J. London Math. Soc. (00) 00 (2017) 1–25
doi:10.1112/jlms.12013

Absolute order in general linear groups

Jia Huang, Joel Brewster Lewis and Victor Reiner

To the memory of J. A. Green, R. Steinberg and A. Zelevinsky

Abstract

This paper studies a partial order on the general linear group $\mathrm{GL}(V)$ called the absolute order, derived from viewing $\mathrm{GL}(V)$ as a group generated by reflections, that is, elements whose fixed space has codimension one. The absolute order on $\mathrm{GL}(V)$ is shown to have two equivalent descriptions: one via additivity of length for factorizations into reflections and the other via additivity of fixed space codimensions. Other general properties of the order are derived, including self-duality of its intervals.

Working over a finite field $\mathbb{F}_q$, it is shown via a complex character computation that the poset interval from the identity to a Singer cycle (or any regular elliptic element) in $\mathrm{GL}_n(\mathbb{F}_q)$ has a strikingly simple formula for the number of chains passing through a prescribed set of ranks.

1. *Introduction*

This paper studies, as a reflection group, the full general linear group $\mathrm{GL}(V) \cong \mathrm{GL}_n(\mathbb{F})$, where V is an n-dimensional vector space over a field $\mathbb{F}$. An element g in $\mathrm{GL}(V)$ is called a *reflection* if its *fixed subspace* $V^g := \{v \in V : \ gv = v\} = \ker(g-1)$ has codimension 1. A *reflection group* is a subgroup of $\mathrm{GL}(V)$ generated by reflections[†]. It is not hard to show that $\mathrm{GL}(V)$ itself is generated by its subset T of reflections, and hence is a reflection group.

Finite, real reflection groups W inside $\mathrm{GL}_n(\mathbb{R}) \cong \mathrm{GL}(V)$ are well studied classically via their *Coxeter presentations* (W,S). Here S is a choice of n generating *simple reflections*, which are the orthogonal reflections across hyperplanes that bound a fixed choice of *Weyl chamber* for W. Recent works of Brady and Watt [**7**] and Bessis [**5**] have focused attention on an alternate presentation, generating real reflection groups W by their subset T of *all* reflections. Their work makes use of the coincidence, first proven by Carter [**9**], between two natural functions $W \to \{0,1,2,\ldots\}$ defined as follows for $w \in W$:

- the *reflection length*[‡] given by $\ell_T(w) := \min\{\ell : w = t_1 t_2 \cdots t_\ell \text{ with } t_i \in T\}$, and
- the *fixed space codimension* given by $\mathrm{codim}(V^w) := n - \dim(V^w)$.

While both of these functions can be defined for all reflection groups, it has been observed (see, for example, Foster-Greenwood [**12**]) that for non-real reflection groups, and even for most finite complex reflection groups, these two functions differ. This leads to two partial orders,

- the *T-prefix order*: $g \leqslant h$ if $\ell_T(g) + \ell_T(g^{-1}h) = \ell_T(h)$ and
- the *fixed space codimension order*: $g \leqslant h$ if $\mathrm{codim}(V^g) + \mathrm{codim}(V^{g^{-1}h}) = \mathrm{codim}(V^h)$.

We discuss some general properties of these orders in Section 2. One of our first results, Proposition 2.16, is the observation that when considering *as a reflection group* the full general

---

Received 12 August 2015; revised 12 April 2016.

2010 *Mathematics Subject Classification* 20G40 (primary), 05E10, 20C33 (secondary).

This work was partially supported by NSF grants DMS-100193, DMS-1148634, and DMS-1401792.

[†]Our definitions here deviate slightly from the literature, where one often insists that a reflection have *finite order*. In particular, by our definition, the determinant of a reflection g need not be a root of unity in $\mathbb{F}^\times$, and $\mathrm{GL}(V) = \mathrm{GL}_n(\mathbb{F})$ is still generated by reflections even when $\mathbb{F}$ is infinite.

[‡]Warning: this is *not* the usual Coxeter group length $\ell_S(w)$ coming from the Coxeter system (W,S).

linear group $\mathrm{GL}(V)$ for a finite-dimensional vector space V over a field, one again has the coincidence $\ell_T(g) = \mathrm{codim}(V^g)$, and hence the two partial orders above give the same order on $\mathrm{GL}(V)$, which we call the *absolute order*.

We proceed to prove two basic enumerative results about this absolute order on $\mathrm{GL}(V)$ when the field $\mathbb{F} = \mathbb{F}_q$ is finite. First, Section 3 uses Möbius inversion to count the elements in $\mathrm{GL}_n(\mathbb{F}_q)$ of a fixed poset rank, that is, those with $\ell_T(g) = \mathrm{codim}(V^g)$ fixed.

Second, in Section 4, we examine the interval $[e, c]$ in the absolute order on $\mathrm{GL}_n(\mathbb{F}_q)$ from the identity element e to a *Singer cycle* c. Singer cycles are the images of cyclic generators for the multiplicative group $\mathbb{F}_{q^n}^\times$, after it has been embedded into $\mathrm{GL}_n(\mathbb{F}_q) \cong \mathrm{GL}_{\mathbb{F}_q}(\mathbb{F}_{q^n})$. In recent years, a close analogy between the Singer cycles in $\mathrm{GL}_n(\mathbb{F}_q)$ and *Coxeter elements* in real reflection groups has been established; see [**32**, § 8–§ 9], [**31**, § 7], [**23**]. The interval from the identity to a Coxeter element in the absolute order on a real reflection group W is a very important and well-behaved poset, called the poset of *noncrossing partitions* for W. Our main result, Theorem 4.2, gives a strikingly simple formula for the *flag f-vector* of $[e, c]$ in $\mathrm{GL}_n(\mathbb{F}_q)$: fixing an ordered composition $\alpha = (\alpha_1, \ldots, \alpha_m)$ of $n = \sum_i \alpha_i$, the number of chains $e = g_0 < g_1 < \cdots < g_{m-1} < g_m = c$ in absolute order having $\ell_T(g_i) - \ell_T(g_{i-1}) = \alpha_i$ is

$$q^{\varepsilon(\alpha)} \cdot (q^n - 1)^{m-1}, \quad \text{where} \quad \varepsilon(\alpha) := \sum_{i=1}^{m} (\alpha_i - 1)(n - \alpha_i).$$

The analogous flag f-vector formulas in real reflection groups are not as simple.

The proof of Theorem 4.2 is involved, using a character-theoretic enumeration technique due to Frobenius, along with information about the complex characters of $\mathrm{GL}_n(\mathbb{F}_q)$ that goes back to Green [**18**] and Steinberg [**37**]. The proof has the virtue of applying not only to Singer cycles in $\mathrm{GL}_n(\mathbb{F}_q)$ but also to elements which are *regular elliptic*; see Section 4 for the definition. Section 5 reformulates the flag f-vector in terms of certain subspace arrangements. We hope that this may lead to a more direct approach to Theorem 4.2. Section 6 collects further questions and remarks.

2. *Length and prefix order*

The next few subsections collect some easy general properties of the length function with respect to a choice of generators for a group and the resulting partial order defined in terms of prefixes of reduced expressions. We borrow heavily from work of Armstrong [**1**, § 2.4], Bessis [**5**, § 0.4], Brady and Watt [**8**], and Foster-Greenwood [**12**], while attempting to clarify the hypotheses responsible for various properties.

2.1. *Generated groups*

DEFINITION 2.1. A *generated group* is a pair (G, T) where G is a group and $T \subseteq G$ a subset that generates G as a monoid: every g in G has at least one *T-word for g*, meaning a sequence $(t_1, t_2, \ldots, t_\ell)$ with $g = t_1 t_2 \cdots t_\ell$. The length function $\ell = \ell_T : G \to \mathbb{N}$ is defined by

$$\ell(g) := \min\{\ell : g = t_1 t_2 \cdots t_\ell \text{ with } t_i \in T\}.$$

That is, $\ell(g)$ is the minimum length of a T-word for g. Words for g achieving this minimum length are called *T-reduced*. Equivalently, $\ell(g)$ is the length of the shortest directed path from the identity e to g in the Cayley graph of (G, T).

It should be clear from this definition that ℓ is *subadditive*, meaning that

$$\ell(gh) \leqslant \ell(g) + \ell(h). \tag{2.1}$$

Understanding the case where equality occurs in (2.1) motivates the next definition.

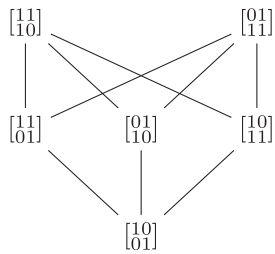
Definition 2.2 (Prefix order). Given a generated group (G, T), define a binary relation $g \leqslant h$ on G by any of the following three equivalent conditions.

(i) Any T-reduced word $(t_1, \ldots, t_{\ell(g)})$ for g extends to a T-reduced word $(t_1, \ldots, t_{\ell(h)})$ for h.
(ii) There is a shortest directed path e to h in the Cayley graph for (G, T) going via g.
(iii) $\ell(g) + \ell(g^{-1}h) = \ell(h)$.

Condition (i) makes the following proposition a straightforward exercise, left to the reader.

Proposition 2.3. *For (G, T) a generated group, the binary relation $\leqslant$ is a partial order on G, with the identity e as minimum element. It is graded by the function $\ell(-)$, in the sense that for any $g < h$, one has $\ell(h) = \ell(g) + 1$ if and only if there is no g' with $g < g' < h$.*

Example 2.4. Taking $G = \mathrm{GL}_2(\mathbb{F}_2)$ and T the set of all reflections in G, the Hasse diagram for $\leqslant$ on G is as follows:



Coincidentally, this is isomorphic to the absolute order on the symmetric group $\mathfrak{S}_3$, since the irreducible reflection representation for $\mathfrak{S}_3$ over $\mathbb{F}_2$ is isomorphic to $\mathrm{GL}_2(\mathbb{F}_2)$.

2.2. *Conjugacy-closed generators*

When (G, T) is a generated group in which T is closed under conjugation by elements of G, one has $\ell(ghg^{-1}) = \ell(h)$ for all g, h in G. This implies, for example, that $\ell(gh) = \ell(g^{-1} \cdot gh \cdot g) = \ell(hg)$.

The next proposition asserts an interesting consequence for the order $\leqslant$ on G, namely that it is *locally self-dual*: each interval is isomorphic to its own opposite as a poset.

Proposition 2.5. *Let (G, T) be a generated group, with T closed under G-conjugacy. Then for any $x \leqslant z$, the bijection $G \to G$ defined by $y \mapsto xy^{-1}z$ restricts to a poset anti-automorphism $[x, z] \to [x, z]$.*

Proof. We first check the bijection restricts to $[x, z]$. By definition, $y \in [x, z]$ if and only if

$$\begin{cases} \ell(y) = \ell(x) + \ell(x^{-1}y), \\ \ell(z) = \ell(y) + \ell(y^{-1}z), \end{cases} \tag{2.2}$$

whereas $xy^{-1}z \in [x, z]$ if and only if

$$\begin{cases} \ell(xy^{-1}z) = \ell(x) + \ell(y^{-1}z), \\ \ell(z) = \ell(xy^{-1}z) + \ell(z^{-1}yx^{-1}z) = \ell(xy^{-1}z) + \ell(yx^{-1}), \end{cases} \tag{2.3}$$

where the last equality in (2.3) uses the conjugacy hypothesis.

To see that (2.2) implies (2.3), note that, assuming (2.2), one has

$$\begin{aligned}\ell(z) &\leqslant \ell(yx^{-1}) + \ell(xy^{-1}z)\\ &\leqslant \ell(x^{-1}y) + \ell(x) + \ell(y^{-1}z)\\ &= (\ell(y) - \ell(x)) + \ell(x) + (\ell(z) - \ell(y)) = \ell(z),\end{aligned}$$

using the conjugacy hypothesis to say $\ell(yx^{-1}) = \ell(x^{-1}y)$. The fact that one has equality at each inequality above implies (2.3). Conversely, assuming (2.3), one has

$$\begin{aligned}\ell(z) &= \ell(x) + \ell(y^{-1}z) + \ell(yx^{-1})\\ &\geqslant \ell(x) + (\ell(z) - \ell(y)) + (\ell(y) - \ell(x)) = \ell(z)\end{aligned}$$

with equality at the inequality implying (2.2).

It remains to show the restricted bijection $[x, z] \to [x, z]$ reverses order. Assume $y_1 \leqslant y_2$ in $[x, z]$. The preceding calculations show that $\ell(xy_i^{-1}z) = \ell(x) - \ell(y_i) + \ell(z)$. Thus

$$\begin{aligned}\ell(xy_1^{-1}z) &= \ell(x) - \ell(y_1) + \ell(z)\\ &= (\ell(x) - \ell(y_2) + \ell(z)) + (\ell(y_2) - \ell(y_1))\\ &= \ell(xy_2^{-1}z) + \ell(y_1^{-1}y_2)\\ &= \ell(xy_2^{-1}z) + \ell(z^{-1}y_2x^{-1}xy_1^{-1}z),\end{aligned}$$

using the conjugacy hypothesis in this last equality. Hence $xy_2^{-1}z \leqslant xy_1^{-1}z$, as desired. $\square$

The following is another important feature of G-conjugacy-closed generators T. Given g, h in G, let $g^h := h^{-1}gh$ and ${}^hg := hgh^{-1}$, and note that

$$g \cdot h = h \cdot g^h = {}^gh \cdot g. \tag{2.4}$$

Definition 2.6 (Hurwitz operators). Given a generated group (G, T) with T closed under G-conjugacy and any T-word

$$\mathbf{t} := (t_1, \ldots, t_{i-1}, t_i, t_{i+1}, t_{i+2}, \ldots, t_m)$$

for $g = t_1 \cdots t_m$, for $1 \leqslant i \leqslant m - 1$ define the *Hurwitz operator* σ_i and its inverse σ_i^{-1} by

$$\begin{aligned}\sigma_i(\mathbf{t}) &:= (t_1, \ldots, t_{i-1}, t_{i+1}, t_i^{t_{i+1}}, t_{i+2}, \ldots, t_m),\\ \sigma_i^{-1}(\mathbf{t}) &:= (t_1, \ldots, t_{i-1}, {}^{t_i}t_{i+1}, t_i, t_{i+2}, \ldots, t_m).\end{aligned}$$

Equation (2.4) shows that $\sigma_i(\mathbf{t})$ and $\sigma_i^{-1}(\mathbf{t})$ are both T-words for g.

Remark 2.7. Although it is not needed in the following, note that $\{\sigma_1, \ldots, \sigma_{m-1}\}$ satisfy the *braid relations* $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ and $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i - j| \geqslant 2$, defining an action of the *braid group* B_m on m strands on the set of all length-m factorizations of g.

Note that the operator σ_i (respectively, σ_i^{-1}) can be used to swap any letter in a word for g one position to the left (respectively, right) *unchanged* at the expense of conjugating the letter with which it swapped; this creates a new word for g of the same length. Armstrong calls this the *shifting property* [**1**, Lemma 2.5.1]. It has the following immediate consequence.

PROPOSITION 2.8 (Subword property). *Let (G,T) be a generated group with T closed under G-conjugacy. Then $g \leqslant h$ if and only there exists a T-reduced word*

$$\mathbf{t} := (t_1, t_2, \ldots, t_{\ell(h)})$$

for h containing as a subword (not necessarily a prefix, nor contiguous) a word

$$\hat{\mathbf{t}} = (t_{i_1}, t_{i_2}, \ldots, t_{i_{\ell(g)}}) \text{ with } 1 \leqslant i_1 < \cdots < i_{\ell(g)} \leqslant \ell(h)$$

that is T-reduced for g.

Proof. The 'only if' direction is direct from condition (i) in Definition 2.2 of $g \leqslant h$. For the 'if' direction, given the T-reduced word $\mathbf{t}$ for h containing the T-reduced subword $\hat{\mathbf{t}}$ for g, one obtains another T-reduced word for h having $\hat{\mathbf{t}}$ as a prefix by repeatedly using Hurwitz operators to first move the letter t_{i_1} leftward (unchanged) to the first position, then moving t_{i_2} leftward (unchanged) to the second position, etc. □

2.3. *Fixed space codimension and reflection groups*

Suppose that the group G is given via a faithful representation, that is, G is a subgroup of $\mathrm{GL}_n(\mathbb{F}) = \mathrm{GL}(V)$, where $V = \mathbb{F}^n$ for some field $\mathbb{F}$. This gives rise to another subadditive function $G \to \mathbb{N}$, namely the fixed space codimension

$$g \mapsto \operatorname{codim}(V^g) = n - \dim(V^g).$$

PROPOSITION 2.9. *One has the subadditivity*

$$\operatorname{codim}(V^{gh}) \leqslant \operatorname{codim}(V^g) + \operatorname{codim}(V^h) \tag{2.5}$$

with equality occurring if and only if both of the following hold:

$$V^g + V^h = V \quad \text{and} \tag{2.6}$$

$$V^g \cap V^h = V^{gh}. \tag{2.7}$$

Proof. One has

$$\dim(V^g) + \dim(V^h) = \dim(V^g + V^h) + \dim(V^g \cap V^h) \leqslant n + \dim(V^g \cap V^h)$$

and hence

$$\operatorname{codim}(V^g) + \operatorname{codim}(V^h) \geqslant n - \dim(V^g \cap V^h) = \operatorname{codim}(V^g \cap V^h),$$

with equality if and only if (2.6) holds. Also, $V^g \cap V^h \subseteq V^{gh}$ and so

$$\operatorname{codim}(V^g \cap V^h) \geqslant \operatorname{codim}(V^{gh}),$$

with equality if and only if (2.7) holds. Hence

$$\operatorname{codim}(V^g) + \operatorname{codim}(V^h) \geqslant \operatorname{codim}(V^g \cap V^h) \geqslant \operatorname{codim}(V^{gh}),$$

with equality if and only if both conditions hold. □

It is natural to compare $\operatorname{codim}(V^g)$ with the length function $\ell(g) = \ell_T(g)$ from before.

DEFINITION 2.10 (Absolute length, absolute order). When a subgroup G of $\mathrm{GL}(V)$ has a subset T generating G as a monoid, so that (G,T) is a generated group, say that $\ell(g) = \ell_T(g)$ is an *absolute length function* if

$$\operatorname{codim}(V^g) = \ell(g) \quad \text{for all } g \text{ in } G. \tag{2.8}$$

In this situation, call the prefix order $\leqslant$ for (G,T) of Definition 2.2 the *absolute order* on G.

PROPOSITION 2.11. *Let (G,T) be a generated group with G a subgroup of $\mathrm{GL}(V)$.*

(i) *If $\ell(g)$ is an absolute length function, then G must be a reflection group and T must be the set of all reflections in G.*

(ii) *Conversely, if G is a reflection group and T its set of all reflections, one at least has*

$$\operatorname{codim}(V^g) \leqslant \ell(g) \quad \text{for all } g \text{ in } G.$$

Proof. Assertion (i) follows as $\operatorname{codim}(V^g) = 1$ if and only if g is a reflection, and $\ell_T(g) = 1$ if and only if g lies in T. For (ii), write $g = t_1 t_2 \cdots t_{\ell(g)}$ and use the subadditivity (2.5). $\square$

EXAMPLE 2.12. Carter showed [**9**, Lemma 2] that one has equality in (2.8) for any finite real reflection group $G \subset \mathrm{GL}_n(\mathbb{R})$.

EXAMPLE 2.13. On the other hand, motivated by considerations from the theory of deformation of skew group rings, Foster-Greenwood [**12**] analyzed the situation for finite *complex* reflection groups $G \subset \mathrm{GL}_n(\mathbb{C})$ that cannot be realized as real reflection groups and showed that in this case it is relatively rare to have equality in (2.8).

For example, the complex reflection group $G = G(4,2,2)$ is the set of monomial matrices in $\mathbb{C}^{2\times 2}$ whose two nonzero entries lie in $\{\pm 1, \pm i\}$ and have product ± 1. It has reflections

$$T = \left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

and different distributions for the functions $\operatorname{codim}(V^g)$ and $\ell(g)$:

$$\sum_{g \in G} t^{\operatorname{codim}(V^g)} = 1 + 6t + 9t^2 \quad \text{and} \quad \sum_{g \in G} t^{\ell_T(g)} = 1 + 6t + 7t^2 + 2t^3.$$

The two scalar matrices $\pm\left[\begin{smallmatrix} i & 0 \\ 0 & i \end{smallmatrix}\right]$ have reflection length 3; neither is a product of two reflections.

REMARK 2.14. Note that whenever G is a reflection group with an absolute length function, so $\ell(g) = \operatorname{codim}(V^g)$, the absolute order relation $\leqslant$ acquires yet another characterization via Proposition 2.9 (in addition to those in Definition 2.2 and Proposition 2.8). Specifically, $g \leqslant h$ if and only if one has both equalities

$$V^g + V^{g^{-1}h} = V \quad \text{and} \tag{2.9}$$

$$V^g \cap V^{g^{-1}h} = V^h. \tag{2.10}$$

EXAMPLE 2.15. Brady and Watt [**8**] considered the order $\leqslant$ defined via Remark 2.14 on real orthogonal groups and complex unitary groups acting on finite-dimensional spaces. They showed [**8**, Corollary 5] that such groups have an absolute length function when considered as reflection groups generated by their subset of reflections.

We come to our first main result, showing that the full general linear group $G = \mathrm{GL}(V)$ always has an absolute length function.

PROPOSITION 2.16. *Let $G = \mathrm{GL}_n(\mathbb{F}) = \mathrm{GL}(V)$ with $V = \mathbb{F}^n$ for some field $\mathbb{F}$ and consider the generated group (G,T), where T is the set of all reflections in G. Then every g in G has*

$$\ell(g) = \operatorname{codim}(V^g).$$

Proof. By Proposition 2.11, it suffices to show that $\ell(g) \leqslant \operatorname{codim}(V^g)$. This follows by induction on $\operatorname{codim}(V^g)$ if one can show that for any g in G other than the identity, there exists some t in T having $V^{gt} \supsetneq V^g$. We construct such a t explicitly. Choose an ordered basis $e_1, \ldots, e_n$ for $V = W \oplus W'$ so that $W' := V^g$ is spanned by $\{e_{m+1}, e_{m+2}, \ldots, e_n\}$. In this basis for V, we have

$$g = \begin{bmatrix} A & 0 \\ B & \mathbf{1}_{n-m} \end{bmatrix}$$

where A in $\mathrm{GL}_m(\mathbb{F})$ expresses the composite $W \overset{i_W}{\hookrightarrow} V \overset{g}{\to} V \overset{\pi_W}{\twoheadrightarrow} W$ in the basis $e_1, \ldots, e_m$.

We claim that by making a change of basis on W, one may assume that $e_m^\top A^{-1} e_m \neq 0$. To see this claim, fix any matrix Q in $\mathrm{GL}_m(\mathbb{F})$ (such as $Q = A^{-1}$) having $e_m^\top Q e_m = 0$. Since $Qe_m \neq \mathbf{0}$, there must exist some j in $\{1, 2, \ldots, m-1\}$ for which $e_j^\top Q e_m \neq 0$. Thus one may define an invertible change of basis P by $P(e_i) = e_i$ for $i \neq j$ and $P(e_j) = e_j + e_m$. Consequently, $P^{-1}(e_m) = e_m$ and $P^\top e_m = e_j + e_m$, so one can calculate that PQP^{-1} satisfies

$$e_m^\top PQP^{-1} e_m = (P^\top e_m)^\top Q e_m = (e_j + e_m)^\top Q e_m = e_j^\top Q e_m + e_m^\top Q e_m = e_j^\top Q e_m \neq 0.$$

Once one has $e_m^\top A^{-1} e_m \neq 0$, define the desired reflection t to fix the hyperplane spanned by $\{e_1, \ldots, e_n\} \setminus \{e_m\}$ and send e_m to $A^{-1}e_m \oplus (-BA^{-1}e_m)$ in $W \oplus W' = V$. One can check that $\det(t) = e_m^\top A^{-1} e_m \neq 0$, so that t does define a reflection in $\mathrm{GL}(V)$. Furthermore, both g and t fix $W' = V^g$ pointwise, so gt also fixes W' pointwise. However, the following shows that gt additionally fixes e_m, and hence $V^{gt} \supsetneq W' = V^g$, as desired:

$$gt(e_m) = g\begin{bmatrix} A^{-1}e_m \\ -BA^{-1}e_m \end{bmatrix} = \begin{bmatrix} A & 0 \\ B & \mathbf{1}_{n-m} \end{bmatrix} \cdot \begin{bmatrix} A^{-1}e_m \\ -BA^{-1}e_m \end{bmatrix} = \begin{bmatrix} A \cdot A^{-1}e_m \\ B \cdot A^{-1}e_m - BA^{-1}e_m \end{bmatrix} = e_m.$$

□

2.4. *Surjection onto subspace lattices*

Consider the lattice $L(V)$ of all $\mathbb{F}$-subspaces of $V = \mathbb{F}^n$ ordered by *reverse inclusion*[†]. For any subgroup G of $\mathrm{GL}(V)$, one has a map

$$\begin{array}{ccc} G & \xrightarrow{\pi} & L(V) \\ g & \longmapsto & V^g \end{array}. \tag{2.11}$$

If G is a reflection group with an absolute length, then Remark 2.14 shows that this map π is order preserving for the absolute order. Orlik and Solomon [**27**, Lemma 4.4] showed that if G is a finite complex reflection group in $\mathrm{GL}_n(\mathbb{C}) = \mathrm{GL}(V)$, then π is a surjection onto the subposet of $L(V)$ consisting of all subspaces that are intersections of reflection hyperplanes. Hence for finite real reflection groups, which have an absolute length, π is an order-preserving surjection onto this subposet. The next observation shows that the same holds for the full general linear groups. The proof is an easy exercise, left to the reader.

Proposition 2.17. *For $G = \mathrm{GL}(V)$ itself, the map (2.11) is an order-preserving surjection.*

Remark 2.18. Brady and Watt [**8**, Theorem 1] showed that the map (2.11) is also surjective, and in fact becomes a bijective order-isomorphism, when one restricts to a lower interval $[e, c]$ between the identity e and an element c having $V^c = \{0\}$ in real orthogonal or complex unitary groups. However, this bijectivity fails for general linear groups, when typically there are many elements below c having the same fixed space. For example, it is a special case of

[†]This matches, for example, the convention common in the theory of geometric lattices.

Theorem 4.2 below that there are $q^{n-2}(q^n-1)$ reflections in $[e,c] \subset \mathrm{GL}_n(\mathbb{F}_q)$, while there are only $(q^n-1)/(q-1)$ hyperplanes in $L(V)$.

Remark 2.19. For finite real reflection groups, orthogonal/unitary groups, and general linear groups, the absolute orders $\leqslant$ are not lattices because they have many incomparable maximal elements.

However, when one restricts to lower intervals $[e,c]$, absolute orders are sometimes lattices. For example, in the case of orthogonal/unitary groups, Brady and Watt's order-isomorphism $[e,c] \cong L(V)$ shows that every lower interval is a lattice. For irreducible finite real reflection groups in the case that c is chosen to be a *Coxeter element*, the fact that $[e,c]$ is a lattice was shown originally via a case-by-case check by Bessis [**5**, Fact 2.3.1] and later with a uniform proof by Reading [**28**, Corollary 8.6].

For the general linear groups $\mathrm{GL}(V) = \mathrm{GL}_n(\mathbb{F})$ with $n \geqslant 3$, the intervals $[e,c]$ are not lattices in general. For example, consider the embedding of $\mathbb{F}_{3^3}^\times$ into $\mathrm{GL}_3(\mathbb{F}_3) \cong \mathrm{GL}_{\mathbb{F}_3}(\mathbb{F}_{3^3})$ generated by the Singer cycle

$$c = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

The interval $[e,c]$ in $\mathrm{GL}_3(\mathbb{F}_3)$ below c contains the two reflections

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ 0 & 2 & 0 \end{bmatrix},$$

both of which are covered by three elements

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 2 & 0 \end{bmatrix}$$

of absolute length 2.

2.5. *Length functions when $T = T^{-1}$*

We close this section on $\ell(-)$ for a generated group (G,T) with two general facts that hold when $T = T^{-1}$, that is, when T is closed under taking inverses. They are reminiscent of properties of Coxeter group length functions.

Proposition 2.20. *For (G,T) a generated group with $T = T^{-1}$, any t in T and g in G have*

$$\ell(g) - 1 \leqslant \ell(tg), \ell(gt) \leqslant \ell(g) + 1.$$

Proof. Subadditivity immediately gives $\ell(gt), \ell(tg) \leqslant \ell(g) + 1$. Meanwhile

$$\ell(g) = \ell(gt \cdot t^{-1}) \leqslant \ell(gt) + 1,$$
$$\ell(g) = \ell(t^{-1} \cdot tg) \leqslant \ell(tg) + 1. \qquad \square$$

Note that $\ell(tg) = \ell(g) = \ell(gt)$ is possible, for example, whenever (G,T) is a reflection group whose set of all reflections T contains reflections t of order 3 or more, so that $\ell(t \cdot t) = \ell(t) = 1$.

PROPOSITION 2.21 (Exchange property). *Let (G,T) be a generated group with $T=T^{-1}$ and T closed under G-conjugation. If $\ell(tg)<\ell(g)$ for some $t\in T$ and g in G, then there is a T-reduced word $g=t_1\cdots t_k$ such that $tg={}^tt_1\cdots{}^tt_{i-1}\cdot t_{i+1}\cdots t_k$.*

Proof. If $\ell(tg)<\ell(g)$ for some $t\in T$ then Proposition 2.20 implies $\ell(tg)=\ell(g)-1$. Hence $t^{-1}\leqslant g$ and the subword property (Proposition 2.8) implies that t^{-1} is a subword of $(t_1,\ldots,t_k)$ for some T-reduced expression $g=t_1\cdots t_k$. If $t_i=t^{-1}$, then

$$tg=tt_1\cdots t_{i-1}t^{-1}t_{i+1}\cdots t_k={}^tt_1\cdots{}^tt_{i-1}\cdot t_{i+1}\cdots t_k. \qquad \square$$

3. *Counting ranks in the absolute order on* $\mathrm{GL}_n(\mathbb{F}_q)$

When the field $\mathbb{F}=\mathbb{F}_q$ is finite, so that $\mathrm{GL}_n:=\mathrm{GL}_n(\mathbb{F}_q)$ is finite, it is easy to give an explicit formula and generating function counting elements at rank k in the absolute order on GL_n, that is, those having fixed space codimension k. Such a formula, equivalent to (3.4) below, was derived[†] in work of Fulman [**14**, Theorem 6(1)] in a probabilistic context.

In the formula and elsewhere, we will use some standard q-analogues:

$$\begin{aligned}
(x;q)_n &:= (1-x)(1-xq)(1-xq^2)\cdots(1-xq^{n-1}),\\
[n]_q &:= 1+q+q^2+\cdots+q^{n-1},\\
[n]!_q &:= [1]_q[2]_q\cdots[n]_q=\frac{(q;q)_n}{(1-q)^n},\\
\begin{bmatrix} n\\ k\end{bmatrix}_q &:= \frac{[n]!_q}{[k]!_q[n-k]!_q}=\frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}=\#\{k\text{-dimensional }\mathbb{F}_q\text{-subspaces of }V=\mathbb{F}_q^n\}.
\end{aligned}$$

We mention for future use the fact that

$$|\mathrm{GL}_n(\mathbb{F}_q)|=(q^n-1)(q^n-q)(q^n-q^2)\cdots(q^n-q^{n-1})=(-1)^nq^{\binom{n}{2}}(q;q)_n \tag{3.1}$$

as well as the q-binomial theorem [**35**, (1.87)]:

$$(x;q)_n=\sum_{k=0}^{n}(-1)^kq^{\binom{k}{2}}\begin{bmatrix} n\\ k\end{bmatrix}_q x^k. \tag{3.2}$$

PROPOSITION 3.1. *The number of g in $\mathrm{GL}_n:=\mathrm{GL}_n(\mathbb{F}_q)$ having rank k in absolute order is*

$$r_q(n,k):=(-1)^kq^{\binom{k}{2}}\begin{bmatrix} n\\ k\end{bmatrix}_q\sum_{j=0}^{k}\begin{bmatrix} k\\ j\end{bmatrix}_q q^{j(n-k)}(q;q)_j \tag{3.3}$$

$$=\frac{|\mathrm{GL}_n|}{|\mathrm{GL}_{n-k}|}\sum_{j=0}^{k}\frac{(-1)^jq^{\binom{j}{2}-j(n-k)}}{|\mathrm{GL}_j|}, \tag{3.4}$$

with generating function

$$1+\sum_{n\geqslant 1}\left(\sum_{0\leqslant k\leqslant n}r_q(n,k)x^{n-k}\right)\frac{y^n}{|\mathrm{GL}_n|}=\frac{1}{1-y}\sum_{n\geqslant 0}\frac{(x;q^{-1})_n}{(q;q)_n}y^n. \tag{3.5}$$

[†]Fulman credits its first proof to unpublished work of Rudvalis and Shinoda [**33**].

Proof. The equivalence of formulas (3.3) and (3.4) is a straightforward exercise using (3.1). Thus we will derive (3.3), and then check that it agrees with (3.5).

By Proposition 2.16, we need to count elements in GL_n whose fixed subspace has codimension k. For a subspace W of $V = \mathbb{F}_q^n$, let

$$g(W) := |\{g \in G : V^g = W\}|,$$
$$f(W) := |\{g \in G : V^g \supseteq W\}| = \sum_{U \supseteq W} g(U),$$

so that if $\mathrm{codim}(W) = k$ one has

$$r_q(n,k) = \begin{bmatrix} n \\ k \end{bmatrix}_q g(W), \tag{3.6}$$
$$f(W) = q^{k(n-k)}|\mathrm{GL}_k| = q^{k(n-k)} \cdot (-1)^k q^{\binom{k}{2}} (q;q)_k.$$

Möbius inversion [**35**, Example 3.10.2] in the lattice of subspaces of $\mathbb{F}_q^n$ gives for $\mathrm{codim}(W) = k$,

$$g(W) = \sum_{U \supseteq W} \mu(W,U) f(U) = \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q (-1)^{k-j} q^{\binom{k-j}{2}} \cdot (-1)^j q^{j(n-j)+\binom{j}{2}} (q;q)_j$$

from which (3.3) follows via (3.6).

To check (3.5), use (3.2) to see that the coefficient of y^n on its right is

$$\sum_{m=0}^{n} \frac{(x;q^{-1})_m}{(q;q)_m} = \sum_{m=0}^{n} \frac{1}{(q;q)_m} \sum_{i=0}^{m} (-1)^i q^{-\binom{i}{2}} \begin{bmatrix} m \\ i \end{bmatrix}_{q^{-1}} x^i.$$

Therefore, the coefficient of $y^n x^{n-k}$ on the right side of (3.5) equals

$$(-1)^{n-k} q^{-\binom{n-k}{2}} \sum_{m=n-k}^{n} \frac{1}{(q;q)_m} \begin{bmatrix} m \\ n-k \end{bmatrix}_{q^{-1}}.$$

Reindexing $j := n - m$ in the summation, and using the fact that

$$\begin{bmatrix} a+b \\ a \end{bmatrix}_{q^{-1}} = q^{-ab} \begin{bmatrix} a+b \\ a \end{bmatrix}_q,$$

one then finds that the coefficient of $y^n x^{n-k}/|\mathrm{GL}_n|$ on the right side of (3.5) equals

$$|\mathrm{GL}_n| \cdot (-1)^{n-k} q^{-\binom{n-k}{2}} \sum_{j=0}^{k} \frac{q^{-(n-k)(k-j)}}{(q;q)_{n-j}} \begin{bmatrix} n-j \\ n-k \end{bmatrix}_q = (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q q^{j(n-k)} (q;q)_j,$$

which agrees with the formula (3.3) for $r_q(n,k)$. $\square$

Remark 3.2. The formula (3.3) for $r_q(n,k)$ is reminiscent of the inclusion–exclusion formula

$$\binom{n}{k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)!$$

counting permutations with $n-k$ fixed points. On the other hand, it seems more natural to think of $r_q(n,k)$ as a q-analogue of $c(n, n-k)$, the *signless Stirling number of the first kind*, counting the permutations in the symmetric group $\mathfrak{S}_n$ having $n-k$ cycles: when $\mathfrak{S}_n$ acts as a real reflection group permuting coordinates in $V=\mathbb{R}^n$, a permutation σ with $n-k$ cycles has $\operatorname{codim}(V^\sigma)=k$. In this sense, Equation (3.5) gives a q-analogue of the formula

$$1+\sum_{n\geqslant 1}\sum_{0\leqslant k\leqslant n} c(n,n-k)x^{n-k}\frac{y^n}{n!}=(1-y)^{-x}=\sum_{k=0}^{\infty}(-1)^k\binom{-x}{k}y^k,$$

particularly when one observes that

$$\frac{(x;q^{-1})_k}{(q;q)_k}=\begin{bmatrix}N\\k\end{bmatrix}_q \text{ if } x=q^N.$$

4. *Counting chains below a Singer cycle in* $\mathrm{GL}_n(\mathbb{F}_q)$

In the theory of finite irreducible real reflection groups, the interval $[e,c]$ in absolute order below a *Coxeter element* c is sometimes called the poset $NC(W)$ of W-*noncrossing partitions*. It is extremely well behaved from several enumerative points of view, including pleasant formulas for its cardinality, its *Möbius function*, and its *zeta polynomial*. In the classical types $A,B/C,D$ one additionally has formulas for the following more refined counts; see Edelman [**11**, Theorem 3.2] for type A, Reiner [**29**, Proposition 7] for types B/C, and Athanasiadis–Reiner [**4**, Theorem 1.2(ii)] for type D.

Definition 4.1. Fix a reflection group G having an absolute order and an element c of G with $\ell(c)=n$. The flag *f-vector* (f_α) of the interval $[e,c]$ has entries $f_\alpha:=f_\alpha[e,c]$ indexed by *compositions* $\alpha=(\alpha_1,\ldots,\alpha_m)$ of $n=\sum_i\alpha_i$ with $\alpha_i>0$. The entry $f_\alpha[e,c]$ is the number of chains

$$e=c_0<c_1<c_2<\cdots<c_{m-1}<c_m=c$$

in which c_i has rank $\alpha_1+\alpha_2+\cdots+\alpha_i$ for each i.

Since $c_{i-1}<c_i$ if and only if $g_i:=c_{i-1}^{-1}c_i$ has $\ell(g_i)=\ell(c_i)-\ell(c_{i-1})$, one can rephrase the definition as

$$f_\alpha[e,c]=\left|\{(g_1,\ldots,g_m)\in G^m : c=g_1\cdots g_m, \text{ and } \ell(g_i)=\alpha_i \text{ for each } i\}\right|.$$

As mentioned in the Introduction, when viewing $\mathrm{GL}_n(\mathbb{F}_q)$ as a finite reflection group, the role analogous to that of a Coxeter element is played by a *Singer cycle* c, which is the image of a multiplicative generator for $\mathbb{F}_{q^n}^\times$ after it is embedded into $\mathrm{GL}_n(\mathbb{F}_q)\cong\mathrm{GL}_{\mathbb{F}_q}(\mathbb{F}_{q^n})$; see [**32**, § 9], [**31**, Theorem 19], [**23**]. Our goal in this section is to prove an unexpectedly simple formula for the flag f-vector $f_\alpha[e,c]$ when c is a Singer cycle; see Theorem 4.2 below. The special case where $\alpha=(1,1,\ldots,1)$ appeared in Lewis, Reiner, and Stanton [**23**], where it was shown that there are exactly $(q^n-1)^{n-1}$ maximal chains in $[e,c]$ (equivalently, minimal factorizations of a Singer cycle into reflections).

In fact, the theorem also confirms a special case† of [**23**, Conjecture 6.3]: it applies not only to a Singer cycle c, but to any element c in $\mathrm{GL}_n(\mathbb{F}_q)$ which is *regular elliptic*, meaning that c stabilizes no proper subspaces in $\mathbb{F}_q^n$. (Equivalently, regular elliptic elements are those that act on $V=\mathbb{F}_q^n$ with characteristic polynomial which is irreducible in $\mathbb{F}_q[x]$; see [**23**, Proposition 4.4] for other equivalent definitions.)

To state the theorem, define the quantity

$$\varepsilon(\alpha):=\sum_{i=1}^{m}(\alpha_i-1)(n-\alpha_i).$$

†Theorem 4.2 confirms the special case [**23**, Conjecture 6.3 at $\ell=n$]. In forthcoming work [**22**], the second author and Morales use the same techniques to confirm [**23**, Conjecture 6.3] in full generality.

THEOREM 4.2. *For any regular elliptic element c in $\mathrm{GL}_n(\mathbb{F}_q)$ and any composition $\alpha = (\alpha_1, \ldots, \alpha_m)$ of n, one has*

$$f_\alpha[e, c] = q^{\varepsilon(\alpha)} \cdot (q^n - 1)^{m-1}. \tag{4.1}$$

In particular, the number of elements of $[e, c]$ of rank k for $1 \leqslant k \leqslant n-1$ is

$$f_{(k,n-k)}[e, c] = q^{2k(n-k)-n} \cdot (q^n - 1). \tag{4.2}$$

We remark that Theorem 4.2 appears very reminiscent of a special case of Goulden and Jackson's *cactus formula*, counting the *genus zero* factorizations $\sigma = \sigma_1 \cdots \sigma_m$ of an n-cycle σ; these are the factorizations that are additive $\sum_{i=1}^m \ell(\sigma_i) = \ell(\sigma)$ for the absolute length function given by $\ell(\tau) = \sum_j (\lambda_j - 1)$ if τ has cycle sizes $(\lambda_1, \lambda_2, \ldots)$. (This is the same length function discussed in Remark 3.2.) To state it, we need the following notation: given a partition $\lambda = 1^{m_1} 2^{m_2} 3^{m_i} \cdots$ having m_i parts of size i and $m := \sum_i m_i$ parts total, we define

$$N(\lambda) = \frac{1}{m} \binom{m}{m_1, m_2, \ldots}.$$

If $\lambda = (\lambda_1, 1^{n-\lambda_1})$ corresponds to a permutation with only one nontrivial cycle then $N(\lambda) = 1$.

THEOREM 4.3 (Cactus formula [**17**, Theorem 3.2]). *For an n-cycle σ in the symmetric group $\mathfrak{S}_n$, the number of factorizations $\sigma = \sigma_1 \cdots \sigma_m$ that*

- *are additive, that is, $\sum_i \ell(\sigma_i) = n - 1 (= \ell(\sigma))$, and*
- *have σ_i with cycle sizes $(\lambda_1^{(i)}, \lambda_2^{(i)}, \ldots) = \lambda^{(i)}$*

is given by

$$n^{m-1} \prod_{i=1}^m N(\lambda^{(i)}).$$

In particular, in the special case where each σ_i has only one nontrivial cycle, the number of factorizations is given by

$$n^{m-1}. \tag{4.3}$$

We currently lack a combinatorial proof of Theorem 4.2; see Question 4.13. Instead, prompted by the similarity between (4.1) and (4.3), we prove the former by following a q-analogue of a proof of the latter due to Zagier; see [**20**, § A.2.4]. We sketch here the steps in Zagier's proof and give the q-analogous steps in the subsections below.

The first step is the same for both proofs, namely a representation-theoretic approach to counting factorizations that goes back to Frobenius; see, for example, [**20**, § A.1.3] for a proof.

DEFINITION 4.4. Given a finite group G, let $\mathrm{Irr}(G)$ be the set of its irreducible ordinary (finite-dimensional, complex) representations U. For each U in $\mathrm{Irr}(G)$, define its *character* $\chi_U(-)$, *degree* $\chi_U(e)$, and *normalized character* $\widetilde{\chi}_U(-)$ by

$$\chi_U(g) := \mathrm{Tr}(g : U \to U),$$

$$\chi_U(e) = \dim_{\mathbb{C}} U,$$

$$\widetilde{\chi}_U(g) := \frac{\chi_U(g)}{\chi_U(e)}.$$

Both functions $\chi_U(-), \widetilde{\chi}_U(-)$ on G extend $\mathbb{C}$-linearly to functions on the *group algebra* $\mathbb{C}[G]$.

In the following, we will frequently conflate a representation U with its character χ_U without comment.

PROPOSITION 4.5 (Frobenius [**13**]). *Let G be a finite group, and let $A_1, \dots, A_m \subset G$ be unions of conjugacy classes in G. Let $z_i := \sum_{g_i \in A_i} g_i$ in $\mathbb{C}[G]$. Then for each g in G,*

$$|\{(g_1, \dots, g_m) \in A_1 \times \cdots \times A_m : g = g_1 \cdots g_m\}| = \frac{1}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} \chi(e)\chi(g^{-1}) \prod_{i=1}^{m} \widetilde{\chi}(z_i). \tag{4.4}$$

Zagier's proof of Theorem 4.3 applies Proposition 4.5 by following these four steps.

Step 1. One observes that, when applying (4.4) to count factorizations of an n-cycle in $G = \mathfrak{S}_n$, the summation is much *sparser* than it looks initially. Irreducible $\mathfrak{S}_n$-characters χ^λ are indexed by partitions λ of n, but the only χ^λ which do not vanish on an n-cycle σ are the *hook shapes*, that is, those of the form $\lambda = (n-d, 1^d)$ for $d = 0, 1, \dots, n-1$. These satisfy

$$\chi^{(n-d,1^d)}(\sigma) = (-1)^d \quad \text{and} \quad \chi^{(n-d,1^d)}(e) = \binom{n-1}{d}.$$

Hence Proposition 4.5 shows that the number of additive factorizations $\sigma = \sigma_1 \cdots \sigma_m$ in which each σ_i has cycle type $\lambda^{(i)}$ is

$$\frac{1}{n!} \sum_{d=0}^{n-1} (-1)^d \binom{n-1}{d} P(d), \quad \text{where} \quad P(d) := \prod_{i=1}^{m} \widetilde{\chi}^{(n-d,1^d)}(z_i) \tag{4.5}$$

and each z_i is the sum in $\mathbb{C}[\mathfrak{S}_n]$ of all permutations of cycle type $\lambda^{(i)}$.

Step 2. One shows that each normalized character value $\widetilde{\chi}^{(n-d,1^d)}(z_i)$ appearing as a factor in (4.5) is the specialization at $x = d$ of a polynomial $P_{\lambda^{(i)}}(x)$ in $\mathbb{Q}[x]$. This polynomial has degree $\sum_j (\lambda_j^{(i)} - 1)$ and a predictable, explicit leading coefficient. Thus the product $P(d)$ is also the specialization of a polynomial $P(x)$ in $\mathbb{Q}[x]$, having degree $n-1$ and a predictable, explicit leading coefficient.

Step 3. Note that the Nth iterate $\Delta^N := \Delta \circ \cdots \circ \Delta$ of the forward difference operator

$$\Delta(f)(x) := f(x+1) - f(x) \tag{4.6}$$

satisfies

$$(\Delta^N f)(x) = \sum_{d=0}^{N} (-1)^d \binom{N}{d} f(x+d). \tag{4.7}$$

Hence the sum (4.5) is the $(n-1)$st forward difference of $P(x)$ evaluated at $x = 0$, that is, $(\Delta^{n-1}P)(0)$.

Step 4. For each integer $m \geqslant 0$, one has

$$\Delta(x^m) = (x+1)^m - x^m = mx^{m-1} + O(x^{m-2}),$$

and so the operator Δ lowers degree by 1 and scales by m the leading coefficient of a degree-m polynomial. Hence the polynomial $P(x)$ from Step 2 has $\Delta^{n-1}P = (\Delta^{n-1}P)(0)$ equal to a constant, namely $(n-1)!$ times the leading coefficient of $P(x)$. Thus our answer (4.5), which is equal to $(\Delta^{(n-1)}P)(0)/n!$ by Step 3, is $(n-1)!/n! = 1/n$ times the leading coefficient of $P(x)$ computed in Step 2.

In the next four subsections, we describe what we view as q-analogues of Steps 1–4, in order to prove Theorem 4.2. As a preliminary step, take $\mathrm{GL}_n := \mathrm{GL}_n(\mathbb{F}_q)$, acting on $V = \mathbb{F}_q^n$, and define for $k = 0, 1, \ldots, n$, the element z_k in $\mathbb{C}[\mathrm{GL}_n]$ to be the sum of all elements g for which $\mathrm{codim}(V^g) = k$. Then Definition 4.1 and Proposition 4.5 show that

$$f_\alpha[e, c] = \frac{1}{|\mathrm{GL}_n|} \sum_{\chi \in \mathrm{Irr}(\mathrm{GL}_n)} \chi(e)\chi(c^{-1}) \prod_{i=1}^{m} \widetilde{\chi}(z_{\alpha_i}). \tag{4.8}$$

4.1. *A q-analogue of Step 1*

Just as in Step 1 above, one observes that for a regular elliptic element c in GL_n, the summation (4.8) is much *sparser* than it looks initially, as many GL_n-irreducibles have $\chi(c^{-1}) = 0$.

To explain this, we begin with a brief outline of some of the theory of complex characters of $\mathrm{GL}_n(\mathbb{F}_q)$. The theory was first developed by Green [**18**], building on Steinberg's work [**37**] constructing the unipotent characters $\chi^{\mathbf{1},\lambda}$. It has been reworked several times, for example, by Macdonald [**24**, Chapters III and IV] and Zelevinsky [**38**, § 11].

Definition 4.6. A key notion is the *parabolic* or *Harish-Chandra induction* $\chi_1 * \chi_2$ of two characters χ_1, χ_2 for $\mathrm{GL}_{n_1}, \mathrm{GL}_{n_2}$ to give a character of GL_n where $n = n_1 + n_2$. To define it, introduce the parabolic subgroup

$$P_{n_1,n_2} := \left\{ \begin{bmatrix} A_1 & B \\ 0 & A_2 \end{bmatrix} \text{ in } \mathrm{GL}_n \right\} \tag{4.9}$$

so that A_i lies in GL_{n_i} for $i = 1, 2$, and B is arbitrary in $\mathbb{F}_q^{n_1 \times n_2}$. Then

$$(\chi_1 * \chi_2)(g) := \frac{1}{|P_{n_1,n_2}|} \sum_{\substack{h \in G: \\ hgh^{-1} \in P_{n_1,n_2}}} \chi_1(A_1)\chi_2(A_2), \tag{4.10}$$

where the element hgh^{-1} of P_{n_1,n_2} has diagonal blocks labeled A_1, A_2 as above. Said differently, $\chi_1 * \chi_2 := (\chi_1 \otimes \chi_2) \Uparrow_{\mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2}}^{P_{n_1,n_2}} \uparrow_{P_{n_1,n_2}}^{\mathrm{GL}_n}$, where

- $(-) \Uparrow_{\mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2}}^{P_{n_1,n_2}}$ is *inflation* of representations of $\mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2}$ into those of P_{n_1,n_2}, by precomposing with the surjection $P \twoheadrightarrow \mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2}$, and
- $(-) \uparrow_{P_{n_1,n_2}}^{\mathrm{GL}_n}$ is *induction* of representations.

The binary operation $(\chi_1, \chi_2) \longmapsto \chi_1 * \chi_2$ turns out [**38**, Chapter III] to define an associative, commutative (!), graded $\mathbb{C}$-algebra structure on $\bigoplus_{n \geqslant 0} \mathrm{Class}(\mathrm{GL}_n)$, where $\mathrm{Class}(\mathrm{GL}_n)$ denotes the $\mathbb{C}$-vector space of class functions on GL_n, with $\mathrm{Class}(\mathrm{GL}_0) := \mathbb{C}$.

Definition 4.7. An irreducible U in $\mathrm{Irr}(\mathrm{GL}_n)$ is called *cuspidal*, with *weight* $\mathrm{wt}(U) = n$, if U is not a constituent of any proper induction $\chi_1 * \chi_2$ for characters χ_i of GL_{n_i} with $n = n_1 + n_2$ and $n_1, n_2 \geqslant 1$.

Denote by Cusp_n the set of weight-n cuspidal characters, and $\mathrm{Cusp} := \sqcup_{n \geqslant 0} \mathrm{Cusp}_n$.

Definition 4.8. An irreducible GL_n-character is called *primary* to the cuspidal U if χ *does* occur as an irreducible constituent of some product $U^{* \frac{n}{s}} = U * U * \cdots * U$, where $\mathrm{wt}(U) = s$.

It turns out that one can parameterize the irreducible GL_n-characters primary to the cuspidal U as $\{\chi^{U,\lambda} : |\lambda| = \frac{n}{s}\}$, parallel to the parameterization of the irreducible $\mathfrak{S}_n$-characters as

$\{\chi^\lambda : |\lambda| = n\}$. In fact, two primary irreducibles $\chi^{U,\mu}, \chi^{U,\nu}$ for $\mathrm{GL}_{n_1}, \mathrm{GL}_{n_2}$ primary to the same cuspidal U have product controlled by the usual *Littlewood–Richardson coefficients*:

$$\chi^{U,\mu} * \chi^{U,\nu} = \sum_\lambda c^\lambda_{\mu,\nu} \chi^{U,\lambda}, \quad \text{where} \quad (\chi^\mu \otimes \chi^\nu) \uparrow^{\mathfrak{S}_{|\mu|+|\nu|}}_{\mathfrak{S}_{|\mu|} \times \mathfrak{S}_{|\nu|}} = \sum_\lambda c^\lambda_{\mu,\nu} \chi^\lambda.$$

Furthermore, the set of *all* irreducibles $\mathrm{Irr}(\mathrm{GL}_n)$ can be indexed as $\{\chi^{\underline{\lambda}}\}$ in which $\underline{\lambda}$ runs through the functions $\underline{\lambda} : U \longmapsto \lambda(U)$ from Cusp to all integer partitions, subject to the restriction $\sum_U \mathrm{wt}(U) \cdot |\lambda(U)| = n$. In this parameterization,

$$\chi^{\underline{\lambda}} = \chi^{U_1,\lambda(U_1)} * \cdots * \chi^{U_m,\lambda(U_m)}$$

if $\{U_1, \ldots, U_m\}$ are the cuspidals having $\lambda(U_i) \neq \varnothing$.

We next recall from [**23**] the sparsity statement analogous to that of Step 1, showing that most irreducible GL_n-characters χ vanish on a regular elliptic element. We also include the character values and a degree formula for certain irreducibles that arise in our computation.

Proposition 4.9 [**23**, Proposition 4.7]. *Let c in GL_n be regular elliptic, for example, a Singer cycle.*

(i) *The irreducible character $\chi^{\underline{\lambda}}$ has vanishing value $\chi^{\underline{\lambda}}(c) = 0$ unless χ is a primary irreducible $\chi^{U,\lambda}$ for some cuspidal U with $\mathrm{wt}(U) = s$ dividing n, and $\lambda = (\frac{n}{s} - d, 1^d)$ is a hook-shaped partition of $\frac{n}{s}$.*

(ii) *If $U = \mathbf{1} = \mathbf{1}_{\mathrm{GL}_1}$ is the trivial cuspidal with $s = \mathrm{wt}(U) = 1$, then*

$$\chi^{\mathbf{1},(n-d,1^d)}(c) = (-1)^d \quad \text{and} \quad \chi^{\mathbf{1},(n-d,1^d)}(e) = q^{\binom{d+1}{2}} \begin{bmatrix} n-1 \\ d \end{bmatrix}_q.$$

4.2. *A q-analogue of Step 2*

Of course, to use (4.8) we also need some character values on the elements z_k. These are provided by the following remarkable result, which was suggested by computations in GAP [**15**].

Proposition 4.10. *One has these normalized character values on z_k for certain $\widetilde{\chi}^{U,\lambda}$.*

(i) *For any primary irreducible GL_n-character $\chi^{U,\lambda}$ with the cuspidal $U \neq \mathbf{1}$ nontrivial,*

$$\widetilde{\chi}^{U,\lambda}(z_k) = (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q.$$

(ii) *For $U = \mathbf{1}$ and $\lambda = (n-d, 1^d)$ a hook, we have*

$$\widetilde{\chi}^{\mathbf{1},(n-d,1^d)}(z_k) = \mathcal{P}_k(q^{-d})$$

where $\mathcal{P}_k(x)$ is the following polynomial in x of degree k:

$$\mathcal{P}_k(x) := (-1)^k q^{\binom{k}{2}} \left(\begin{bmatrix} n \\ k \end{bmatrix}_q + \frac{1-q^n}{[n-k]!_q} \sum_{j=1}^{k} \frac{[n-j]!_q}{[k-j]!_q} q^{j(n-k)} x \cdot (xq^{n-j+1}; q)_{j-1} \right). \tag{4.11}$$

Remark 4.11. Taking $d = 0$ in Proposition 4.10(ii), the character $\widetilde{\chi}^{\mathbf{1},(n)}$ is the trivial character $\mathbf{1}_{\mathrm{GL}_n}$. Hence $\widetilde{\chi}^{\mathbf{1},(n)}(z_k) = r_q(n,k)$ is the kth rank size for the absolute order on GL_n, as computed in Proposition 3.1. It is not hard to check that the formula for $r_q(n,k)$ given there is consistent with the $d = 0$ case of Proposition 4.10(ii), that is, with $\mathcal{P}_k(1)$.

The proof of Proposition 4.10, by necessity, involves some technical $\mathrm{GL}_n(\mathbb{F}_q)$-character computations. Some readers may wish to skip ahead to the q-analogue of Step 3 (Section 4.3).

Proof of Proposition 4.10. We begin the proof of both assertions (i) and (ii) with a Möbius function calculation as in the proof of Proposition 3.1.

Fix a character χ. Since χ is a class function, one has for any fixed subspace X in V of codimension k that

$$\widetilde{\chi}(z_k) := \sum_{\substack{g\in\mathrm{GL}_n:\\ \mathrm{codim}(V^g)=k}} \widetilde{\chi}(g) = \begin{bmatrix} n\\ k\end{bmatrix}_q F(X) \quad \text{where } F(X) := \sum_{\substack{g\in\mathrm{GL}_n:\\ V^g=X}} \widetilde{\chi}(g).$$

Rather than $F(X)$, it is more convenient to compute

$$G(X) := \sum_{\substack{g\in\mathrm{GL}_n:\\ V^g\supseteq X}} \widetilde{\chi}(g) = \sum_{Y: X\subseteq Y\subseteq V} F(Y).$$

Then by Möbius inversion [**35**, Example 3.10.2] on the lattice of subspaces of $\mathbb{F}_q^n$, we have

$$F(X) = \sum_{Y: X\subseteq Y\subseteq V} (-1)^{\dim Y - \dim X} q^{\binom{\dim Y-\dim X}{2}} G(Y).$$

It follows that

$$\widetilde{\chi}(z_k) = \begin{bmatrix} n\\ k\end{bmatrix}_q \sum_{j=0}^{k} (-1)^{k-j} q^{\binom{k-j}{2}} \begin{bmatrix} k\\ j\end{bmatrix}_q G(Y), \tag{4.12}$$

where $Y := Y_j$ is any particular subspace of codimension j. Thus it only remains to compute $G(X)$, where X is a particular codimension-k subspace; for concreteness, we take X to be the span of the first $n-k$ standard basis vectors in V.

If $k=0$ then $X=V$ and $G(X)=\widetilde{\chi}(e)=1$. Thus, in what follows we assume $k\geqslant 1$.

Abbreviate a tower of groups

$$\begin{array}{ccccc} \mathrm{GL}_n & \supset & P & \supset & H \\ & & \| & & \| \\ & & \left\{\begin{bmatrix} A_1 & B\\ 0 & A_2\end{bmatrix}\right\} & & \left\{\begin{bmatrix} \mathbf{1}_{n-k} & B\\ 0 & A_2\end{bmatrix}\right\} \end{array}$$

in which P is the parabolic (block upper triangular) subgroup stabilizing X (*not* necessarily pointwise), and H is the subgroup of P that fixes X pointwise. Also recall that inside P one finds the block-diagonal product group $\mathrm{GL}_{n-k}\times\mathrm{GL}_k$. Still fixing a GL_n-character χ, we compute

$$G(X) = \sum_{h\in H} \widetilde{\chi}(h) = \frac{|H|}{\chi(e)} \left\langle \chi\downarrow_H^{\mathrm{GL}_n},\quad \mathbf{1}_H \right\rangle_H = \frac{|H|}{\chi(e)} \left\langle \chi,\quad \mathbf{1}_H\uparrow_H^{\mathrm{GL}_n} \right\rangle_{\mathrm{GL}_n}$$

via Frobenius Reciprocity for induction $(-)\uparrow_H^{\mathrm{GL}_n}$ and restriction $(-)\downarrow_H^{\mathrm{GL}_n}$. The map sending

$$p = \begin{bmatrix} A_1 & B\\ 0 & A_2\end{bmatrix} \longmapsto A_1$$

induces a bijection $P/H \to \mathrm{GL}_{n-k}$ showing that the left-translation action of p on cosets P/H is isomorphic to the left-regular action of A_1 on GL_{n-k}. Hence

$$\mathbf{1}_H\uparrow_H^P = (\mathbb{C}\mathrm{GL}_{n-k}\otimes \mathbf{1}_{\mathrm{GL}_k}) \Uparrow_{\mathrm{GL}_{n-k}\times\mathrm{GL}_k}^P,$$

where $\mathbb{C}\mathrm{GL}_{n-k}$ is the regular representation of GL_{n-k}, and recall that $(-)\Uparrow^{P}_{\mathrm{GL}_{n-k}\times\mathrm{GL}_k}$ denotes inflation of a $\mathrm{GL}_{n-k}\times\mathrm{GL}_k$-representation to a P-representation by precomposing with the surjection $P \twoheadrightarrow \mathrm{GL}_{n-k}\times\mathrm{GL}_k$. Hence, via transitivity of induction, one can rewrite

$$\mathbf{1}_H\uparrow_H^{\mathrm{GL}_n} = \left(\mathbf{1}_H\uparrow_H^P\right)\uparrow_P^{\mathrm{GL}_n} = (\mathbb{C}\mathrm{GL}_{n-k}\otimes\mathbf{1}_{\mathrm{GL}_k})\Uparrow^{P}_{\mathrm{GL}_{n-k}\times\mathrm{GL}_k}\uparrow_P^{\mathrm{GL}_n} = \mathbb{C}\mathrm{GL}_{n-k} * \mathbf{1}_{\mathrm{GL}_k}.$$

To apply (4.12), we need to compute for $\mathrm{codim}(X)=k\geqslant 1$ the values

$$\begin{aligned} G(X) &= \frac{|H|}{\chi(e)}\langle\chi,\quad \mathbb{C}\mathrm{GL}_{n-k} * \mathbf{1}_{\mathrm{GL}_k}\rangle_{\mathrm{GL}_n} \\ &= \frac{|H|}{\chi(e)}\sum_{\underline{\lambda}}\chi^{\underline{\lambda}}(e)\left\langle\chi,\quad \chi^{\underline{\lambda}} * \chi^{\mathbf{1}_{\mathrm{GL}_1},(k)}\right\rangle_{\mathrm{GL}_n} \end{aligned} \tag{4.13}$$

with $\chi^{\underline{\lambda}}$ running through $\mathrm{Irr}(\mathrm{GL}_{n-k})$. We compute this now for χ as in assertions (i), (ii).

Assertion (i) Here $\chi=\chi^{U,\lambda}$ with $U\neq\mathbf{1}_{\mathrm{GL}_1}$. In this case, $\langle\chi^{U,\lambda}\quad \chi^{\underline{\lambda}} * \chi^{\mathbf{1}_{\mathrm{GL}_1},(k)}\rangle_{\mathrm{GL}_n}$ always vanishes, since $\chi^{\underline{\lambda}} * \chi^{\mathbf{1}_{\mathrm{GL}_1},(k)}$ cannot have the primary irreducible $\chi^{U,\lambda}$ as a constituent: its irreducible constituents $\chi^{\underline{\mu}}$ must each have μ assigning the cuspidal $\mathbf{1}_{\mathrm{GL}_1}$ a partition of weight at least k, and hence are not irreducibles primary to U. Consequently, (4.12) gives the desired answer

$$\widetilde{\chi}^{U,\lambda}(z_k) = \begin{bmatrix} n\\ k\end{bmatrix}_q (-1)^k q^{\binom{k}{2}}\begin{bmatrix} k\\ 0\end{bmatrix}_q\cdot 1 = (-1)^k q^{\binom{k}{2}}\begin{bmatrix} n\\ k\end{bmatrix}_q.$$

(ii) Here $\chi=\chi^{\mathbf{1}_{G_1},(n-d,1^d)}$. We claim that $k>0$ and *Pieri's rule* [**24**, (5.16)] for expanding the induction product of χ^{λ} and $\chi^{(k)}$ imply that almost every $\chi^{\underline{\lambda}}$ in $\mathrm{Irr}(\mathrm{GL}_{n-k})$ has the inner product $\langle\chi^{\mathbf{1}_{\mathrm{GL}_1},(n-d,1^d)},\quad \chi^{\underline{\lambda}} * \chi^{\mathbf{1}_{\mathrm{GL}_1},(k)}\rangle_{\mathrm{GL}_n}$ vanishing, unless both

- $k\leqslant n-d$, and
- $\chi^{\underline{\lambda}}=\chi^{\mathbf{1}_{\mathrm{GL}_1},\lambda}$ for either $\lambda=(n-k-d,1^d)$ or $(n-k-d+1,1^{d-1})$,

in which case the inner product is 1. Hence, starting with (4.13), we compute

$$\begin{aligned} G(X) &= \frac{|H|}{\chi^{\mathbf{1}_{\mathrm{GL}_1},(n-d,1^d)}(e)}\sum_{\underline{\lambda}}\chi^{\underline{\lambda}}(e)\left\langle\chi^{\mathbf{1}_{\mathrm{GL}_1},(n-d,1^d)},\quad \chi^{\underline{\lambda}} * \chi^{\mathbf{1}_{\mathrm{GL}_1},(k)}\right\rangle_{\mathrm{GL}_n} \\ &= \frac{|H|}{\chi^{\mathbf{1}_{\mathrm{GL}_1},(n-d,1^d)}(e)}\left(\chi^{\mathbf{1}_{\mathrm{GL}_1},(n-k-d,1^d)}(e)+\chi^{\mathbf{1}_{\mathrm{GL}_1},(n-k-d+1,1^{d-1})}(e)\right) \\ &= \frac{|H|}{q^{\binom{d+1}{2}}\begin{bmatrix} n-1\\ d\end{bmatrix}_q}\left(q^{\binom{d+1}{2}}\begin{bmatrix} n-k-1\\ d\end{bmatrix}_q + q^{\binom{d}{2}}\begin{bmatrix} n-k-1\\ d-1\end{bmatrix}_q\right) \\ &= (-1)^k(q;q)_k q^{\binom{k}{2}+k(n-k)-d}\begin{bmatrix} n-k\\ d\end{bmatrix}_q \Big/ \begin{bmatrix} n-1\\ d\end{bmatrix}_q \end{aligned}$$

Plugging this result into (4.12), after separating out the $j=0$ summand, gives

$$\widetilde{\chi}^{\mathbf{1},(n-d,1^d)}(z_k) = (-1)^k q^{\binom{k}{2}}\begin{bmatrix} n\\ k\end{bmatrix}_q\left(1+\sum_{j=1}^{\min(k,n-d)}(-1)^j q^{j(n-k)-d}\begin{bmatrix} k\\ j\end{bmatrix}_q\frac{(-1)^j(q;q)_j\begin{bmatrix} n-j\\ d\end{bmatrix}_q}{\begin{bmatrix} n-1\\ d\end{bmatrix}_q}\right)$$

$$= (-1)^k q^{\binom{k}{2}} \left(\begin{bmatrix} n \\ k \end{bmatrix}_q + \frac{1-q^n}{[n-k]!_q} \sum_{j=1}^{\min(k,n-d)} q^{j(n-k)-d} (q^{n-j-d+1};q)_{j-1} \frac{[n-j]!_q}{[k-j]!_q} \right)$$

$$= \mathcal{P}_k(q^{-d}). \qquad \square$$

4.3. *A q-analogue of Step* 3

We are now well equipped to analyze the summation in (4.8) by breaking it into two pieces:

$$f_\alpha[e,c] = \frac{1}{|\mathrm{GL}_n|} \sum_{\substack{\chi \in \mathrm{Irr}(\mathrm{GL}_n): \\ \chi(c^{-1}) \neq 0}} \chi(e)\chi(c^{-1}) \prod_{i=1}^m \widetilde{\chi}(z_{\alpha_i}) = \frac{1}{|\mathrm{GL}_n|}(A+B) \tag{4.14}$$

where A is the sum over primary irreducibles $\chi^{U,\lambda}$ with $U \neq \mathbf{1} = \mathbf{1}_{\mathrm{GL}_1}$ and B is the sum over primary irreducibles of the form $\chi^{\mathbf{1},\lambda}$. By Proposition 4.10(i), one has

$$A = \prod_{i=1}^m (-1)^{\alpha_i} q^{\binom{\alpha_i}{2}} \begin{bmatrix} n \\ \alpha_i \end{bmatrix}_q \sum_{\substack{\chi^{U,\lambda} \in \mathrm{Irr}(\mathrm{GL}_n): \\ U \neq \mathbf{1}}} \chi^{U,\lambda}(e)\chi^{U,\lambda}(c^{-1}).$$

However, Proposition 4.9(i) lets one rewrite this last summation as

$$\sum_{\substack{\chi^{U,\lambda} \in \mathrm{Irr}(\mathrm{GL}_n): \\ U \neq \mathbf{1}}} \chi^{U,\lambda}(e)\chi^{U,\lambda}(c^{-1}) = \sum_{\substack{\chi \in \mathrm{Irr}(\mathrm{GL}_n): \\ \chi(c^{-1}) \neq 0}} \chi(e)\chi(c^{-1}) - \sum_{d=0}^{n-1} \chi^{\mathbf{1},(n-d,1^d)}(e)\chi^{\mathbf{1},(n-d,1^d)}(c^{-1}).$$

The first sum on the right side is the character of the *regular representation* for GL_n evaluated at c, and hence is equal to 0. By Proposition 4.9(ii) and the q-binomial theorem (3.2), the second sum on the right side is

$$\sum_{d=0}^{n-1} (-1)^d q^{\binom{d+1}{2}} \begin{bmatrix} n-1 \\ d \end{bmatrix}_q = (q;q)_{n-1}.$$

Thus one concludes that

$$A = -(q;q)_{n-1} \prod_{i=1}^m (-1)^{\alpha_i} q^{\binom{\alpha_i}{2}} \begin{bmatrix} n \\ \alpha_i \end{bmatrix}_q . \tag{4.15}$$

Next we analyze the sum B in (4.14). For a composition α, define $\mathcal{P}_\alpha(x) = \prod_i \mathcal{P}_{\alpha_i}(x)$. By Propositions 4.9 and 4.10 and the definition of B, we may rewrite

$$B = \sum_{d=0}^{n-1} (-1)^d q^{\binom{d+1}{2}} \begin{bmatrix} n-1 \\ d \end{bmatrix}_q \mathcal{P}_\alpha(q^{-d}). \tag{4.16}$$

We identify B in terms of the $(n-1)$st iterate of a q-difference operator Δ_q. This operator is the q-analogue of (4.6) defined by

$$\Delta_q(f)(x) = \frac{f(qx) - f(x)}{qx - x} = \frac{f(qx) - f(x)}{(q-1)x}.$$

One can check via the q-Pascal recurrence

$$\begin{bmatrix} N \\ d \end{bmatrix}_q = \begin{bmatrix} N-1 \\ d \end{bmatrix}_q + q^{N-d} \begin{bmatrix} N-1 \\ d-1 \end{bmatrix}_q$$

and induction that for $N \geqslant 0$, the Nth iterate $\Delta_q^N = \Delta_q \circ \cdots \circ \Delta_q$ has the following expression:

$$\Delta_q^N(f)(x) = q^{-\binom{N}{2}}(q-1)^{-N} \sum_{d=0}^{N} (-1)^d q^{\binom{d}{2}} \begin{bmatrix} N \\ d \end{bmatrix}_q \frac{f(q^{N-d}x)}{x^N}. \tag{4.17}$$

(This is q-analogous to (4.7).) Taking $N = n-1$ in (4.17) and applying the operator to $\mathcal{P}_\alpha(x)/x$ gives

$$\begin{aligned}\left[\Delta_q^{n-1}\left(\frac{\mathcal{P}_\alpha(x)}{x}\right)\right]_{x=q^{1-n}} &= q^{-\binom{n-1}{2}}(q-1)^{1-n} \sum_{d=0}^{n-1} (-1)^d q^{\binom{d}{2}} \begin{bmatrix} n-1 \\ d \end{bmatrix}_q \left[\frac{1}{x^{n-1}} \frac{\mathcal{P}_\alpha(q^{n-1-d}x)}{(q^{n-1-d}x)}\right]_{x=q^{1-n}} \\ &= q^{-\binom{n-1}{2}+(n-1)^2}(q-1)^{1-n} \sum_{d=0}^{n-1} (-1)^d q^{\binom{d+1}{2}} \begin{bmatrix} n-1 \\ d \end{bmatrix}_q \mathcal{P}_\alpha(q^{-d}).\end{aligned}$$

Combining with (4.16) gives

$$B = (q-1)^{n-1} q^{-\binom{n}{2}} \left[\Delta_q^{n-1}\left(\frac{\mathcal{P}_\alpha(x)}{x}\right)\right]_{x=q^{1-n}}. \tag{4.18}$$

4.4. *A q-analogue of Step 4*

We process the expression (4.18) for B further. It is easily verified by induction on $N \geqslant 0$ that for any m,

$$\Delta_q^N(x^m) = \frac{(q^m-1)(q^{m-1}-1)\cdots(q^{m-N+1}-1)}{(q-1)^N} \cdot x^{m-N} = \frac{(q^{m-N+1};q)_N}{(1-q)^N} \cdot x^{m-N}.$$

In particular, for integer m one has

$$\Delta_q^N(x^m) = \begin{cases} 0 & \text{if } N > m \geqslant 0, \\ [m]!_q & \text{if } N = m \geqslant 0, \\ (-1)^N q^{\binom{N+1}{2}} [N]!_q \cdot x^{-N-1} & \text{if } m = -1. \end{cases} \tag{4.19}$$

Proposition 4.12. *For any composition $\alpha = (\alpha_1, \ldots, \alpha_m)$ of n, the function $\mathcal{P}_\alpha(x) = \prod_i \mathcal{P}_{\alpha_i}(x)$*

- *is a polynomial in x of degree n,*
- *has leading coefficient equal to $q^{\varepsilon(\alpha)+n(n-1)} \cdot (q^n-1)^m$, and*
- *has constant coefficient equal to $-A/(q;q)_{n-1}$.*

Proof. Note from the definition (4.11) of $\mathcal{P}_k(x)$ that it is a polynomial in x of degree k, with constant coefficient $(-1)^k q^{\binom{k}{2}} \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q$. Hence $\mathcal{P}_\alpha(x)$ is polynomial in x of degree $\sum_i \alpha_i = n$ with constant coefficient

$$\prod_{i=1}^{m} (-1)^{\alpha_i} q^{\binom{\alpha_i}{2}} \begin{bmatrix} n \\ \alpha_i \end{bmatrix}_q = \frac{-A}{(q;q)_{n-1}},$$

where the last equality uses (4.15). One sees that in (4.11), the x^k coefficient in $\mathcal{P}_k(x)$ is entirely accounted for by the $j = k$ summand and is equal to

$$(-1)^k q^{\binom{k}{2}+k(n-k)+\sum_{j=n-k+1}^{n-1} j} \cdot (q^n-1) = q^{k(n-k)+n(k-1)} \cdot (q^n-1).$$

Therefore, the product $\mathcal{P}_\alpha(x) = \prod_i \mathcal{P}_{\alpha_i}(x)$ has leading coefficient

$$q^{\sum_i \alpha_i(n-\alpha_i)+n(\alpha_i-1)} \cdot (q^n-1)^m = q^{\varepsilon(\alpha)+n(n-1)} \cdot (q^n-1)^m. \qquad \square$$

As $\mathcal{P}_\alpha(x)$ has degree n in x, the quotient $\mathcal{P}_\alpha(x)/x$ is a Laurent polynomial with top degree $n-1$ and bottom degree -1. Therefore, combining Proposition 4.12 with (4.19) gives

$$\begin{aligned}\Delta_q^{n-1}\left(\frac{\mathcal{P}_\alpha(x)}{x}\right) &= (-1)^{n-1}q^{-\binom{n}{2}}[n-1]!_q \cdot x^{-n} \cdot \frac{-A}{(q;q)_{n-1}} + [n-1]!_q q^{\varepsilon(\alpha)+n(n-1)} \cdot (q^n-1)^m \\ &= [n-1]!_q \left((-1)^{n-1}q^{-\binom{n}{2}} \frac{-A}{x^n \cdot (q;q)_{n-1}} + q^{\varepsilon(\alpha)+n(n-1)} \cdot (q^n-1)^m\right).\end{aligned}$$

Plugging this into (4.18) and using $(q-1)^{n-1}[n-1]!_q = (-1)^{n-1}(q;q)_{n-1}$ gives

$$\begin{aligned}B &= (-1)^{n-1}q^{-\binom{n}{2}}(q;q)_{n-1}\left((-1)^{n-1}q^{-\binom{n}{2}} \frac{-A}{q^{-n(n-1)}(q;q)_{n-1}} + q^{\varepsilon(\alpha)+n(n-1)} \cdot (q^n-1)^m\right) \\ &= -A + (-1)^{n-1}(q;q)_{n-1}q^{\varepsilon(\alpha)+\binom{n}{2}} \cdot (q^n-1)^m.\end{aligned}$$

Using (3.1), one can finally compute from (4.14) that

$$f_\alpha[e,c] = \frac{1}{|\mathrm{GL}_n|}(A+B) = \frac{(-1)^{n-1}(q;q)_{n-1}q^{\varepsilon(\alpha)+\binom{n}{2}}}{(-1)^n(q;q)_n q^{\binom{n}{2}}} \cdot (q^n-1)^m = q^{\varepsilon(\alpha)} \cdot (q^n-1)^{m-1}.$$

This concludes the proof of Theorem 4.2.

The preceding proof is computational and unenlightening. This prompts the following questions.

Question 4.13. Biane [**6**] has given a short, inductive proof of (4.3) not relying on any auxiliary objects (trees, maps, etc.). Is there an analogous proof of Theorem 4.2?

Question 4.14. Is there a reasonable q-analogue of the cactus formula (Theorem 4.3) in full generality, not just in the special case (4.3)?

We currently have no conjectural candidate for such a q-analogue.

5. *Reformulating the flag f-vector*

The goal of this section is to prove Proposition 5.2, a linear algebraic reformulation of $f_\alpha[e,c]$ when $V^c = 0$. We hope that this reformulation may be more amenable to combinatorial counting methods. In particular, we show below that it helps recover somewhat more directly the rank sizes for $[e,c]$ given in (4.2).

Definition 5.1. Fix a field $\mathbb{F}$, and let V be an n-dimensional $\mathbb{F}$-vector space.

Given a sequence $g_\bullet := (g_0, g_1, \ldots, g_{m-1}, g_m)$ with g_i in $\mathrm{GL}(V)$, define a sequence of subspaces $\varphi(g_\bullet) := (V_1, \ldots, V_m)$ via

$$V_i := V^{g_{i-1}} \cap V^{g_i^{-1}g_m}.$$

Fix c in $\mathrm{GL}(V)$. Given an ordered vector space decomposition $V_\bullet = (V_i)_{i=1}^m$ of V, so that

$$V = \underbrace{V_1 \oplus V_2 \oplus \cdots \oplus V_i}_{=:V_{\leqslant i}} \oplus \underbrace{V_{i+1} \oplus V_{i+2} \oplus \cdots \oplus V_m}_{=:V_{>i}},$$

define a sequence $\psi(V_\bullet) := (g_0, g_1, \ldots, g_{m-1}, g_m)$ of $\mathbb{F}$-linear maps $g_i : V \to V$ by

$$g_i(x+y) = c(x) + y \quad \text{for } x, y \text{ in } V_{\leqslant i}, V_{>i}, \text{ respectively.}$$

Proposition 5.2. *Let $V=\mathbb{F}^n$ for a field $\mathbb{F}$, and let c lie in $G:=\mathrm{GL}(V)$ with $V^c=0$. Then the maps φ,ψ restrict to inverse bijections between these two sets:*

(a) *multichains $g_\bullet := (e=g_0\leqslant g_1\leqslant\cdots\leqslant g_{m-1}\leqslant g_m=c)$ in absolute order on G and*
(b) *decompositions $V_\bullet=(V_i)_{i=1}^m$ satisfying $V=c(V_{\leqslant i})\oplus V_{>i}$ for every $i=0,1,\ldots,m$.*

Moreover, they satisfy $\dim V_i=\ell(g_i)-\ell(g_{i-1})$.

In particular, when $\mathbb{F}=\mathbb{F}_q$ is finite, for any composition $\alpha=(\alpha_1,\ldots,\alpha_m)$ of n, the flag number $f_\alpha[e,c]$ counts decompositions $V_\bullet$ as in (b) having $\dim V_i=\alpha_i$ for $i=1,2,\ldots,m$.

Proof. Given $g_\bullet$ as in (a), we wish to show that $\phi(g_\bullet)=(V_1,\ldots,V_m)$ is as in (b). First note that Proposition 2.5 and $e\leqslant g_{i-1}\leqslant g_i\leqslant c$ imply $g_i^{-1}c\leqslant g_{i-1}^{-1}c$. Thus, from Remark 2.14 we have

$$\begin{array}{ccccc} V & = & V^{g_i} & \oplus & V^{g_i^{-1}c} \\ & & \cap & & \cup \\ V & = & V^{g_{i-1}} & \oplus & V^{g_{i-1}^{-1}c}. \end{array} \tag{5.1}$$

As a first goal, we show $V=\bigoplus_{i=1}^m V_i$ via induction on m, with the base case $m=1$ being trivial. In the inductive step, remove g_1 from $g_\bullet$ to give $g'_\bullet=(e\leqslant g_2\leqslant\cdots\leqslant g_{m-1}\leqslant c)$. Then $\varphi(g'_\bullet)=(U_2,V_3,V_4,\ldots,V_m)$ satisfies $V=U_2\oplus(\bigoplus_{i=3}^m V_i)$ by induction. Moreover, note that

$$U_2=V\cap V^{g_2^{-1}c}=(V^{g_1^{-1}c}\oplus V^{g_1})\cap V^{g_2^{-1}c}=V^{g_1^{-1}c}\oplus(V^{g_1}\cap V^{g_2^{-1}c})=V_1\oplus V_2,$$

where the second-to-last equality uses $V^{g_1^{-1}c}\subset V^{g_2^{-1}c}$ from (5.1). Hence $V=\bigoplus_{i=1}^m V_i$.

We also claim $V_{\leqslant i}=V^{g_i^{-1}c}$ and $V_{>i}=V^{g_i}$. To see this, note that for each $j\leqslant i$ one has

$$V_j=(V^{g_{j-1}}\cap V^{g_j^{-1}c})\subset V^{g_j^{-1}c}\subset V^{g_i^{-1}c}$$

by (5.1), and hence $V_{\leqslant i}\subseteq V^{g_i^{-1}c}$; a similar argument shows that $V_{>i}\subseteq V^{g_i}$. But then

$$V=V_{\leqslant i}\oplus V_{>i}=V^{g_i^{-1}c}\oplus V^{g_i}$$

forces the claimed equalities, as well as $\dim(V_i)=\ell(g_i)-\ell(g_{i-1})$. Lastly, applying g_i to the decomposition in (5.1), one obtains the final desired property for (b):

$$V=g_iV=g_i(V^{g_i^{-1}c}\oplus V^{g_i})=g_iV^{g_i^{-1}c}\oplus g_iV^{g_i}=cV^{g_i^{-1}c}\oplus V^{g_i}=cV_{\leqslant i}\oplus V_{>i}.$$

Conversely, given $V_\bullet$ as in (b), we must show that $\psi(V_\bullet)=g_\bullet$ is as in (a). The assumption that $V=cV_{\leqslant i}\oplus V_{>i}$ shows that $g_iV=V$, and hence each g_i is invertible.

We claim $V^c=0$ shows $V^{g_i}=V_{>i}$: expressing $v=x+y$ uniquely with x,y in $V_{\leqslant i},V_{>i}$, one has v in V^{g_i} if and only if $c(x)+y=x+y$ if and only if $c(x)=x$ if and only if $x=0$. Similarly, $V^{g_{i-1}^{-1}g_i}=V_{\leqslant i-1}\oplus V_{>i}$. Hence

$$\ell(g_{i-1})+\ell(g_{i-1}^{-1}g_i)=\dim V_{\leqslant i-1}+\dim V_i=\dim(V_{\leqslant i})=\ell(g_i).$$

Thus $g_{i-1}<g_i$ and so $g_\bullet$ satisfies (a).

Finally, one can check ϕ and ψ are inverses to each other. □

Alternate proof of Equation (4.2), *via Proposition* 5.2. Choose c in $\mathrm{GL}(V)$ regular elliptic. By Proposition 2.5, it is enough to show that for $1\leqslant k\leqslant n/2$, there are

$$f_{(k,n-k)}[e,c]=q^{\varepsilon((k,n-k))}\cdot(q^n-1)=q^{2k(n-k)-n}\cdot(q^n-1)$$

elements g in $[e,c]$ of rank k. By Proposition 5.2, these elements are in bijection with direct sum decompositions

$$V=\mathbb{F}_q^n=U\oplus W=cU\oplus W$$

where $\dim U = k$. Count such decompositions by first choosing U, and then choosing W complementary to both U and cU. The number of choices of W depends only on $k = \dim U$ and $d := \dim(U \cap cU)$, and thus it helps to have the following very special case of a general formula due to Chen and Tseng [**10**, p. 28]: for a regular elliptic element c in $\mathrm{GL}_n(\mathbb{F}_q)$, there are

$$g(n,k,d) := \frac{[n]_q}{[k]_q} \begin{bmatrix} n-k-1 \\ k-d-1 \end{bmatrix}_q \begin{bmatrix} k \\ d \end{bmatrix}_q q^{(k-d)(k-d-1)}$$

subspaces U of $\mathbb{F}_q^n$ for which $\dim U = k$ and $\dim(U \cap cU) = d$, assuming $0 \leqslant d < k < n$.

Given two k-dimensional subspaces U_1, U_2 of V with $\dim(U_1 \cap U_2) = d$ (such as $U_1 = U$ and $U_2 = cU$ above), it is a straightforward exercise to check that when $0 \leqslant d \leqslant k \leqslant n/2$ there are

$$f(n,k,d) := q^{k(n-k) - \binom{k-d+1}{2}} (-1)^{k-d} (q;q)_{k-d} \tag{5.2}$$

subspaces W with $V = U_1 \oplus W = U_2 \oplus W$. Thus

$$\begin{aligned} f_\alpha[e,c] &= \sum_{d=0}^{k-1} g(n,k,d) f(n,k,d) \\ &= \sum_{d=0}^{k-1} \frac{[n]_q}{[k]_q} \begin{bmatrix} n-k-1 \\ k-d-1 \end{bmatrix}_q \begin{bmatrix} k \\ d \end{bmatrix}_q q^{(k-d)(k-d-1)} \cdot q^{k(n-k) - \binom{k-d+1}{2}} (-1)^{k-d} (q;q)_{k-d} \\ &= (q^n - 1) q^{k(n-k)-1} \sum_{d=0}^{k-1} \begin{bmatrix} k-1 \\ d \end{bmatrix}_q (q^{n-k-1} - 1)(q^{n-k-1} - q) \cdots (q^{n-k-1} - q^{k-d-2}). \end{aligned}$$

Finally, we apply the special case

$$q^{ab} = \sum_{d=0}^{a} \begin{bmatrix} a \\ d \end{bmatrix}_q (q^b - 1)(q^b - q) \cdots (q^b - q^{a-d-1})$$

of the q-Chu–Vandermonde identity [**16**, II.6] with $(a,b) = (k-1, n-k-1)$ to conclude. $\square$

Remark 5.3. Both the Chen–Tseng result and the needed case of the q-Chu–Vandermonde identity have elementary proofs: in the former case by a complicated recursive argument, and in the latter case by counting matrices in $\mathbb{F}_q^{a \times b}$ by their row spaces (see, for example, [**21**]).

6. *Final remarks and questions*

It was shown by Athanasiadis, Brady, and Watt [**2**] that the noncrossing partition posets $[e,c]$ for Coxeter elements c in real reflection groups are EL-shellable; this was extended to well-generated complex reflection groups by Mühle [**26**]. In particular, the open intervals (e,c) are homotopy Cohen–Macaulay. They also have predictable Euler characteristics, that is, Möbius functions $\mu(e,c)$.

Analogously, Theorem 4.2 allows one to compute for regular elliptic elements c in $\mathrm{GL}_n(\mathbb{F}_q)$ that the interval $[e,c]$ in the absolute order on $\mathrm{GL}_n(\mathbb{F}_q)$ has

$$\mu(e,c) = \sum_{\alpha = (\alpha_1, \ldots, \alpha_m)} (-1)^m f_\alpha[e,c] = \sum_{\alpha = (\alpha_1, \ldots, \alpha_m)} (-1)^m q^{\varepsilon(\alpha)} \cdot (q^n - 1)^{m-1}.$$

We do not suggest any simplifications for this last expression.

QUESTION 6.1. Is the open interval (e, c) in the absolute order on $\mathrm{GL}_n(\mathbb{F}_q)$ homotopy Cohen–Macaulay? Is it furthermore shellable?

Homotopy Cohen–Macaulayness would imply two weaker conditions:

(i) $(-1)^{\ell(y)-\ell(x)}\mu(x,y) \geqslant 0$ for all $x \leqslant y$ in $[e,c]$, and
(ii) for $i < n-2$, one has vanishing reduced homology $\tilde{H}_i((e,c),\mathbb{Z}) = 0$.

Condition (i) is easily seen to hold for $n = 2$ or $n = 3$ and any q; in addition, we have checked by direct computation that it holds for $n = 4$ if $q = 2$ or 3.

Condition (ii) is trivial for $n = 2$. For $n = 3$, it amounts to connectivity of the bipartite graph which is the Hasse diagram for (e, c), and one can give a direct proof (using Proposition 5.2) that this graph is connected. For $n = 4$ and $q = 2$, we have checked in Sage [**36**] that $\tilde{H}_i((e,c),\mathbb{Z}) = 0$ for $i = 0, 1$ and $\tilde{H}_2((e,c),\mathbb{Z}) = \mathbb{Z}^{|\mu(e,c)|} = \mathbb{Z}^{1034}$.

Similarly, it was shown by Athanasiadis and Kallipoliti [**3**] that, after removing the bottom element e, the absolute order on all of $\mathfrak{S}_n$ gives a *constructible* simplicial complex, and hence also this poset is homotopy Cohen–Macaulay. In type B_n, it is open whether removing the bottom element from the absolute order gives a homotopy Cohen–Macaulay complex; however, Kallipoliti [**19**] showed that when one restricts to the order ideal which is the union of all intervals below Coxeter elements, one obtains a homotopy Cohen–Macaulay complex.

QUESTION 6.2. After removing the bottom element from the absolute order on all of $\mathrm{GL}(V)$, say for $V = \mathbb{F}_q^n$, does one obtain a homotopy Cohen–Macaulay simplicial complex? What about the order ideal which is the union of all intervals below Singer cycles?

For example, for $\mathrm{GL}_3(\mathbb{F}_2)$, every maximal element in the absolute order is already a Singer cycle, so that the two simplicial complexes in Question 6.2 are the same. Both have reduced simplicial homology vanishing in dimensions $0, 1$, and isomorphic to $\mathbb{Z}^{838}$ in dimension 2.

In terms of Sperner theory, the poset $[e, c]$ is rank-symmetric and rank-unimodal by (4.2) and is self-dual by Proposition 2.5. This raises a question, suggested by Kyle Petersen.

QUESTION 6.3. For every Singer cycle c in $\mathrm{GL}_n(\mathbb{F}_q)$, does the absolute order interval $[e, c]$ have a symmetric chain decomposition?

The local self-duality proven in Proposition 2.5 also implies that, for any c in $\mathrm{GL}_n(\mathbb{F}_q)$, the *Ehrenborg quasi-symmetric function* encoding the flag f-vector of the ranked poset $[e, c]$ will actually be a symmetric function; see [**34**, Theorem 1.4]. When c is regular elliptic, Theorem 4.2 lets one compute this symmetric function explicitly, but we did not find the results suggestive.

Lastly, we ask how the poset $[e, c]$ in $\mathrm{GL}_n(\mathbb{F}_q)$ depends upon the choice of Singer cycle c.

QUESTION 6.4. Do all Singer cycles c in $\mathrm{GL}_n(\mathbb{F}_q)$ have isomorphic posets $[e, c]$?

Certainly $[e, c]$ and $[e, c']$ are poset-isomorphic whenever c, c' are conjugate, and whenever $c' = c^{-1}$. However, not all Singer cycles can be related by conjugacy and taking inverses. A similar issue arises for Coxeter elements c in finite reflection groups W. For real reflection groups, all Coxeter elements *are* W-conjugate. For well-generated complex reflection groups, they are all related by what Marin and Michel [**25**] call *reflection automorphisms*, and these give rise to the desired poset isomorphisms $[e, c] \cong [e, c']$; see Reiner, Ripoll, and Stump [**30**].

REMARK 6.5. In spite of Theorem 4.2, within some $\mathrm{GL}_n(\mathbb{F}_q)$ there exist *regular elliptic* elements c' and Singer cycles c for which $[e, c'] \ncong [e, c]$. For example, the Singer cycles in $\mathrm{GL}_4(\mathbb{F}_2)$ are the elements c with characteristic polynomial $t^4 + t + 1$ or $t^4 + t^3 + 1$, whereas the elements c' having characteristic polynomial $1 + t + t^2 + t^3 + t^4$ are regular elliptic but not Singer cycles; such c' have multiplicative order $5 \neq 15 = 2^4 - 1 = |\mathbb{F}_{2^4}^\times|$. One can check that $[e, c] \not\cong [e, c']$, for example by computing the determinants of the $\{0, 1\}$-incidence matrices between ranks 1 and 3 for the two intervals.

Acknowledgements. The authors thank Christos Athanasiadis, Valentin Féray, Alejandro Morales, Kyle Petersen, Dennis Stanton, and an anonymous referee for helpful remarks, suggestions, and references.

References

1. D. Armstrong, 'Generalized noncrossing partitions and combinatorics of Coxeter groups', *Mem. Amer. Math. Soc.* 202 (2009) x+159.

2. C. A. Athanasiadis, T. Brady and C. Watt, 'Shellability of noncrossing partition lattices', *Proc. Amer. Math. Soc.* 135 (2007) 939–949 (electronic).

3. C. A. Athanasiadis and M. Kallipoliti, 'The absolute order on the symmetric group, constructible partially ordered sets and Cohen-Macaulay complexes', *J. Combin. Theory Ser. A* 115 (2008) 1286–1295.

4. C. A. Athanasiadis and V. Reiner, 'Noncrossing partitions for the group D_n', *SIAM J. Discrete Math.* 18 (2004) 397–417 (electronic).

5. D. Bessis, 'The dual braid monoid', *Ann. Sci. École Norm. Sup.* (4) 36 (2003) 647–683.

6. P. Biane, 'Minimal factorizations of a cycle and central multiplicative functions on the infinite symmetric group', *J. Combin. Theory Ser. A* 76 (1996) 197–212.

7. T. Brady and C. Watt, '$K(\pi, 1)$'s for Artin groups of finite type', In *Proceedings of the Conference on Geometric and Combinatorial Group Theory*, Part I (Haifa, 2000) 94 (2002) 225–250.

8. T. Brady and C. Watt, 'A partial order on the orthogonal group', *Comm. Algebra* 30 (2002) 3749–3754.

9. R. W. Carter, 'Conjugacy classes in the Weyl group', *Compos. Math.* 25 (1972) 1–59.

10. E. Chen and D. Tseng, 'The splitting subspace conjecture', *Finite Fields Appl.* 24 (2013) 15–28.

11. P. H. Edelman, 'Chain enumeration and noncrossing partitions', *Discrete Math.* 31 (1980) 171–180.

12. B. Foster-Greenwood, 'Comparing codimension and absolute length in complex reflection groups', *Commun. Algebra* 42 (2014) 4350–4365.

13. F. G. Frobenius, 'Uber gruppencharacktere', *Gesammelte Abhandlungen. Bände III*, (Herausgegeben von J.-P. Serre; Springer-Verlag, Berlin, 1968).

14. J. Fulman, 'A probabilistic approach toward conjugacy classes in the finite general linear and unitary groups', *J. Algebra* 212 (1999) 557–590.

15. The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.7.7*, 2015.

16. G. Gasper and M. Rahman, *Basic hypergeometric series, volume 96 of Encyclopedia of Mathematics and its Applications.* (Cambridge University Press, Cambridge, UK, 2nd edn, 2004).

17. I. P. Goulden and D. M. Jackson, 'The combinatorial relationship between trees, cacti and certain connection coefficients for the symmetric group', *Eur. J. Combin.* 13 (1992) 357–365.

18. J. A. Green, 'The characters of the finite general linear groups', *Trans. Amer. Math. Soc.* 80 (1955) 402–447.

19. M. Kallipoliti, 'The absolute order on the hyperoctahedral group', *J. Algebraic Combin.* 34 (2011) 183–211.

20. S. K. Lando and A. K. Zvonkin, *Graphs on surfaces and their applications, volume 141 of Encyclopaedia of Mathematical Sciences* (Springer-Verlag, Berlin, 2004).

21. G. Landsberg, 'Ueber eine Anzahlbestimmung und eine damit zusammenhängende Reihe', *J. Reine Ang. Math.* 111 (1893) 87–88.

22. J. B. Lewis and A. Morales, '$GL(n, q)$-analogues of factorization problems in the symmetric group', *Eur. J. Combin.* 58 (2016) 75–95.

23. J. B. Lewis, V. Reiner and D. Stanton, 'Reflection factorizations of Singer cycles', *J. Algebraic Combin.* 40 (2014) 663–691.

24. I. G. Macdonald, *Symmetric functions and Hall polynomials.* Oxford Mathematical Monographs (The Clarendon Press, Oxford University Press, New York, 2nd edn, 1995).

25. I. Marin and J. Michel, 'Automorphisms of complex reflection groups', *Represent. Theory* 14 (2010) 747–788.

26. H. Mühle, 'EL-shellability and noncrossing partitions associated with well-generated complex reflection groups', *Eur. J. Combin.* 43 (2015) 249–278.

27. P. Orlik and L. Solomon, 'Unitary reflection groups and cohomology', *Invent. Math.* 59 (1980) 77–94.

28. N. Reading, 'Noncrossing partitions and the shard intersection order', *J. Algebraic Combin.* 33 (2011) 483–530.

29. V. Reiner, 'Non-crossing partitions for classical reflection groups', *Discrete Math.* 177 (1997) 195–222.

30. V. Reiner, V. Ripoll and C. Stump, 'On non-conjugate Coxeter elements in well-generated reflection groups', Preprint, arxiv:1404.5522, 2014.

31. V. Reiner, D. Stanton and P. Webb, 'Springer's regular elements over arbitrary fields', *Math. Proc. Cambridge Philos. Soc.* 141 (2006) 209–229.

32. V. Reiner, D. Stanton and D. White, 'The cyclic sieving phenomenon', *J. Combin. Theory Ser. A* 108 (2004) 17–50.
33. A. Rudvalis and K. Shinoda, 'An enumeration in finite classical groups', 1988. UMass Amherst Department of Mathematics Technical Report.
34. R. P. Stanley, 'Flag-symmetric and locally rank-symmetric partially ordered sets', *Electron. J. Combin.* 3 (1996) Research Paper 6, approx. 22 pp. (electronic). The Foata Festschrift.
35. R. P. Stanley, *Enumerative combinatorics*. Volume 1, Cambridge Studies in Advanced Mathematics 49 (Cambridge University Press, Cambridge, 2nd edn, 2012).
36. W. A. Stein, *et al.*, *Sage Mathematics Software* (*Version 6.4.1*), The Sage Development Team, 2014. http://www.sagemath.org.
37. R. Steinberg, 'A geometric approach to the representations of the full linear group over a Galois field', *Trans. Amer. Math. Soc.* 71 (1951) 274–282.
38. A. V. Zelevinsky, *Representations of finite classical groups, a Hopf algebra approach*, Lecture Notes in Mathematics 869 (Springer-Verlag, Berlin, 1981).

Jia Huang
Department of Mathematics and Statistics
University of Nebraska at Kearney
Kearney, NE 68849
USA

huangj2@unk.edu

Joel Brewster Lewis and Victor Reiner
School of Mathematics
University of Minnesota
Minneapolis, MN 55455
USA

jblewis@math.umn.edu
reiner@math.umn.edu