NOTE ON THE EXPECTED NUMBER OF YANG-BAXTER MOVES APPLICABLE TO REDUCED DECOMPOSITIONS

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Consider the symmetric group $\mathfrak{S}_n$ as a Coxeter group generated by the adjacent transpositions $\{s_1, \ldots, s_{n-1}\}$. Its *longest element* w_0 is the permutation sending i to $n+1-i$ for each i. A *reduced decomposition* for w_0 is an expression $w_0 = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ where $\ell = \binom{n}{2}$. See [3] and the references therein for more on these notions, and for undefined terms below.

For any value $k = 1, 2, \ldots, \ell - 2$, say that a reduced decomposition $s_{i_1} s_{i_2} \cdots s_{i_\ell}$ for w_0 supports a *Yang-Baxter move in position* k if

$$\begin{array}{ccc} (i_k, i_{k+1}, i_{k+2}) & = & (\quad j, \quad j+1, \quad j \quad) \\ \text{or} & & (\quad j+1, \quad j, \quad j+1 \quad) \end{array}$$

for some $j = 1, 2, \ldots, n-2$.

Let X_n be the random variable on a reduced decomposition for w_0 in $\mathfrak{S}_n$ (chosen from the uniform probability distribution on all reduced decompositions) which counts the number of positions in which it supports a Yang-Baxter move. Surprisingly, its expectation turns out to be independent of n.

Theorem 1. *For all $n \geq 3$, one has $\mathbf{E}(X_n) = 1$.*

Proof. Write X_n as the sum of the indicator random variables $X_n^{(k,j)}$ for the event that the reduced decomposition supports a Yang-Baxter move in position k and with value j as described above. The fact that $s_i w_0 s_{n-i} = w_0$ leads to a $\mathbb{Z}/\ell\mathbb{Z}$-action by cyclic rotation on the set of reduced decompositions:

$$s_{i_1} s_{i_2} s_{i_3} \cdots s_{i_\ell} \mapsto s_{i_2} s_{i_3} \cdots s_\ell s_{n-i_1}.$$

This symmetry implies that the distribution of $X_n^{(k,j)}$ is independent of k, so one only needs to compute $\mathbf{E}(X_n^{(1,j)})$. Note that this is the same

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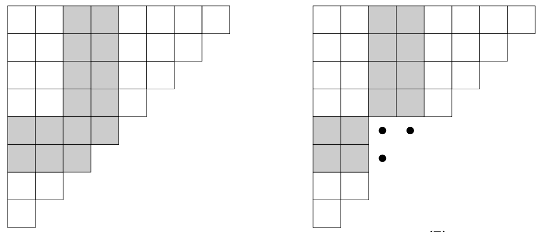

FIGURE 1. For $n = 9, j = 5$, the staircase partition δ_n and the almost-staircase partition $\delta_n^{(j)}$. Cells in which the hook-lengths for the two diagrams will differ are highlighted.

as the probability that the reduced decomposition for w_0 is of either form

$$s_j s_{j+1} s_j \cdot s_{i_4} s_{i_5} \cdots s_{i_\ell} \quad \text{or} \quad s_{j+1} s_j s_{j+1} \cdot s_{i_4} s_{i_5} \cdots s_{i_\ell}.$$

In either case, this means that $s_{i_4} s_{i_5} \cdots s_{i_\ell}$ is a reduced decomposition for $s_j s_{j+1} s_j w_0$, so $\mathbf{E}(X_n^{(1,j)})$ is twice the quotient of the cardinalities of the set of reduced decompositions for $s_j s_{j+1} s_j w_0$ and for w_0. Since these two permutations w_0 and $s_j s_{j+1} s_j w_0$ are both *vexillary* (that is, they both satisfy the conditions of [3, Corollary 4.2]), their numbers of reduced decompositions are the numbers $f_{\delta_n}, f_{\delta_n^{(j)}}$ of *standard Young tableaux* for the staircase and "almost-staircase" Ferrers diagrams δ_n and $\delta_n^{(j)}$ illustrated in Figure 1.

Using the *hook-length formula* [2, Cor. 7.21.6] for f_λ, and the fact that most of the corresponding cells in these two diagrams have the same hook-length, one can then compute

$$\begin{aligned} \mathbf{E}(X_n) &= \sum_{k=1}^{\ell-2} \sum_{j=1}^{n-2} \mathbf{E}(X_n^{(k,j)}) = (\ell-2) \sum_{j=1}^{n-2} \mathbf{E}(X_n^{(1,j)}) \\ &= (\ell-2) \sum_{j=1}^{n-2} 2 \frac{f_{\delta_n^{(j)}}}{f_{\delta_n}} = \binom{\ell}{2}^{-1} \frac{1}{3} \sum_{j=1}^{n-2} c_j c_{n-j-1} \end{aligned} \tag{1}$$

where

$$c_j := \frac{3 \cdot 5 \cdots (2j+1)}{2 \cdot 4 \cdots (2j-2)} \quad \text{for } j \geq 2, \text{ and } \quad c_1 := 3.$$

This last sum is easy to evaluate, for example by noting that

$$\sum_{j \geq 1} c_j x^j = \frac{3x}{(1-x)^{\frac{5}{2}}}.$$

Using this, and letting $[x^m]f(x)$ denote the coefficient of x^m in a formal power series $f(x)$, one has

$$\begin{aligned} \sum_{j=1}^{n-2} c_j c_{n-j-1} &= [x^{n-1}] \left(\sum_{j \geq 1} c_j x^j \right)^2 \\ &= [x^{n-1}] \frac{9x^2}{(1-x)^5} = 9 \binom{n+1}{4} = 3 \binom{\ell}{2}. \end{aligned}$$

Combining this with (1) gives $\mathbf{E}(X_n) = 1$. □

The referee suggests a nice alternate proof ending: the Murnaghan-Nakayama rule [2, §7.17] shows $\sum_{j=1}^{n-2} \frac{f_{\delta_n^{(j)}}}{f_{\delta_n}} = -\frac{\chi^{\delta_n}(\pi)}{\chi^{\delta_n}(\mathrm{id})}$ where π is a 3-cycle. Now use known explicit formulas for such characters (e.g. [1, 4]).

Conjecture 2. *As n approaches infinity, the distribution of X_n approaches that of a Poisson random variable with mean* 1*. That is, for each $k = 0, 1, 2, \ldots$, one has* $\lim_{n \to \infty} \mathbf{Prob}(X_n = k) = \frac{1}{e \cdot k!}$.

The following conjecture on the variance of X_n was suggested by computations for $n = 4, 5, 6$, and is consistent with Conjecture 2.

Conjecture 3. *For all $n \geq 4$, one has* $\mathbf{Var}(X_n) = \frac{\ell - 4}{\ell - 2}$, *where* $\ell = \binom{n}{2}$.

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