NOTES ON POINCARÉ SERIES OF FINITE AND AFFINE COXETER GROUPS

VICTOR REINER

ABSTRACT. There are two famous formulae relating the Poincaré series of a finite/affine Weyl group to the degrees of fundamental invariants for the finite Weyl group. We review the classical proof due to Solomon of the finite formula that uses the Coxeter complex, and sketch Steinberg's analogous proof of the affine (Bott) formula using the "toroidal" Coxeter complex.

1. RECALLING THE POINCARÉ SERIES

Let W be a finite reflection group acting on a n-dimensional real vector space V. Then W also acts on the polynomial algebra $S := \mathrm{Sym}(V^*)$, and a famous theorem due to Shephard and Todd and to Chevalley asserts that the invariant ring S^W is a polynomial subalgebra. If one picks a minimal set of homogeneous algebra generators $f_1, \ldots, f_n$ for S^W, that is, $S^W = \mathbb{R}[f_1, \ldots, f_n]$, then their degrees $d_1, \ldots, d_n$ are called the *fundamental degrees*, and the *Hilbert series* for S^W has the following expression: $\mathrm{Hilb}(S^W, q) = \prod_{i=1}^n \frac{1}{1-q^{d_i}}$. The same degrees then enter into two famous Poincaré series formulae for Coxeter groups.

Let S be any set of *simple reflections* for W, that is, reflections through walls of any fixed chamber in the decomposition of V by the reflecting hyperplanes of reflections in W. Let $\ell(w)$ denotes the length of w with respect to the generators S. Then (W, S) becomes a *Coxeter system*, whose Poincaré series is defined to be $W(q) := \sum_{w \in W} q^{\ell(w)}$ where $\ell(w)$ denotes the length of w with respect to the generators S.

Theorem 1.1. *(Chevalley-Solomon) For any finite reflection group,*

$$\begin{aligned} W(q) &= \prod_{i=1}^n (1 + q + q^2 + \cdots q^{d_i - 1}) \\ &= \prod_{i=1}^n \frac{1 - q^{d_i}}{1 - q} = \frac{\mathrm{Hilb}(S, q)}{\mathrm{Hilb}(S^W, q)}. \end{aligned}$$

When W is a finite *Weyl* group, that is, W stabilizes a rank n lattice inside V, there is an associated *affine Weyl group* $\tilde{W}$. When W is irreducible, one has a Coxeter system $(\tilde{W}, \tilde{S})$, where $\hat{S} = S \cup \{s_0\}$ in which s_0 is the affine reflection in V through a hyperplane normal to the highest root for W.

Theorem 1.2. *(Bott) For any irreducible finite Weyl group W, the associated affine Weyl group $\tilde{W}$ has*

$$\begin{aligned}\tilde{W}(q) &= \frac{1}{(1-q)^n} \prod_{i=1}^{n} \frac{1-q^{d_i}}{1-q^{d_i-1}} = \mathrm{Hilb}(S,q) \prod_{i=1}^{n} \frac{1-q^{d_i}}{1-q^{d_i-1}} \\ &= W(q) \prod_{i=1}^{n} \frac{1}{1-q^{d_i-1}}\end{aligned}$$

Our goal here is to sketch proofs of Theorems 1.1 and 1.2 that have similar natures. The proof for Theorem 1.1 is due to Solomon, and appears in many places, such as Humphreys [4]. The proof given here for Theorem 1.2 is not Bott's original one, but rather one that is less well-known, due to Steinberg[1]. A more detailed exposition can now be found in the Masters Thesis of Leonid Grau [2].

We also remark at the end on some comparison with other styles of proof.

2. Proofs of Theorems 1.1 and 1.2

Both proofs compare an easy Coxeter group recursion for $W(q)$ with a recursion derived from the Hopf trace formula applied to the W-action on some nice simplicial complex. We review some of the relevant notions before embarking on the proofs.

2.1. **The descent induction.** The discussion in this subsection is essentially the same as [4, §5.12].

For any Coxeter system (W, S), and any subset $J \subseteq S$, the *parabolic subgroup* W_J generated by J turns out to form a Coxeter system (W, J) in its own right, with its length function inherited from that of (W, S). One also has the set W^J of minimum length coset representatives for W/W_J, characterized by this property: $x \in W^J$ if and only if its *right descent set*

$$D_R(x) := \{s \in S : \ell(xs) < \ell(x)\}$$

lies entirely in $S \backslash J$. There is a well-known unique factorization result: $W = W^J W_J$, and if $w = xy$ with $x \in W^J, y \in W_J$ then their lengths satisfy $\ell(w) = \ell(x) + \ell(y)$. This shows that $W(q) = W^J(q)\, W_J(q)$ and hence

$$W^J(q) = \sum_{w \in W : D_R(w) \subseteq S \backslash J} q^{\ell(w)} = \frac{W(q)}{W_J(q)}.$$

Inclusion-exclusion then gives the crucial formula

$$\sum_{J \subseteq S} (-1)^{|J|} \frac{W(q)}{W_J(q)} = \sum_{w \in W : D_R(w) = S} q^{\ell(w)} = \begin{cases} q^{\ell(w_0)} & \text{if } W \text{ is finite,} \\ 0 & \text{if } W \text{ is infinite.} \end{cases} \tag{2.1}$$

Here w_0 denotes the unique element of maximum length when W is finite, which is also the unique element having $D_R(w_0) = S$. We will later need the fact that its length $\ell(w_0)$ is the number of reflections in W.

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[1]Thanks to John Stembridge for telling me of the existence of this gorgeous proof, and suggesting that I buy the *The Collected Papers of Robert Steinberg*, the best 37 AMS points I ever spent.

One can recast this as a recursion, dividing through by $W(q)$, and bringing the $J = S$ term in the sum to the right side:

$$\sum_{J \subsetneq S} (-1)^{|J|} \frac{1}{W_J(q)} = f(q) \frac{1}{W(q)}, \tag{2.2}$$

where

$$f(q) = \begin{cases} q^{\ell(w_0)} - (-1)^{|S|} & \text{if } W \text{ is finite,} \\ -(-1)^{|S|} & \text{if } W \text{ is infinite.} \end{cases}$$

Note this recursion shows, by induction on $n = |S|$, that $W(q)$ is a rational function in q.

2.2. **The Hopf trace formula.** When Δ is a simplicial complex carrying the action of a group W by simplicial automorphisms, one has the following equality of virtual characters of W-representations over $\mathbb{C}$:

$$\sum_i (-1)^i C_i(\Delta) = \sum_i (-1)^i H_i(\Delta). \tag{2.3}$$

Here the C_i are the simplicial chain groups for Δ, and H_i the simplicial homology groups for Δ, both taken with coefficients in $\mathbb{C}$. The summation indices in both sums should run

- over $i \geq 0$ when using ordinary (nonreduced) simplicial homology, which will be relevant for the affine case,
- over $i \geq -1$ when using reduced homology, which will be relevant for the finite case.

2.3. **Proof of Theorem 1.1 via induction.** Given our finite reflection group W and its Coxeter system (W, S), consider the *Coxeter complex* $\Delta := \Delta(W, S)$, which has two alternate descriptions.

On one hand, Δ is the simplicial decomposition of the unit $(n-1)$-sphere in V cut out by the reflecting hyperplanes for W.

On the other hand, Δ is the abstract simplicial complex whose faces are indexed by cosets wW_J of parabolic subgroups, with the inclusion order on faces given by reverse inclusion of cosets. A typical face in Δ, say the one indexed by the coset wW_J, will have W-stabilizer wW_Jw^{-1}, and the stabilizer will fix the face *pointwise.*

The first description tells us that Δ has reduced homology concentrated in dimension $n-1$, with $H_{n-1}(\Delta) \cong \mathbb{C}$, carrying the W-action by the *sign* or *determinant* character $\det : W \to \mathbb{C}^\times$.

The second description tells us that $C_i(\Delta)$ is a direct sum of coset representations $1 \uparrow_{W_J}^W$ for parabolic subgroups W_J in which $|J| = n - 1 - i$.

Consequently the Hopf trace formula (2.3) applied to the reduced homology of Δ yields this equality of virtual characters:

$$\sum_{J \subseteq S} (-1)^{|J|} \; 1 \uparrow_{W_J}^W \cong \det . \tag{2.4}$$

Now consider the polynomial algebra $S = \mathrm{Sym}(V^*)$ with its W-action as a *graded* W-representation. We wish to apply to both sides of (2.4) the map that sends a virtual W-character χ to the generating function for its intertwining numbers with the graded components S_d of S, that is,

$$\chi \mapsto \sum_{d \geq 0} \langle \chi, S_d \rangle \, q^d.$$

Note that for any W-representation M, there is an isomorphism of intertwining spaces $\mathrm{Hom}_{\mathbb{C}[W]}(1 \uparrow_{W_J}^{W}, M) \cong M^{W_J}$, where M^{W_J} are the W_J-invariants in M, e.g. by Frobenius reciprocity. One similarly has an isomorphism of intertwining spaces $\mathrm{Hom}_{\mathbb{C}[W]}(\det, M) \cong M^{W,\det}$ where

$$M^{W,\det} := \{m \in M : w(m) = \det(w)\, m \text{ for all } w \in W\}$$

are the det-*relative invariants of* M. Consequently, one obtains

$$\sum_{J \subseteq S} (-1)^{|J|} \mathrm{Hilb}(S^{W_J}, q) = \mathrm{Hilb}(S^{W,\det}, q) = q^{\ell(w_0)} \mathrm{Hilb}(S^W). \tag{2.5}$$

Here it is important to note that we are considering all subgroups W_J as acting by restriction from W on the *same space* V, and hence on $S = \mathrm{Sym}(V^*)$. The last equality in (2.5) follows from the structure of $S^{W,\det}$: it is a free S^W-submodule inside S of rank one, consisting of all polynomials in S divisible by the *Jacobian* $\prod_H \ell_H$. This Jacobian is the product of all of the linear forms ℓ_H in S that define reflecting hyperplanes for W, and has degree $\ell(w_0)$.

Dividing (2.5) by $\mathrm{Hilb}(S, q)$ and bringing the $J = S$ term in the sum to the right side, one obtains a recursion:

$$\sum_{J \subsetneq S} (-1)^{|J|} \frac{\mathrm{Hilb}(S^{W_J}, q)}{\mathrm{Hilb}(S, q)} = (q^{\ell(w_0)} - (-1)^{|S|}) \frac{\mathrm{Hilb}(S^W, q)}{\mathrm{Hilb}(S, q)}.$$

Comparing this with (2.2), one sees that $\frac{1}{W(q)}$ and $\frac{\mathrm{Hilb}(S^W,q)}{\mathrm{Hilb}(S,q)}$ satisfy the same recursion on $|S|$. Since both are 1 when $|S| = 0$, they also satisfy the same initial condition, and hence are equal, proving Theorem 1.1.

2.4. **Steinberg's proof of Bott's formula.** Steinberg's proof of Theorem 1.2 is modelled on the proof of Theorem 1.1 just given[2], and also uses this theorem as a lemma.

One replaces the *spherical* Coxeter complex associated to the finite Coxeter system (W, S) with the *toroidal* quotient space $\Delta := \Delta(\tilde{W}, \tilde{S})/L$ of the affine Coxeter complex $\Delta(\tilde{W}, \tilde{S})$ by the coroot lattice L. Here we are thinking of L as the translation subgroup inside the affine Weyl group $\tilde{W} = W \ltimes L$. It is again true that Δ has two descriptions, which one compares in order to apply the Hopf trace formula.

On one hand, since $\Delta(\tilde{W}, \tilde{S})$ triangulates $V \cong \mathbb{R}^n$, the complex Δ triangulates the n-torus $V/L \cong \mathbb{R}^n/L$. Since the finite Weyl group W stabilizes the lattice L, it acts on the quotient Δ. The cohomology of the torus Δ can be identified via deRham theory with the exterior algebra $\wedge(V^*)$ on the differential forms $\{dx_i\}_{i=1}^n$, and consequently, the W-action on the cohomology is the same as that on $\wedge(V^*)$.

On the other hand, as an abstract simplicial complex, $\Delta(\tilde{W}, \tilde{S})$ again has faces indexed by cosets $w\tilde{W}_J$ of parabolic subgroups, with the inclusion order on faces

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[2]Steinberg actually had bigger fish to fry, using this method to prove an amazing "twisted" version of Theorem 1.2. The twisting involves any linear automorphism σ of V that permutes the set of affine simple reflections $\tilde{S}$ when acting on them by conjugation. However, we found this σ slightly distracting in following the proof, and have omitted it in our discussion.

given by reverse inclusion of cosets. The face indexed by $w\tilde{W}_J$ has dimension $n-|J|$. Faces of the quotient Δ are indexed by *double cosets* $Lw\tilde{W}_J$. To understand the action of the finite Weyl group W on these faces in the quotient Δ, first note that each proper parabolic subgroup $\tilde{W}_J$ is finite, and is the $\tilde{W}$-stabilizer of some face in the fundamental alcove. It turns out that when one applies to $\tilde{W}_J$ the quotient map $\tilde{W} = W \ltimes L \twoheadrightarrow W$ which mods out by L, the image $W_J^\pi := \pi(\tilde{W}_J)$ within the finite Weyl group W has three crucial properties, explained carefully by L. Grau in [2, Chap. 5]:

(a) When the finite Weyl group W acts on the torus

$$\Delta = \Delta(\tilde{W}, \tilde{S})/L = \bigsqcup_{J \subsetneq S} L\backslash\tilde{W}/\tilde{W}_J,$$

for each $J \subsetneq S$, the W-action on faces represented by double cosets of the form $\{Lw\tilde{W}_J\}_{w\in\tilde{W}}$ is isomorphic to the W-action on cosets W/W_J^π.

(b) This image W_J^π is a *reflection subgroup*[3] of the finite Weyl group W.

(c) There exists a translation in V which will conjugate this image subgroup $W_J^\pi \subset W \subset \tilde{W}$ to the parabolic subgroup $\tilde{W}_J \subset \tilde{W}$.

Consequently, via (a) and the Hopf trace formula (2.3) applied to the (nonreduced) homology of Δ, one obtains this equality of virtual W-characters:

$$\sum_{J\subsetneq\tilde{S}} (-1)^{|J|}\ 1\uparrow_{W_J^\pi}^W \cong (-1)^n \sum_{i=0}^n (-1)^i \wedge^i (V) \tag{2.6}$$

As before, one considers the polynomial algebra $S = \mathrm{Sym}(V^*)$ with its W-action as a graded W-representation, and applies to both sides of (2.6) the map that sends a virtual W-character χ to the generating function for its intertwining numbers with the graded components S_d of S. This gives

$$\begin{aligned}\sum_{J\subsetneq\tilde{S}} (-1)^{|J|}\mathrm{Hilb}(S^{W_J^\pi}, q) &= (-1)^n \sum_{i=0}^n (-1)^i \sum_{d\geq 0} \langle S_d, \wedge^i(V)\rangle q^d \\ &= (-1)^n \sum_{i=0}^n (-1)^i \mathrm{Hilb}(S\otimes\wedge^i(V^*))^W, q).\end{aligned} \tag{2.7}$$

One then uses a theorem of Solomon [5] that describes the invariant subalgebra $(S\otimes\wedge(V^*))^W$: it is a free S^G-module with S^G-basis given by the differentials $\{df_i\}_{i=1,\ldots,n}$. This shows that

$$\sum_{i=0}^n \mathrm{Hilb}\left(\left(S\otimes\wedge^i(V^*)\right)^W, q\right) u^i = \prod_{i=1}^n \frac{1+uq^{d_i-1}}{1-q^{d_i}}.$$

Hence plugging in $u=-1$ allows us to rewrite the last equation in (2.7) as

$$\sum_{J\subsetneq\tilde{S}} (-1)^{|J|}\mathrm{Hilb}(S^{W_J^\pi}, q) = (-1)^n \prod_{i=1}^n \frac{1-q^{d_i-1}}{1-q^{d_i}}. \tag{2.8}$$

[3] W_J^π may not be a *standard parabolic* subgroup of W_K, nor a conjugate wW_Kw^{-1}, contrary to an incorrect assertion in a previous version of these notes; thanks to L. Grau for the correction.

Dividing by $\mathrm{Hilb}(S,q)$, one obtains

$$\sum_{J \subsetneq \tilde{S}} (-1)^{|J|} \frac{\mathrm{Hilb}(S^{W_J^\pi}, q)}{\mathrm{Hilb}(S,q)} = \frac{(-1)^n}{\mathrm{Hilb}(S,q)} \prod_{i=1}^{n} \frac{1-q^{d_i-1}}{1-q^{d_i}}.$$

By property (b) above for W_J^π, one can apply Theorem 1.1 to each summand on the left. However, one must first choose generating reflections that give a Coxeter presentation for W_J^π, and property (c) above tells us that it affords such a presentation isomorphic to that of the Coxeter system $(\tilde{W}_J, J)$. Hence one has

$$\sum_{J \subsetneq \tilde{S}} (-1)^{|J|} \frac{1}{\tilde{W}_J(q)} = \frac{(-1)^n}{\mathrm{Hilb}(S,q)} \prod_{i=1}^{n} \frac{1-q^{d_i-1}}{1-q^{d_i}}.$$

By (2.2) applied to $(\tilde{W}, \tilde{S})$, the left side here is $(-1)^n/\tilde{W}(q)$, giving Theorem 1.2.

3. Remarks on the Schubert basis proofs

Both Theorems 1.1 and 1.2 have proofs, discussed in Hiller's book [3] that use something like a Schubert basis in various forms. We briefly mention these here.

For Theorem 1.1, a little commutative algebra allows one to interpret the right side $\frac{\mathrm{Hilb}(S,q)}{\mathrm{Hilb}(S^W,q)}$ as the Hilbert series for the *coinvariant algebra* $S/(S^W_+)$. Here (S^W_+) denotes the ideal within S generated by the elements S^W_+ having positive degree inside S^W. One must then explain why, for a finite reflection group W, one has

$$W(q) = \mathrm{Hilb}(S/(S^W_+), q). \tag{3.1}$$

There are two related approaches to this. The (historically) first works when W is a *Weyl* group. If G is its associated complex reductive algebraic group, with a choice of Borel subgroup B, the Bruhat decomposition $G = \amalg_{w \in W} BwB$ gives rise to a decomposition of the *generalized flag manifold* $G/B = \amalg_{w\in W} BwB/B$ into the (relatively open) *Schubert cells* $X_w^o := BwB/B$. This X_w^o is isomorphic to a real $2\ell(w)$-dimensional cell, and together these Schubert cells give a CW-decomposition for G/B, whose cellular boundary maps are all zero. Hence the ordinary homology/cohomology of G/B have no torsion, and have $\mathbb{Z}$-bases indexed by W, with Poincaré series given by $W(q^2)$. On the other hand, Borel showed that there is a (degree-doubling) isomorphism between the coinvariant algebra and the cohomology of G/B:

$$A_W \cong H^*(G/B, \mathbb{Z}).$$

This then implies the desired equality (3.1). Kostant argued analogously, substituting Lie algebra homology and harmonic representatives; see [3, §5.2].

Hiller's book, in Chapter 6, discusses in detail the Demazure/Bernstein-Gelfand-Gelfand approach, which applies even when W is a non-crystallographic finite reflection group, in spite of the lack of a flag manifold G/B. One exhibits a Schubert-like basis for the coinvariant algebra A_W, dual to the collection of divided difference operators $\{\partial_w\}_{w\in W}$ acting on S. In the case where W is crystallographic, one can identify these divided difference operators with the homology classes dual to the Schubert basis in the cohomology of G/B.

In a similar vein (see [3, Chap. 5 §6]), Bott's original proof of his Theorem 1.2 uses a Schubert cell decomposition, this time for the *loop group* ΩG. The Schubert cells in this case are indexed by the elements of the lattice $L(= \tilde{W}/W)$, so that the cohomology $H^*(\Omega G)$ has Poincaré series $L(q^2) = \sum_{w \in L} q^{2\ell(w)}$. On the other hand, Theorem 1.2 is equivalent to the assertion that

$$L(q) \left(= \frac{\tilde{W}(q)}{W(q)} \right) = \frac{1}{\prod_{i=1}^{n} 1 - q^{d_i - 1}}. \tag{3.2}$$

Hence Bott only needed to show that cohomology $H^*(\Omega G)$ has Poincaré series equal to the right side of (3.2). He achieves this using the Leray-Serre spectral sequence applied to the path-fibration $\Omega G \to PG \to G$, in which the path space PG is contractible and the group G has known cohomology, involving the degrees d_i.

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School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA

E-mail address: `reiner@math.umn.edu`