcd}[column sep=small]
& s_2 \arrow[dl, dash] \arrow[dr, dash] & & \\
s_1 \arrow[dr, dash] & & s_3 \arrow[dl, dash] \arrow[dr, dash] & \\
& s_2 & & s_4
\end{tikzcd}
\longrightarrow
\begin{array}{l}
w = s_2s_1s_4s_3s_2 = s_2s_4s_1s_3s_2 = s_4s_2s_1s_3s_2 \\
\quad = s_2s_4s_3s_1s_2 = s_4s_2s_3s_1s_2
\end{array}
$$

Note in (2) that the map $\epsilon : E \to \Gamma$ from the heap poset E to the Dynkin diagram Γ is just the labeling of poset elements by simple reflections. Additionally, each node s_i and edge $\{s_i, s_j\}$ of the Dynkin diagram Γ have inverse images $\epsilon^{-1}(s_i), \epsilon^{-1}(\{s_i, s_j\})$ which are totally ordered (*chains*) in E, and these chains *characterize* the poset E as the weakest partial order extending all of these chain orders.

The latter axiomatic properties of heaps over Dynkin diagrams are the starting point for Green's book. After the quick introductory Chapter 1 on classical Lie algebras and Weyl groups, Chapter 2 dives into the seemingly innocuous theory of heaps over graphs and over Dynkin diagrams, and their *order ideals* (sets closed under going downward, the complements of order filters). It introduces the crucial notion of a *full heap* over a Dynkin diagram. Chapter 3 shows how the Weyl group W acts on the subset of *proper* order ideals of any full heap over its Dynkin diagram.

Chapters 4–7 are the technical heart of the book. They review more Lie algebra theory, Weyl groups, Chevalley bases, and eventually show how minuscule representations come from heaps. The upshot comes in § 6.6, asserting that every minuscule representation of a simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ can be constructed from a *principal subheap* of a full heap over an affine Dynkin diagram.

Remark. Here 'technical' does not mean hard, but that at times it requires case-by-case checks, folding arguments for non-simply laced cases, and can get a bit grungy. In particular, § 7.4 is notable for a long sequence of lemmas beginning either with 'Maintain the notation of Definition. . . ' or 'Maintain the hypotheses of Proposition. . . '.

Chapter 8 is a fascinating excursion into properties of W acting on $W\lambda$, including

(i) the poset isomorphism between $W\lambda$ and the subposet of positive roots lying above the unique simple root α non-orthogonal to λ;

(ii) a description of the W-orbits on $W\lambda \times W\lambda$ and its relation to *branching rules* for minuscule representations; and
(iii) structural results on the *weight polytope* which is the convex hull of $W\lambda$.

In Chapters 9 and 10, the author really warms to his subject. It is a detailed exposition of the relation between exceptional minuscule representations (two of dimension 27 in type E_6, one of dimension $56 = 2 \cdot 28$ in type E_7) and the classical incidence geometry of special configurations: the 27 lines on a smooth cubic surface, the 28 pairs of lines on a degree 2 del Pezzo surface or bitangents to a non-singular plane quartic curve, Schläfli's double sixes, Steiner complexes, generalized quadrangles, etc. The discussion relies on algebro–geometric results of du Val and the recent work of Dolgachev [1] to get the ball rolling, then proceeds into incredible detail. As a neophyte in these topics, I was overwhelmed, but impressed.

The final Chapter 11 returns to briefly treat more standard topics, mainly related to the *enumerative* combinatorics of minuscule heaps as posets, such as the product formula for their rank generating function. Most proofs are omitted in this chapter. It is an old story, well-told by Proctor [3], and Stembridge [7] (see also Stanley's exercises related to *pleasant* and *Gaussian posets* in [6, Chapter 3, Exercises 170, 172]) how one can deduce these results by combining the q-analog of *Weyl's dimension formula* with Seshadri's *standard monomial theory* for the coordinate rings of the *minuscule flag variety* G/P_{S-s}. Here P_{S-s} is the parabolic subgroup associated to W_{S-s} inside the complex reductive Lie group G associated to $\mathfrak{g}$. Chapter 11 also discusses Stembridge's application of minuscule posets to his $q=-1$ *phenomenon* for self-complementary plane partitions [7], along with very recent work of Rush and Shi on the *cyclic sieving phenomenon* for the *Fon-Der-Flaass* or *rowmotion* action on minuscule posets [5]. Finally, it discusses recent results of Thomas and Yong [9], applying Proctor's theory [4] of *jeu-de-taquin* on the minuscule heap to do Schubert calculus for the minuscule flag varieties G/P_{S-s}. In spite of its brevity and lack of proofs, I found Chapter 11 to be an excellent collection of my favorite minuscule topics, and a very useful update to the older surveys such as Proctor [3] or Hiller [2, Chapter 5].

I like Green's book. The Introduction sets out the plan well, the end notes for each chapter are helpful, the bibliography is thorough and up-to-date, and it is the most thorough source for the heap-theoretic approach. It has many good exercises, although I might hesitate to use it as a course text, for fear of exhausting some students. I did have a few minor terminology quibbles, listed in the Remark below. Nevertheless, I think it will serve as a very useful updated reference for combinatorialists and Lie theorists who use minuscule representations in their work.

REMARK. In § 2.1, the computer science terminology 'trace' for elements of commutation monoids conflicts with trace of matrices. The notation $T_{p,q}$ in § 3.1 confused me, as q is a Hecke algebra parameter, while p is a Dynkin diagram node; it looks like q, p should play symmetric roles. The terminology 'complemented' in § 11.2 for posets with an order-reversing involution is non-standard.

References

1. I. V. Dolgachev, *Classical algebraic geometry: a modern view* (Cambridge University Press, Cambridge, 2012).
2. H. Hiller, *Geometry of Coxeter groups*, Research Notes in Mathematics 54 (Pitman (Advanced Publishing Program), Boston, MA, 1982).
3. R. A. Proctor, 'Bruhat lattices, plane partition generating functions, and minuscule representations', *European J. Combin.* 5 (1984) 331–350.
4. R. A. Proctor, 'd-Complete Posets Generalize Young Diagrams for the Jeu de Taquin Property', Preprint, 2009, arXiv:0905.3716.
5. D. B. Rush and X. Shi, 'On orbits of order ideals of minuscule posets', *J. Algebraic Combin.* 37 (2013) 545–569.

6. R. P. Stanley, *Enumerative combinatorics*, vol. 1, 2nd edn, Cambridge Studies in Advanced Mathematics 49 (Cambridge University Press, Cambridge, 2012).
7. J. R. Stembridge, 'On minuscule representations, plane partitions and involutions in complex Lie groups', *Duke Math. J.* 73 (1994) 469–490.
8. J. R. Stembridge, 'Minuscule elements of Weyl groups', *J. Algebra* 235 (2001) 722–743.
9. H. Thomas and A. Yong, 'A combinatorial rule for (co)minuscule Schubert calculus', *Adv. Math.* 222 (2009) 596–620.

Victor Reiner
School of Mathematics
University of Minnesota
Minneapolis, MN 55455
USA

reiner@math.umn.edu