JACOBI-TRUDI DETERMINANTS OVER FINITE FIELDS

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ABSTRACT. We study the basis of the ring of symmetric functions by constructing random ring homomorphisms from the ring of symmetric functions to finite fields $\mathbb{F}_q$. In particular, we aim to answer the question: what is the probability of some Schur function mapping to zero as we map each elementary symmetric function to $\mathbb{F}_q$ uniformly at random. In this paper, We show that this probability is always at least $1/q$ and is asymptotically $1/q$. Moreover, we give a complete classification of all shapes that can achieve probability $1/q$. In addition, we further research some conditional probabilities and the probabilities of some Schur functions mapping to other values in $\mathbb{F}_q$.

1. INTRODUCTION

1.1. Motivation and Problem Statement.

A symmetric polynomial is defined to be a polynomial which stays unchanged by any permutation of its variables. This definition can be adapted to generic functions, but we are only concerned about polynomial functions as little theory has been established on symmetric non-polynomial functions. The study of symmetric functions has never stopped as it has extremely wide applications to many areas of math.

Let Λ denote the algebra of symmetric functions over $\mathbb{Z}$, the ring of integers. The structure of Λ, as a $\mathbb{Z}$-module, is generally well-understood by some of its famous bases: the monomial symmetric functions $\{m_\lambda\}$, the elementary symmetric functions $\{e_\lambda\}$, the complete homogeneous symmetric functions $\{h_\lambda\}$ and the Schur functions $\{s_\lambda\}$ [1], where these functions are indexed by partitions λ. And as a $\mathbb{Z}$-algebra, the elementary symmetric functions $\{e_i\}_{i\in\mathbb{Z}_{\geq 1}}$ and the complete homogeneous symmetric functions $\{h_i\}_{i\in\mathbb{Z}_{\geq 1}}$ are both bases of Λ. We will give definitions to these functions in Section 1.2. Transforming from one basis to another carries a lot of combinatorial meanings, as shown in [1].

As a standard method to understand a complicated structure, we will try to study the relations between these standard bases by forming ring homomorphisms from Λ to $\mathbb{F}_q$, the finite field of order q. In particular, we pose the following question.

Question. If we take a random ring homomorhpism $\Lambda \to \mathbb{F}_q$ by picking the image of $\{h_i\}_{i\in\mathbb{Z}_{\geq 1}}$ (or $\{e_i\}_{i\in\mathbb{Z}_{\geq 1}}$) uniformly at random, what is the probability that $s_\lambda \mapsto 0$, for some certain partition shape λ?

The Jacobi-Trudi identity (Theorem 1.8) [1] gives us a way to write s_λ as a determinant of h_i's (e_i's). As a result, our problem is essentially equivalent to studying certain random matrices over the finite field $\mathbb{F}_q$. The literature of random matrices is vast on its own.

In this paper, we show that the specific matrices we are looking at, which we call "Jacobi-Trudi matrices", have very nice properties in terms of the probability that the determinants

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go to zero. In particular, we show that the probability of s_λ mapping to zero is at least $1/q$ (Theorem 3.1) and is asymptotically $1/q$ as $q \to \infty$ (Theorem 3.5). Moreover, we give a complete classification of all shapes λ such that the probability of its Jacobi-Trudi matrix being singular is $1/q$ (Theorem 5.8). In fact, they are hooks (Definition 1.3), rectangles (Definition 1.5) and staircases (Definition 1.6). In addition, we look at some conditional probability of these shapes (Section 6) and the probability that s_λ maps to other values in $\mathbb{F}_q$ (Section 7).

1.2. **Review of Basic Definitions.**

We review some of the related definitions and theorems. Readers can refer to [1] for details.

A (polynomial) function $f(x_1, \cdots, x_m)$ is called symmetric if for any $\sigma \in \mathcal{S}_m$, we have $f(x_1, \cdots, x_m) = f(x_{\sigma(1)}, \cdots, x_{\sigma(m)})$.

Definition 1.1. For any positive integer k, the elementary symmetric function e_k is defined as

$$e_k(x_1, \cdots, x_n) = \sum_{i_1 < \cdots < i_n} x_{i_1} \cdots x_{i_k}$$

Definition 1.2. For any positive integer k, the complete homogeneous symmetric function h_k is defined as

$$h_k(x_1, \cdots, x_n) = \sum_{i_1 \leq \cdots \leq i_n} x_{i_1} \cdots x_{i_k}$$

For example, $e_2(x_1, x_2) = x_1x_2$, while $h_2(x_1, x_2) = x_1^2 + x_1x_2 + x_2^2$.

The elementary symmetric functions are symmetric functions and they form an algebraic basis for Λ, the algebra of symmetric functions, and so do the complete homogeneous symmetric functions.

A partition λ of a positive integer n is a sequence of positive integers $(\lambda_1, \lambda_2, \cdots, \lambda_k)$ where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ and $\sum_{i=1}^{k} \lambda_k = n$. For each i, the integer λ_i is called the i^{th} part of λ. We call n the size of λ, and denote by $|\lambda| = n$. We call k the length of λ, and denote by $l(\lambda) = k$. For simplicity, we use the notation b^n as an abbreviation for the partition $\underbrace{(b, \cdots, b)}_{n \text{ many } b\text{'s}}$.

We give some partitions of certain forms special names:

Definition 1.3. A **hook shape** is a partition λ of the form $\lambda = (a, 1^m)$.

Definition 1.4. A **fattened hook** is a partition λ of the form $\lambda = (a^n, b^m)$.

Definition 1.5. A **rectangle** is a partition λ of the form $\lambda = (b^m)$.

Definition 1.6. A **staircase** is a partition λ of the form $\lambda = (k, k-1, \ldots, 1)$.

For any partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ of n we can associate a Young diagram, which is a collection of left-justified boxes with λ_i many boxes in the i^{th} row from the top for each i. By abuse of notation we will also call this Young diagram λ.

For any partition λ of n, if we look at its Young diagram and write down the number of boxes in each column from left to right, then we obtain a new partition λ' of n. We call it the transpose of λ.

For example, for $\lambda = (4, 2, 1)$, the corresponding Young diagram is

$$\ydiagram{4,2,1},$$

and its transpose is $\lambda' = (3, 2, 1, 1)$.

A semi-standard Young tableau of shape λ and size n is a filling of the boxes of λ with positive integers such that the entries weakly increase across rows and strictly increase down columns. To each of the semi-standard Young tableaux T of shape λ and size n we may associate a monomial x^T given by

$$x^T = \prod_{i \in \mathbb{N}^+} x_i^{m_i},$$

where m_i is the number of times the integer i appears as an entry in T.

To illustrate, take $\lambda = (4, 2, 1)$. Then a semi-standard Young tableau of shape λ is given by

$$T = \begin{ytableau} 1 & 1 & 2 & 4 \\ 4 & 6 \\ 5 \end{ytableau},$$

and the corresponding monomial is $x^T = x_1^2 x_2 x_4^2 x_5 x_6$.

With these, we can define the Schur function as follows.

Definition 1.7. The Schur function s_λ is defined as

$$s_\lambda = \sum_T x^T,$$

where the sum is across all semi-standard Young tableaux of shape λ.

It is well-known that the Schur functions are symmetric functions and they form a linear basis for the algebra of symmetric functions.

We have the following Jacobi-Trudi identity that connects Schur functions with elementary symmetric functions and complete homogeneous symmetric functions.

Theorem 1.8 (Jacobi-Trudi Identity)**.** *For any partition $\lambda = (\lambda_1, \cdots, \lambda_k)$ and its transpose λ', we have*

$$s_\lambda = \det\left(h_{\lambda_i - i + j}\right)_{i,j=1}^k,$$

$$s_{\lambda'} = \det\left(e_{\lambda_i - i + j}\right)_{i,j=1}^k,$$

where we define $h_0 = e_0 = 1$ and $h_m = e_m = 0$ for $m < 0$.

2. Preliminaries

2.1. **Definition of Certain Matrices.**

We begin by giving some terminology on matrices and polynomials.

Definition 2.1. Let $A = (a_{ij})$ be a square matrix of size n, for each $1 \le k \le 2n-1$, define *the k^{th} diagonal* to be the collection of all entries a_{ij} with $i - j = n - k$. We call the n^{th} diagonal *the main diagonal.* Similarly, we define the *the k^{th} antidiagonal* of A to be the collection of all entries a_{ij} with $i + j = k + 1$. We call the n^{th} antidiagonal *the main antidiagonal.*

Definition 2.2. Let $x_1, x_2, \cdots, x_m$ be free variables. For a polynomial in the form $x_k - f_{k-1}$ where f_{k-1} is a polynomial in $x_1, \cdots, x_{k-1}$, we call k the **label** of this polynomial. For a nonzero constant, we define its label to be 0. We leave the label of 0 undefined.

It is handy to generalize the kind of square matrices arising from Jacobi-Trudi identities for Schur functions. In particular, we define three types of square matrices, namely, general Schur matrix, reduce general Schur matrix, and special Schur matrix, each containing the next.

Definition 2.3. An $n \times n$ matrix $M = (M_{ij})_{i,j=1}^{n}$ is called a **general Schur matrix** of size n with m variables $x_1, \cdots, x_m$ if it satisfies the following conditions:

(a) For each $1 \le i \le n$, the i^{th} row is in the form $(\underbrace{0, \cdots, 0}_{d_i \text{ many } 0\text{'s}}, \underbrace{M_{i(d_i+1)}, \cdots, M_{in}}_{\text{nonzero entries}})$, and the number of zeros are weakly increasing across rows, i.e., we have $0 \le d_1 \le \cdots \le d_n \le n$.
(b) Every nonzero entry is either a nonzero constant in $\mathbb{F}_q$ or a polynomial in the form $x_k - f$ where $k \in [m]$ and f is a polynomial in $x_1, \cdots, x_{k-1}$ with coefficients in $\mathbb{F}_q$.
(c) The labels of the nonzero entries are strictly increasing across rows and strictly decreasing down columns. So in particular, the label of the upper right entry is the largest.

Example 2.4. For example, this is a general Schur matrix M of size 6 with 13 free variables $x_1, \cdots, x_m$:

$$\begin{bmatrix} x_4 & x_5 & x_6 - x_1x_3 & x_8 - x_5^2 & x_{10} & x_{13} + 4 \\ x_2 & x_3 & x_5 - x_3 & x_7 - x_5^2 & x_9 & x_{12} - x_{11} + 3 \\ 0 & x_2 - x_1 & x_4 & x_5 & x_8 - x_1 - x_2 & x_{11} \\ 0 & 3 & x_3 - 3x_1x_2 & x_4 & x_7 & x_{10} - x_7x_9 \\ 0 & 0 & x_2 - x_1 & x_3 - x_2 & x_6 - 4x_2 + 4 & 2x_9 \\ 0 & 0 & 0 & x_1 & x_2 - 2 & x_8 \end{bmatrix}$$

If we strengthen the condition a bit such that we don't allow nonzero constants as entries, then we obtain a reduced general Schur matrix:

Definition 2.5. Let M be a general Schur matrix of size n with m free variables $x_1, \cdots, x_m$. It is called a **reduced general Schur matrix** if it has the additional property that no entry is a nonzero constant.

Definition 2.6. A reduced general Schur matrix of size n with m free variables $x_1, \cdots, x_m$ is called a **special Schur matrix** if

(a) none of its entries is 0;
(b) none of its entries has a nonzero constant term;
(c) for any 2×2 submatrix of it, the sum of labels of the two entries on the main diagonal equals the sum of labels of the two entries on the main antidiagonal.

Example 2.7. Here is an example of a special Schur matrix of size 4:

$$\begin{bmatrix} x_5 & x_6 & x_8 - x_5^2 & x_9 - x_3x_6 \\ x_4 & x_5 - x_2 & x_7 & x_8 \\ x_2 & x_3 & x_4 & x_5 \\ x_1 & x_2 & x_3 & x_4 \end{bmatrix}$$

2.2. **Definition of Certain Operations.**

Now we define two operations ψ and φ that are going to be useful later. Intuitively, ψ can be thought of as doing row and column reductions in the flavor of Gaussian elimination, whereas φ can be viewed as making assignments and applying ψ together step by step.

Definition 2.8. Let M be a general Schur matrix of size n with m variables. Define an operation ψ that takes general Schur matrices to reduced general Schur matrices:

(a) If M has no nonzero constants as entries, then $\psi(M) = M$.
(b) If M has $k \geq 1$ many nonzero constant entries, then from top to bottom, for each of these k entries we use it as a pivot to turn all the other entries in its column into zero by subtracting multiple of its row from each of the rows above. Then we further use these nonzero constants to turn all the other entries in the their rows into zero by column operations. After that we delete the rows and columns with these nonzero constants and obtain a reduced general Schur matrix M'. Define $\psi(M) = M'$ in this case.

In this way, $\psi(M)$ is either an empty matrix, or a nonempty reduced general Schur matrix of size at most n with at most m variables. Notice that as long as there is one row or column of M that doesn't have a nonzero constant, then $\psi(M)$ will be nonempty, in which case the determinant of $\psi(M)$ equals a nonzero constant times the determinant of M. So if we assign the variables to numbers in $\mathbb{F}_q$ randomly, we have $P(\det M \mapsto 0) = P(\det \psi(M) \mapsto 0)$.

We now give two examples to illustrate the operation ψ.

Example 2.9. Here is an example of how ψ works:

$$M_1 = \begin{bmatrix} 0 & 2x_2 & x_4 & x_5 \\ 0 & 1 & 4x_3 & x_4 \\ 0 & 0 & x_1 & x_3 - x_2 \\ 0 & 0 & 0 & x_2 \end{bmatrix}$$

$$\xrightarrow{\text{use nonzero constants to do row and column operations}} \begin{bmatrix} 0 & 0 & x_4 - 8x_2x_3 & x_5 - 2x_2x_4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & x_1 & x_3 - x_2 \\ 0 & 0 & 0 & x_2 \end{bmatrix}$$

$$\xrightarrow{\text{delete the rows and columns with nonzero constants}} \begin{bmatrix} 0 & x_4 - 8x_2x_3 & x_5 - 2x_2x_4 \\ 0 & x_1 & x_3 - x_2 \\ 0 & 0 & x_2 \end{bmatrix} = \psi(M_1)$$

Let M_2 be a Jacobi-Trudi matrix corresponding to $\lambda = (4, 4, 2, 2)$. Then we have

$$M_2 = \begin{bmatrix} h_4 & h_5 & h_6 & h_7 \\ h_3 & h_4 & h_5 & h_6 \\ 1 & h_1 & h_2 & h_3 \\ 0 & 1 & h_1 & h_2 \end{bmatrix}$$

$$\xrightarrow{\psi} \begin{bmatrix} h_6 - h_1h_5 - h_2h_4 + h_1^2h_4 & h_7 - h_2h_5 - h_3h_4 + h_1h_2h_4 \\ h_5 + h_1^2h_3 - h_2h_3 - h_1h_4 & h_6 - h_2h_4 - h_3^2 + h_1h_2h_3 \end{bmatrix} = \psi(M_2)$$

Remark 2.10. For any Schur function, its Jacobi-Trudi matrix M is a general Schur matrix. If we apply ψ to it, then we obtain a reduced general Schur matrix $\psi(M)$. Further, notice that by the Jacobi-Trudi identity, any 0 in M must appear in a row with a 1 in it, and this row is deleted when we apply ψ. Therefore, $\psi(M)$ does not have any 0's and hence satisfies property (a) for special Schur matrices. In addition, notice that M satisfies property (b) and (c) for special Schur matrices, and a simple induction shows that $\psi(M)$ also satisfies these properties. Hence $\psi(M)$ is actually a special Schur matrix.

Definition 2.11. Let M be a reduced general Schur matrix of size n with m variables. Define recursively an operation φ that takes general Schur matrices and a set of assignments to reduced general Schur matrices:

(a) $\varphi(\emptyset; \text{any assignment}) = \emptyset$, where $\emptyset$ denotes the empty matrix.

(b) $\varphi(M; x_1 = a_1) = \psi(M(x_1 = a_1))$, where $M(x_1 = a)$ denotes the matrix obtained from M by assigning value a_1 to x_1.

(c) $\varphi(M; x_1 = a_1, \cdots, x_i = a_i) = \varphi(\varphi(M; x_1 = a_1, \cdots, x_{i-1} = a_{i-1}); x_i = a_i)$ for $i \geq 2$.

In this way, $\varphi(M; x_1 = a_1, \cdots, x_i = a_i)$ is either empty or a reduced general Schur matrix.

Notice that if $M' = \varphi(M; x_1 = a_1, \cdots, x_i = a_i)$ is empty, then $P(\det M \mapsto 0 | x_1 = a_1, \cdots, x_i = a_i) = 0$. If instead M' is nonempty, then by how ψ works we have $P(\det M \mapsto 0 | x_1 = a_1, \cdots, x_i = a_i) = P(\det M' \mapsto 0)$. Also notice that as long as M has one row or column such that its entries all have labels strictly larger than i, then M' will be nonempty, which can be easily shown by induction. These two observations will be useful in many proofs below.

3. General Results

3.1. **Lower bound on the probability.**

Theorem 3.1. *Let matrix M be a reduced general Schur matrix of size n with m free variables $x_1, \cdots, x_m$. Then assigning the variables to numbers in $\mathbb{F}_q$ randomly, we have $P(\det M \mapsto 0) \geq 1/q$.*

Proof. By induction on the number of variables m.

Base case. If the number of variables is 0, then for any $n > 0$ the matrix M is the zero matrix, and the conclusion trivially holds.

If the number of variables is 1, then by the constraints we know all entries except the $(1, n)^{\text{th}}$ entry is 0. Hence $P(\det M \mapsto 0)$ equals $1/q$ if $n = 1$ and equals 1 if $n \geq 2$. In any case, $P(\det M \mapsto 0) \geq 1/q$.

Induction step. Suppose for any $m < k$, where $k \geq 2$, we have $P(\det M \mapsto 0) \geq 1/q$ for any matrix M with k many free variables.

Then for $m = k$, for a matrix M with k free variables, consider the smallest label among the entries. Let it be some $i \geq 1$. If there are n many entries with label i, then by definition of reduced general Schur matrix we know M is forced to be an upper triangular matrix and all the entries on the main diagonal have label i. Then for any assignment of the variables other than x_i, we have at least one way to assign x_i so that at least one of the diagonal entries equals 0, which makes the determinant equal 0. In this case we have $P(\det M \mapsto 0) \geq 1/q$.

If instead there are at most $(n-1)$ many entries with label i, then for any list of constants $a_1, \cdots, a_i$, $M' = \varphi(M; x_1 = a_1, \cdots, x_i = a_i)$ is a reduced general Schur matrix of size at least 1 with at most $k - i$ variables. By the induction hypothesis, we have $P(\det M' \mapsto 0) \geq 1/q$. Since $P(\det M' \mapsto 0) = P(\det M \mapsto 0 | x_1 = a_1, \cdots, x_i = a_i)$, combining all the conditional probabilities from the different assignments we get $P(\det M \mapsto 0) \geq 1/q$.

Hence we have that $P(\det M \mapsto 0) \geq 1/q$, as desired. □

Corollary 3.2. *$P(s_\lambda \mapsto 0) \geq 1/q$ for all shape λ.*

Proof. Let M be the matrix for s_λ. It is a general Schur matrix. Since M has at most $n-1$ many 1's, $\psi(M)$ is a nonempty reduced general Schur matrix. Apply the previous theorem we obtain $P(\det M \mapsto 0) = P(\det \psi(M) \mapsto 0) \geq 1/q$. □

This corollary gives us the lower bound on the probability $P(\det M \mapsto 0)$, which is $1/q$. In the following sections, we will investigate which shapes will have exactly the probability $1/q$, and give a complete characterization of them. Further, we will show that the probability is asymptotically $1/q$ for all shapes when q approaches infinity.

3.2. **Asymptotic Bound.**

In this section we show the asymptotic bound of $P(\det M \mapsto 0)$ is $1/q$ as q approaches infinity. To do that, we first find an upper bound for the probability that a reduced general Schur matrix is singular.

Lemma 3.3. *For a reduced general Schur matrix M of size n with 0's strictly below the main diagonal, we have $P(\det M \mapsto 0) \leq n/q$.*

Proof. By induction on size n.

Base case. When $n = 1$, M consists of a single entry in the form $x_j - f_{j-1}$ for some positive integer j, and $P(\det M \mapsto 0) = P(x_j = f_{j-1}) = 1/q \leq n/q$.

Induction step. Suppose the lemma holds for all $k \leq n$. Then for n, let the smallest label among all the entries on the main diagonal be i. It suffices to show each of the conditional probabilities from assigning $x_1, \cdots, x_{i-1}$ is at most n/q.

Assign $x_1, \cdots, x_{i-1}$ by some values $a_1, \cdots, a_{i-1}$. Consider $M' = \varphi(M; x_1 = a_1, \cdots, x_{i-1} = a_{i-1})$ with size k. We know $k \geq 1$ since all the entries in the last column of M have label at least i, and 0's are still strictly below the main diagonal in M'. If $k < n$, by the induction hypothesis we know that $P(\det M' \mapsto 0) \leq k/q \leq n/q$.

Hence we only need to consider the case when $k = n$. This means we deleted no rows or columns when applying φ. Let the number of diagonal entries with label i be l. Notice the diagonal entries with label i are all in the form $x_i - a$ for some constant a, and let the number of different a's be $r \leq l$. We can assign x_i to be one of these r numbers, and the probability of $\det M' = 0$ in this sub-case is always no greater than 1. If we instead assign x_i to be none of these r numbers, then we obtain l many nonzero constants on the main diagonal. Applying ψ we obtain a reduced general Schur matrix of size at most $n - l$, and the conditional probability in this sub-case is at most $(n-l)/q$ by our induction hypothesis. Combining the two sub-cases, we get

$$\begin{aligned}P(\det M \mapsto 0 | x_1 = a_1, \cdots, x_{i-1} = a_{i-1}) &= P(\det M' \mapsto 0)\\ &\le \frac{r}{q} \cdot 1 + \frac{q-r}{q} \cdot \frac{n-l}{q}\\ &= \frac{n}{q} - \frac{l-r}{q} - \frac{r(n-l)}{q^2}\\ &\le \frac{n}{q}\end{aligned}$$

□

Lemma 3.4. *Let M be a reduced general Schur matrix of size $n \ge 2$ with 0's strictly below the $(n-1)^{th}$ diagonal. Let N be the $(n-1) \times (n-1)$ submatrix on its lower left corner. Then $P(\det M \mapsto 0 \,\&\, \det N \mapsto 0) \le n(n-1)/q^2$.*

Proof. By induction on n.

Base case. When $n = 2$, denote M by $(M_{ij})_{i,j=1}^2$. $\det N = M_{21}$ has a $1/q$ probability to be 0. Given $\det N = 0$, we have $\det M = M_{11}M_{22}$. Each of M_{11} and M_{22} has a conditional probability of $1/q$ to be zero, so combining gives at most $2/q$ to make $\det M = 0$. Hence

$$P(\det M \mapsto 0 \ \&\ \det N \mapsto 0) \le 1/q \cdot 2/q = n(n-1)/q^2.$$

Induction step. Suppose the lemma holds for all $2 \le k < n$. Then for n, let the smallest among all the labels of the entries on the $(n-1)^{\text{th}}$ diagonal be i. It suffices to show each of the conditional probabilities from assigning $x_1, \cdots, x_{i-1}$ is at most $n(n-1)/q$.

Assign $x_1, \cdots, x_{i-1}$ by some values $a_1, \cdots, a_{i-1}$. Consider the square matrix $M' = \varphi(M; x_1 = a_1, \cdots, x_{i-1} = a_{i-1})$ of size k. Notice $k \ge 2$, and 0's are still strictly below the $(k-1)^{\text{th}}$ diagonal. Since the determinants of M' and its $(k-1) \times (k-1)$ submatrix at the lower left corner N' are just some nonzero constant times the determinants of M and N, respectively, the conditional probability in this case equals $P(\det M' \mapsto 0 \ \&\ \det N' \mapsto 0)$. If $k < n$, then by the induction hypothesis, the conditional probability is at most $k(k-1)/q^2 \le n(n-1)/q^2$.

We are only left with the case when $k = n$. Let the number of entries on the $(n-1)^{\text{th}}$ diagonal with label i be $l \le n-1$. Each of these l entries is in the form $x_i - a$ for some constant a, and let the number of different a's be $r \le l$. We can assign x_i to be one of these r numbers, and by the previous lemma the conditional probability of $\det M = 0$ in this sub-case is at most n/q. If we instead assign x_i to be none of these r numbers, then we get l many nonzero constants on the $(n-1)^{\text{th}}$ diagonal of M'. Applying ψ we obtain a reduced general Schur matrix of size at most $n-l$, and in this sub-case the condition probability is at most $(n-l)(n-l-1)/q^2$ by our induction hypothesis. Combining the two sub-cases, we get

$$
\begin{aligned}
&P(\det M \mapsto 0 \;\&\; \det N \mapsto 0 | x_1 = a_1, \cdots, x_{i-1} = a_{i-1}) \\
\le& \frac{r}{q}\frac{n}{q} + \frac{q-r}{q} \cdot \frac{(n-l)(n-l-1)}{q^2} \\
=& \frac{n(n-1)}{q^2} - \frac{n(l-r)}{q^2} - \frac{l(n-1-l)}{q^2} - \frac{r[l^2-(2n+1)l+n(n-1)]}{q^3} \\
\le& \frac{n(n-1)}{q^2} - \frac{r[(n-1)^2-(2n-1)(n-1)+n(n-1)]}{q^3} \\
\le& \frac{n(n-1)}{q^2}
\end{aligned}
$$

□

Theorem 3.5. *For any shape λ, we have $P(s_\lambda \mapsto 0) = 1/q + O(1/q^2)$.*

Proof. We show equivalently that $q \cdot P(s_\lambda \mapsto 0) \to 1$ as $q \to \infty$.

Let M be the Jacobi-Trudi matrix for s_λ and let $M' = \psi(M)$. We have $P(s_\lambda \mapsto 0) = P(\det M' \mapsto 0)$.

If M' has size 1, then $P(s_\lambda \mapsto 0) = 1/q$ and we automatically have the result.

If M' has size $n \ge 2$, denote the label of the upper right entry of M' by k and the $(n-1)\times(n-1)$ submatrix on its lower right corner by N'. If $\det N' \ne 0$. Expansion across the first row gives $\det M' = (-1)^{n+1} \det N' h_k + P(h_1, \cdots, h_{k-1})$ where $P(h_1, \cdots, h_{k-1})$ is a polynomial not involving h_k. Thus for any assignment of $h_1, \cdots, h_{k-1}$, we have precisely one way to assign h_k to achieve $\det M' = 0$. Hence $P(\det M' \mapsto 0 | \det N' \not\mapsto 0) = 1/q$.

We thus have

$$
\begin{aligned}
P(\det M' \mapsto 0) &= P(\det M' \mapsto 0 | \det N' \mapsto 0) P(\det N' \mapsto 0) \\
&\quad + P(\det M' \mapsto 0 | \det N' \not\mapsto 0) P(\det N' \not\mapsto 0) \\
&= P(\det M' \mapsto 0 \;\&\; \det N' \mapsto 0) + 1/q \cdot P(\det N' \not\mapsto 0) \\
&\le n(n-1)/q^2 + 1/q \cdot 1 \\
&= 1/q + n(n-1)/q^2
\end{aligned}
$$

where the inequality follows from the previous lemma.

Since by Theorem 3.2, $P(s_\lambda \mapsto 0) \ge 1/q$, we have that

$$1 = q \cdot 1/q \le q \cdot P(s_\lambda \mapsto 0) \le q \cdot [1/q + n(n-1)/q^2] = 1 + n(n-1)/q.$$

Taking $q \to \infty$ gives us the desired result. □

3.3. **General Form of the Probability.**

For any partition shape λ, the probability $P(s_\lambda \mapsto 0)$ is given by the number of assignments that maps s_λ to 0 over the total number of assignments, where the total number of assignments is always a power of q. The nice results in the following sections may mislead the reader into assuming that for a given Jacobi-Trudi matrix, the number of singular matrices is always counted by a polynomial in q and the probability is thus always a rational function in q. However, as the next proposition points out, a counterexample shows that this is not always the case.

Proposition 3.6. *For $\lambda = (4, 4, 2, 2)$, we have*

$$P(s_\lambda \mapsto 0) = \begin{cases} \frac{q^4+(q-1)(q^2-q)}{q^5} & \text{if } q \equiv 0 \mod 2 \\ \frac{q^4+(q-1)(q^2-q+1)}{q^5} & \text{if } q \equiv 1 \mod 2 \end{cases}$$

Proof. By Jacobi-Trudi, we have

$$s_\lambda = \begin{vmatrix} h_4 & h_5 & h_6 & h_7 \\ h_3 & h_4 & h_5 & h_6 \\ 1 & h_1 & h_2 & h_3 \\ 0 & 1 & h_1 & h_2 \end{vmatrix}.$$

Denote this Jacobi-Trudi matrix by M. We count the number of assignments of the variables such that M is singular.

Apply ψ to M gives

$$\psi(M) = \begin{bmatrix} h_6 - h_1h_5 - h_2h_4 + h_1^2h_4 & h_7 - h_2h_5 - h_3h_4 + h_1h_2h_4 \\ h_5 + h_1^2h_3 - h_2h_3 - h_1h_4 & h_6 - h_2h_4 - h_3^2 + h_1h_2h_3 \end{bmatrix} = \begin{bmatrix} h_6 - a & h_7 - b \\ h_5 - c & h_6 - d \end{bmatrix}$$

The question now turns to count how many assignments will make $\psi(M)$ singular. We have $\det\psi(M) = (h_6 - a)(h_6 - d) - (h_7 - b)(h_5 - c)$. If $h_5 \neq c$, then $a \neq 0$, so once we have chosen other variables, there is exactly one choice of h_7 to make the matrix singular. Choosing the variables in order gives $q^4 \cdot (q-1) \cdot q \cdot 1 = q^6 - q^5$ many singular matrices in this case.

If instead $h_5 = c$, then $\det\psi(M) = (h_6 - a)(h_6 - d)$. $\det\psi(M) = 0$ gives either $h_6 = a$ or $h_6 = d$. The equality $h_5 = c$ gives $a - d = 2h_1h_2h_3 - h_1^3h_3 - h_3^2$. Depending on whether q has characteristic 2, we have two cases:

<u>q has characteristic 2.</u> Then $a - d = -h_3(h_3 + h_1^3)$.

If $a - d = 0$, then we have exactly one choice $h_6 = a = d$ to make the matrix singular once we have chosen other variables. In this sub-case either both h_1 and $h - 3$ are zero or h_1 nonzero and $h - 3$ equals 0 or $-x_1^3$, resulting in a total of $1 + (q-1) \cdot 2 = 2q - 1$ choices of h_1 and h_3. Hence choosing the x_1, x_3 pair, and x_2, x_4, x_5, x_7, x_6 in order gives $q \cdot (2q-1) \cdot q \cdot 1 \cdot q \cdot 1 = 2q^4 - q^3$ many singular matrices.

If $a - d \neq 0$, then we have two choices of h_6, namely, a and d, to make the matrix singular. In this sub-case we have a total of $q^2 - (2q-1) = q^2 - 2q + 1$ choices of h_1 and h_3. Hence choosing the x_1, x_3 pair, and x_2, x_4, x_5, x_7, x_6 in order gives $q \cdot (q^2 - 2q + 1) \cdot q \cdot 1 \cdot q \cdot 2 = 2q^5 - 4q^4 + 2q^3$ many singular matrices.

Adding these two gives $2q^5 - 2q^4 + q^3$ many singular matrices in this case.

<u>q doesn't have characteristic 2.</u> Then $a - d = -h_3[h_3 - h_1(2h_2 - h_1^2)]$.

If $a - d = 0$, then we have exactly one choice of h_6 to make the matrix singular. Discussion on h_1, h_2, h_3 gives a total of $2q^2 - 2q + 1$ choices of them to make $a - d = 0$. Hence choosing the x_1, x_2, x_3 triple, and x_4, x_5, x_7, x_6 in order gives $(2q^2 - 2q + 1) \cdot q \cdot 1 \cdot q \cdot 1 = 2q^4 - 2q^3 + q^2$ many singular matrices.

If $a - d \neq 0$, then we have two choices of h_6 to make the matrix singular. In this sub-case we have a total of $q^3 - (2q^2 - 2q + 1) = q^3 - 2q^2 + 2q - 1$ choices of h_1, h_2, h_3 to make $a - d \neq 0$. Hence choosing the x_1, x_2, x_3 triple, and x_4, x_5, x_7, x_6 in order gives $(q^3 - 2q^2 + 2q - 1) \cdot q \cdot 1 \cdot q \cdot 2 = 2q^5 - 4q^4 + 4q^3 - 2q^2$ many singular matrices.

Adding these two gives $2q^5 - 2q^4 + 2q^3 - q^2$ many singular matrices in this case.

Therefore, if q has characteristic 2, then we have a total of $q^6 - q^5 + 2q^5 - 2q^4 + q^3$ many singular matrices; if q doesn't have characteristic 2, then we instead get $q^6 - q^5 + 2q^5 - 2q^4 + 2q^3 - q^2$ many of them. Dividing these two polynomials by q^7 respectively gives us the desired probabilities in the two cases. $\square$

We call a function $g : \mathbb{Z} \to \mathbb{C}$ a **quasi-polynomial** if there exists an integer $N > 0$ and polynomials $g_0, g_1, \cdots, g_{N-1}$ such that $g(n) = g_i(n)$ if $n \equiv i \mod N$.

Remark 3.7. We have two other counterexamples. For $\lambda_1 = (4, 4, 3, 3)$, the data are given below

$\lambda_1 = (4,4,3,3)$								
q	2	3	4	5	7	8	9	11
$f(q)$	82	891	4852	18145	132013	290872	583929	1918081
P	$41/2^6$	$11/3^4$	$1213/4^6$	$3629/5^6$	$18859/7^6$	$36359/8^6$	$89/9^4$	$174371/11^6$

We have a quasi-polynomial as the numerator for $P(s_{\lambda_1} \mapsto 0)$ which fits the data we have:

$$P(s_{\lambda_1} \mapsto 0) = \begin{cases} \frac{q^5+(q-1)q^3}{q^6} = \frac{q^2+q-1}{q^3} & \text{if } q \equiv 0 \mod 3 \\ \frac{q^5+(q-1)(q^3-1)}{q^6} & \text{if } q \equiv 1 \mod 3 \\ \frac{q^5+(q-1)(q^3+1)}{q^6} & \text{if } q \equiv 2 \mod 3 \end{cases}$$

Another counterexample is $\lambda_2 = (4, 4, 3, 2)$. The numerical data are

$\lambda_2 = (4,4,3,2)$								
q	2	3	4	5	7	8	9	11
$f(q)$	84	909	5008	18425	133525	293952	588465	1928861
P	$21/2^5$	$101/3^5$	$313/4^5$	$737/5^5$	$2725/7^5$	$4593/8^5$	$7265/9^5$	$15941/11^5$

We also have a quasi-polynomial as the numerator for $P(s_{\lambda_2} \mapsto 0)$ which fits the data above:

$$P(s_{\lambda_2} \mapsto 0) = \begin{cases} \frac{q^4+(q-1)(q^2+q-1)}{q^5} & \text{if } q \equiv 0 \mod 2 \\ \frac{q^4+(q-1)(q^2+q-2)}{q^5} & \text{if } q \equiv 1 \mod 2 \end{cases}$$

We can show that every partition with less than 4 parts yields a rational function for the probability. We can also show that of all partitions of 4 parts, these three examples we give here are the smallest in terms of the first part of the partition to make the probability not a rational function in q.

Based on these, we have the following conjecture:

Conjecture 3.8. For a partition λ, $P(s_\lambda \mapsto 0)$ is always in the form $f(q)/q^k$, where k is some positive integer and $f(q)$ is a quasi-polynomial depending on the residue class of q modulo some integer.

3.4. Conjecture on the Upper Bound.

For any given k, we consider a special case such that the partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ has its rows being far apart, i.e., $\lambda_i - \lambda_{i+1} \geq k - 1$ for each $i < k$ and $\lambda_k \geq k$. In this case, there is no constant or repeated variable in the Jacobi-Trudi matrix, hence we are just counting the number of singular matrices in the general linear group $GL_k(\mathbb{F}_q)$.

Proposition 3.9. *Let $\lambda = (\lambda_1, \dots, \lambda_k)$, where $\lambda_i - \lambda_{i+1} \geq k-1$ and $\lambda_k \geq k$. Then*

$$P(s_\lambda \mapsto 0) = 1 - \frac{GL_k(\mathbb{F}_q)}{q^{k^2}} = \frac{1}{q^{k^2}} \left(q^{k^2} - \prod_{j=0}^{k-1}(q^k - q^j) \right).$$

All the other conditions being equal, if we have $\lambda_k < k$ instead, then

$$P(s_\lambda \mapsto 0) = 1 - \frac{GL_{k-1}(\mathbb{F}_q)}{q^{(k-1)^2}} = \frac{1}{q^{(k-1)^2}} \left(q^{(k-1)^2} - \prod_{j=0}^{k-2}(q^{k-1} - q^j) \right)$$

Proof. We have

$$s_\lambda = \begin{vmatrix} h_{\lambda_1} & h_{\lambda_1+1} & \cdots & h_{\lambda_1+k-1} \\ h_{\lambda_2-1} & h_{\lambda_2} & \cdots & h_{\lambda_2+k-2} \\ \vdots & & \ddots & \vdots \\ h_{\lambda_k-k+1} & & \cdots & h_{\lambda_k} \end{vmatrix}.$$

The condition $\lambda_i - \lambda_{i+1} \geq k-1$ ensures that the h_j are all distinct, and the condition $\lambda_k \geq k$ ensures that no entry in the above determinant is constant. The number of choices of the h_j such that the above determinant is zero is thus equal to the number of singular $k \times k$ matrices over $\mathbb{F}_q$, which is

$$q^{k^2} - |\mathrm{GL}_k(\mathbb{F}_q)| = q^{k^2} - \prod_{j=0}^{k-1}(q^k - q^j).$$

The result follows.

Suppose now we have $\lambda_k < k$ instead. We will count the number of invertible matrices of the given form. Since we have $\lambda_{k-1} - \lambda_k \geq k-1$, $\lambda_{k-1} \geq \lambda_k + k - 1 \geq 1 + k - 1 \geq k$. Hence the first $k-1$ rows only have h_j's in them, while the last row has λ_k many h_j's, 1 one, and $k - 1 - \lambda_k$ zeros. Since row k has a nonzero entry thus cannot be the zero vector, we may choose the λ_k many noncosntant entries in row k freely. Thus there are q^{λ_k} number of ways to choose row k. We can choose the other rows from bottom to top successively, and for row $k-i$ where $1 \leq< k$, there are $q^k - q^i$ ways to choose so that row $k-i$ is not in the span of the rows previously chosen. So we have in total $q^{\lambda_k} \prod_{i=1}^{k-1}(q^k - q^i)$ many ways to get an invertible matrix. There are $(k-1)k + \lambda_k$ many h_j's, so there are in total $q^{(k-1)k+\lambda_k}$ many matrices. Therefore

$$\begin{aligned}
P(s_\lambda \mapsto 0) &= 1 - \frac{q^{\lambda_k} \prod_{i=1}^{k-1}(q^k - q^i)}{q^{(k-1)k+\lambda_k}} \\
&= \frac{1}{q^{k^2-k}} \left(q^{k^2-k} - \prod_{i=1}^{k-1}(q^k - q^i) \right) \\
&= \frac{1}{q^{k^2-k}} \left(q^{k^2-k} - q^{k-1} \prod_{i=0}^{k-2}(q^{k-1} - q^i) \right) \\
&= \frac{1}{q^{(k-1)^2}} \left(q^{(k-1)^2} - \prod_{i=0}^{k-2}(q^{k-1} - q^i) \right) \\
&= 1 - \frac{|GL_{k-1}(\mathbb{F}_q)|}{q^{(k-1)^2}}
\end{aligned}$$

□

Conceivably, if we have repeated variables or 1's in the Jacobi-Trudi matrix, then it would be harder to make some of the rows to be linearly dependent, which in turns decreases probability that the Schur function is mapped to zero. This suggests that for a partition λ with k parts, $1 - GL_k(\mathbb{F}_q)/q^{k^2}$ provides an upper bound for the probability $P(s_\lambda \mapsto 0)$. Numerical calculations so far seems to support this. Hence we have the following conjecture.

Conjecture 3.10 (Upper Bound)**.** For any partition λ with k parts, we have $P(s_\lambda \mapsto 0) \leq 1 - GL_k(\mathbb{F}_q)/q^{k^2}$.

4. Hooks, Rectangles and Staircases

In this section, we show that three kind of shapes, namely, hooks, rectangles and staircases, have the probability being exactly $1/q$.

4.1. **Hooks.**

Proposition 4.1. *Let $\Lambda := \{\lambda^{(k)}\}_{k\in\mathbb{N}}$ be a collection of hook shapes such that $|\lambda^{(k)}| = k$ for all k. Then the distributions of values of the collection $\{s_{\lambda^{(k)}}\}_k$ is uniform and independent of each other.*

Proof. Let $\lambda = (a, 1^b)$ be a hook. Note that $n := a + b = |\lambda|$. By Jacobi-Trudi, we have

$$s_\lambda = \begin{vmatrix} h_a & h_{a+1} & \cdots & h_n \\ 1 & h_1 & \cdots & h_{n-1} \\ 0 & \ddots & \ddots & \vdots \\ 0 & 0 & 1 & h_1 \end{vmatrix}$$

Using the cofactor expansion about the first row, we see that

$$s_\lambda = (-1)^n h_n + f(h_1, \dots, h_{n-1}),$$

for some polynomial f. For any fixed assignment of $\{h_1, \dots, h_{n-1}\}$, the values of s_λ are uniformly distributed. Hence the distribution of s_λ is itself uniform.

For independence, observe that $s_{\lambda^{(k)}}$ is uniformly distributed for every set of values $\{h_1, \ldots, h_{k-1}, h_{k+1}, \ldots\}$. So the variables s_{λ_k} are independent of one another. □

Corollary 4.2. *Let $f : \{e_n\}_{n\in\mathbb{N}} \to \mathbb{F}_q$ and $g : \{h_n\}_{n\in\mathbb{N}} \to \mathbb{F}_q$ be arbitrary. Consider the unique extension of f and g to homomorphisms from the algebra of symmetric functions to $\mathbb{F}_q$. Then*

$$P\big(f(s_\lambda) = a\big) = P\big(g(s_\lambda) = a\big)$$

for all $a \in \mathbb{F}_q$.

Corollary 4.3. *Let $\mathcal{H}$ be a collection of hooks satisfying the condition of Proposition 4.1. Then $\{s_\lambda\}_{\lambda\in\mathcal{H}}$ forms an algebraic basis of the space of symmetric functions.*

Proof. We use without proof that $\{h_n\}_{n\leq N}$ form an algebraic basis of the space of symmetric functions of degree $\leq N$. Write $\mathcal{H} = \{\lambda_n\}_{n\in\mathbb{N}}$ with $|\lambda_n| = n$.

We prove by induction on N that $\{s_{\lambda_n}\}_{n\leq N}$ is algebraically equivalent to $\{h_n\}_{n\leq N}$. The case $N = 1$ is trivial, since λ_1 is necessarily a single block. Hence $s_{\lambda_1} = h_1$.

Now suppose that $\{s_{\lambda_n}\}_{n\leq N}$ is algebraically equivalent to $\{h_n\}_{n\leq N}$. Recall from Proposition 4.1 that

$$s_{\lambda_{n+1}} = (-1)^{n+1}h_{n+1} + f(h_n, \ldots, h_1),$$

for some polynomial f. By hypothesis, we may write $f(h_n, \ldots, h_1) = g(s_{\lambda_1}, \ldots, s_{\lambda_n})$ for some polynomial g. Plugging this in above, we may write h_{n+1} as a polynomial in $\{s_{\lambda_1}, \ldots, s_{\lambda_{n+1}}\}$. Directly by Jacobi-Trudi, $s_{\lambda_{n+1}}$ can be obtained as a polynomial in $\{h_1, \ldots, h_{n+1}\}$. So the two are algebraically equivalent.

Hence since $\{h_n\}_{n\in\mathbb{N}}$ forms an algebraic basis for the space of symmetric functions, so too does $\{s_\lambda\}_{\lambda\in\mathcal{H}}$. □

4.2. **Rectangles.**

Let $\lambda = (a^n)$ be a rectangle shape, where $a, n \geq 1$. We will first deal with the case where $a \geq n$ and then we will see that the case $a < n$ will follow from the exact same argument. Assume that $\lambda = a^n$ with $a \geq n \geq 1$, then the Jacobi-Trudi matrix corresponding to it, after renaming the variables, can be written as $A = (x_{j-i+n})_{1\leq i,j\leq n}$.

Lemma 4.4. *Let $A = (x_{j-i+n})_{1\leq i,j\leq n}$ be a square matrix of size n, where $x_1, \ldots, x_{2n-1}$ are variables. Then $\varphi(A; x_1 = a_1, \ldots, x_r = a_r)$, for any $a_1, \ldots, a_r \in \mathbb{F}_q$, is either empty; or a matrix of size $n' \geq 1$ such that all its first n' diagonals contain only zeroes; or a matrix of size $n' \geq 1$ with $0 \leq k \leq 2n'$ such that all entries in the i^{th} diagonal are 0 for $i < k$, all entries in the k^{th} diagonal are exactly the same and all entries in the i^{th} diagonal have label $2n - 2n' + i$.*

Notice that if the lower left corner of $\varphi(A; x_1 = a_1, \ldots, x_r = a_r)$ has positive label, then the lemma is trivially correct with $k = 0$.

Intuitively, this lemma is saying that no matter how we assign variables and do row and column operations, the matrix is behaving nicely. It is the core for the rectangle case and will be extensively used for the rest of the paper.

Example 4.5. An example related to Lemma 4.4.

$$A = \begin{bmatrix} x_4 & x_5 & x_6 & x_7 \\ x_3 & x_4 & x_5 & x_6 \\ x_2 & x_3 & x_4 & x_5 \\ x_1 & x_2 & x_3 & x_4 \end{bmatrix} \xrightarrow{\varphi(x_1=1)} \begin{bmatrix} x_5 - x_2x_4 & x_6 - x_3x_4 & x_7 - x_4^2 \\ x_4 - x_2x_3 & x_5 - x_3^2 & x_6 - x_3x_4 \\ x_3 - x_2^2 & x_4 - x_2x_3 & x_5 - x_2x_4 \end{bmatrix}$$

$$\xrightarrow{\varphi(x_2=2)} \begin{bmatrix} x_5 - 2x_4 & x_6 - x_3x_4 & x_7 - x_4^2 \\ x_4 - 2x_3 & x_5 - x_3^2 & x_6 - x_3x_4 \\ x_3 - 4 & x_4 - 2x_3 & x_5 - 2x_4 \end{bmatrix} \xrightarrow{\varphi(x_3=4)} \begin{bmatrix} x_5 - 2x_4 & x_6 - 4x_4 & x_7 - x_4^2 \\ x_4 - 8 & x_5 - 16 & x_6 - 4x_4 \\ 0 & x_4 - 8 & x_5 - 2x_4 \end{bmatrix}$$

$$\xrightarrow{\varphi(x_4=8)} \begin{bmatrix} x_5 - 16 & x_6 - 32 & x_7 - 64 \\ 0 & x_5 - 16 & x_6 - 32 \\ 0 & 0 & x_5 - 16 \end{bmatrix}$$

Notice that in this example, we clearly see that each intermediate matrix satisfies the condition mentioned in Lemma 4.4. In other words, these matrices have zeroes in the first few diagonals and exactly same entries on the next nonzero diagonal.

To prove Lemma 4.4, we need to know exactly how each entry changes after some row and column operations. And here comes the next definition and lemma.

Definition 4.6. Suppose that A is a matrix of size n. Define $A[S,T]$ to be the $k \times k$ submatrix of A by selecting the i^{th} row and j^{th} column of A for all $i \in S$ and $j \in T$ where S and T are subsets of $[n]$ of carnality k.

Lemma 4.7. *The determinant of $A[S,T]$ does not change if we perform row and column operations that only involve subtracting by rows in S and by columns in T.*

Proof. Looking at $A[S,T]$ as a matrix by itself will give us the desired result. $\square$

Now we are ready to state the proof for Lemma 4.4.

Proof.[Proof for Lemma 4.4] We will use proof by induction on r.

When $r = 0$, the lemma is trivially correct.

Assume that the lemma is correct for $r-1$ and consider the matrix $B = \varphi(A; x_1 = a_1, \ldots, x_{r-1} = a_{r-1})$. If B is empty, then by assigning any $a_r \in \mathbb{F}_q$ to x_r, we will also get an empty matrix so the inductive step holds trivially. Similarly, if all entries in B below or on the main diagonal are zero, the inductive step will also hold. Also, if the smallest positive label of entries in B is strictly greater than r, assigning a_r to x_r won't change any labels so the inductive step holds. If the labels of the main diagonal entries are r, since these entries are the same, we can assume that they are $x_r - b$ for some $b \in \mathbb{F}_q$. Then after we apply $\varphi(x_r = a_r)$, if $a_r \neq b$, the matrix will become empty after ψ and if $a_r = b$, the matrix will have only zeroes on the main diagonal and the diagonals below the main diagonal.

Now we deal with the main case. Suppose that the first nonzero diagonal of B has entries $x_r - b$ by induction hypothesis. And suppose that the next diagonal contains $x_{r+1} -$

$f_1, \dots, x_{r+1} - f_\ell$ where f_i's are polynomials in x_r.

$$B = \begin{bmatrix} \vdots & & & & & & \\ x_{r+1} - f_1 & & & & & & \\ x_r - b & x_{r+1} - f_2 & & & & & \\ 0 & x_r - b & \ddots & & & & \\ \vdots & 0 & \ddots & \ddots & & & \\ 0 & \ddots & 0 & x_r - b & \ddots & & \\ 0 & 0 & \cdots & 0 & x_r - b & x_{r+1} - f_\ell & \cdots \end{bmatrix}$$

If we assign $a_r \neq b$ to x_r and do ψ to this matrix, we will use the $a_r - b$'s to cancel out the corresponding rows and columns. The remaining matrix after applying $\varphi(B; x_r = a_r)$ contains no constant so the lemma holds. Therefore, the only case we need to consider is that $a_r = b$. It suffices to show that when $x_r = b$, $x_{r+1} - f_i$ is the same for each i.

Let $m = n - n'$, where n' is the size of B. According to the definition of φ and induction hypothesis, we can use row and column operations involving only subtracting by the last m rows and the first m columns to get from $A' := A(x_1 = a_1, \dots, x_r = b)$ to $\begin{bmatrix} 0 & B' \\ M & 0 \end{bmatrix}$, where $B' = B(x_r = b)$ and M can be written as blocks of nonzero multiples of identity matrices from lower left to upper right. Specifically,

$$M = \begin{bmatrix} & & & c_w I_w \\ & & \iddots & \\ & c_2 I_2 & & \\ c_1 I_1 & & & \end{bmatrix}, \tag{1}$$

where $c_i \in \mathbb{F}_q^\times$. We know that the size of M is m. Let $d = \det M \neq 0$.

Suppose that the first k diagonals of B' are all zeroes and the $(k+1)^{\text{th}}$ diagonal of B' has entries of label $r + 1$ where $k \leq n - m$. In other words, $B'_{i,j} = 0$ for all $i, j \in \mathbb{Z}_{\geq 1}$ such that $i - j \geq n - m - k$. And then we want to show that all entries on the $(k+1)^{\text{th}}$ diagonal are the same.

Let $S = \{n - m + 1, n - m + 2, \dots, n - 1, n\}$ and $T = \{1, 2, \dots, m\}$. They are both subsets of $[n]$ of cardinality m. Lemma 4.7 shows that the determinant of $A'[S, T]$ does not change after row and column operations from A' to $\begin{bmatrix} 0 & B' \\ M & 0 \end{bmatrix}$. So $\det A'[S, T] = \det M = d \neq 0$.

Since $\det A'[S, T] \neq 0$, $\{(a_t, a_{t+1}, \dots, a_{t+m-1})^T : t = 1, \dots, m\}$ forms a linear basis of $\mathbb{F}_q^m$. There exists a row vector $\mathbf{v} \in \mathbb{F}_q^m$ such that $\mathbf{v} \cdot A[S, T] = (a_{m+1}, a_{m+2}, \dots, a_{2m})$.

We know that for any $j = 1, \dots, k - 1$, $\det A'\big[S \cup \{n - m\}, T \cup \{m + j\}\big] = 0$. For this $(m+1) \times (m+1)$ submatrix, if we subtract the first row by $\mathbf{v} \cdot A'\big[S, T \cup \{m + 1\}\big]$, which is a linear combination of the last m rows, the first row becomes 0 for the first m entries and $a_{2m+j} - \mathbf{v} \cdot (a_{m+j}, a_{m+j-1}, \cdots, a_{2m+j-1})$ for the last entry. Since this $(m+1) \times (m+1)$ submatrix has zero determinant, we have

$$\det A'[S, T]\Big(a_{2m+j} - \mathbf{v} \cdot (a_{m+j}, a_{m+j-1}, \cdots, a_{2m+j-1})^T\Big) = 0.$$

As $\det A'[S,T] = d \neq 0$, we know $a_{2m+j} = \mathbf{v}\cdot(a_{m+j}, a_{m+j-1}, \cdots, a_{2m+j-1})$ for $j = 1, \ldots, k-1$. To encode this piece of information in matrix form, let us define an $m \times m$ matrix

$$V := \left(\begin{array}{c|c} 0 & I_{m-1} \\ \hline \multicolumn{2}{c}{\mathbf{v}} \end{array} \right)$$

where the "0" here represents an $(m-1) \times 1$ array with all zeroes. Consequently, we have

$$V \cdot (a_t, a_{t+1}, \ldots, a_{t+m-1})^T = (a_{t+1}, a_{t+2}, \ldots, a_{t+m})^T \text{ for all } 1 \leq t \leq m+k-1.$$

A simple inductive argument will give us, for all $t + h \leq m + k$,

$$V^h \cdot (a_t, a_{t+1}, \ldots, a_{t+m-1})^T = (a_{t+h}, a_{t+h+1}, \ldots, a_{t+h+m-1})^T. \tag{2}$$

At the same time, for all $t + h \leq m + k - 1$, $t \geq 1$, $h \geq 0$, we have

$$\mathbf{v} \cdot V^h \cdot (a_t, a_{t+1}, \ldots, a_{t+m-1})^T = \mathbf{v} \cdot (a_{t+h}, a_{t+h+1}, \ldots, a_{t+h+m-1})^T = a_{t+h+m}. \tag{3}$$

Now we have enough tools to deal with $B'_{i,j}$ for $i - j = n - m - k - 1$. For convenience of notation, we will fix $i + j = k + 1$ (i.e., change $n - m - i$ to i) and deal with $B'_{n-m-i,j}$. Consider $A'\big[S \cup \{n - m - i + 1\}, T \cup \{m + j\}\big]$. For this $(m + 1) \times (m + 1)$ submatrix, to calculate its determinant, we subtract the first row by $(\mathbf{v}V^{i-1}) \cdot A'\big[S, T \cup \{m + j\}\big]$, which is a linear combination of its last m rows. Then, the first m entries of its first row become 0 because of Equation (3) with $h = i-1$ and $t = 1, \ldots, m$, noticing that $m+i-1 \leq m+k-1$. The last entry of the first row is

$$\begin{aligned} &x_{2m+k-1} - v \cdot V^{i-1} \cdot (a_{m+j}, a_{m+j-1}, \ldots, a_{2m+j-1})^T \\ =&x_{2m+k-1} - v \cdot V^{i-1} \cdot V^{m+j-1} \cdot (a_1, a_2, \ldots, a_m)^T \\ =&x_{2m+k-1} - v \cdot V^{m+k-1} \cdot (a_1, a_2, \ldots, a_m)^T \end{aligned} \tag{4}$$

The first equality comes from Equation (2) with $t = 1$ and $h = m + j - 1 \leq m + k - 1$.

With this, it is clear that

$$\begin{aligned} \det A'\big[S \cup \{n - m - i\}, T \cup \{m + j\}\big] =&\det A'[S,T] \cdot \big(x_{2m+k-1} - v \cdot V^{m+k-1} \cdot (a_1, a_2, \ldots, a_m)^T\big) \\ =&d\big(x_{2m+k-1} - v \cdot V^{m+k-1} \cdot (a_1, a_2, \ldots, a_m)^T\big). \end{aligned}$$

Then, according to the form of M and B' and Lemma 4.7,

$$\begin{aligned} B'_{n-m-i,j} =&\frac{\det A'\big[S \cup \{n - m - i\}, T \cup \{m + j\}\big]}{\det M} \\ =&x_{2m+k-1} - v \cdot V^{m+k-1} \cdot (a_1, a_2, \ldots, a_m), \end{aligned}$$

which is an expression that is only dependent on $i + j$.

A simple index counting shows that $r + 1 = 2m + k - 1$. And thus, the inductive step is complete as desired. □

Corollary 4.8. *For any rectangular shape λ, $P(s_\lambda \mapsto 0) = 1/q$.*

Proof. We will use an inductive style argument to show this. Let A be the Jacobi-Trudi matrix corresponding to shape $\lambda = b^n$. First assume $b \geq n \geq 1$. We claim that if $B = \varphi(A; x_1 = a_1, \ldots, x_r = a_r)$ is not empty and if it's entries in the main diagonal and entries below the main diagonal are not all 0, then $P(\det B \mapsto 0) = 1/q$.

We will induct on the number of free variables, i.e. $2n-1-r$. According to Lemma 4.4, if B is not empty and if entries on and below the main diagonal are not all zero, suppose that B has size n'. There exists $k \leq n'$ such that all entries in diagonal i are zero, for all $i < k$ and all entries in diagonal k are the same and are in the form of $x_t - f_{t-1}$ for some $t > 0$ and f_{t-1} being a polynomial depends only on $x_1, \ldots, x_{t-1}$. If $k = n'$, then $\det B = (x_t - f_{t-1})^{n'}$ and clearly $\det B = 0$ iff $x_t = f_{t-1}$, giving a probability of $1/q$ as desired. If $k < n'$, then

$$P(\det B \mapsto 0) = \frac{1}{q} \sum_{c \in \mathbb{F}_q} P\big(\varphi(A; x_1 = a_1, \ldots, x_r = a_r, x_{r+1} = c) \mapsto 0\big).$$

We can use induction hypothesis on each term on the right because (1) we have less free variables; (2) the matrix $\varphi(A; x_1 = a_1, \ldots, x_r = a_r, x_{r+1} = c)$ is not empty since at most $k < n'$ rows and columns will be removed due to row and column operations from B; (3) the matrix either has size smaller than B with no zeroes or has size equal to B with only the first $k-1$ or $k+1 \leq n'$ being 0. Thus, $P(\det B \mapsto 0) = \frac{1}{q}(q \cdot \frac{1}{q}) = 1/q$.

Therefore, the inductive step is completed.

If $b < n$, let A be the Jacobi-Trudi matrix corresponds to it. It is evident that there exists a rectangle shape $\lambda' = t^t$ with Jacobi-Trudi matrix A' such that $A'(x_1 = 0, \ldots, x_{k-1} = 0, x_k = 1)$ equals A with a change of variable. Therefore, $P(\det A \mapsto 0) = 1/q$ by the claim above. $\square$

4.3. **Staircases.**

Now we turn our attention to the staircase shapes, $\lambda = (k, k-1, \cdots, 1)$.

Theorem 4.9. *Let $\lambda = (k, k-1, \ldots, 1)$ be a staircase, then*

$$P(s_\lambda \mapsto 0) = 1/q.$$

Proof. By Jacobi-Trudi,

$$\begin{aligned} s_\lambda &= \begin{vmatrix} h_k & \cdots & h_{2k-4} & h_{2k-3} & h_{2k-2} & h_{2k-1} \\ \vdots & & & \ddots & & \vdots \\ 0 & \cdots & h_2 & h_3 & h_4 & h_5 \\ 0 & \cdots & 1 & h_1 & h_2 & h_3 \\ 0 & \cdots & 0 & 0 & 1 & h_1 \end{vmatrix} \\ &= (-1)^l \left| \begin{array}{ccc|ccc} \cdots & h_{2k-4} & h_{2k-2} & \cdots & h_{2k-3} & h_{2k-1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \cdots & h_2 & h_4 & \cdots & h_3 & h_5 \\ \cdots & 1 & h_2 & \cdots & h_1 & h_3 \\ \cdots & 0 & 1 & \cdots & 0 & h_1 \end{array} \right|. \end{aligned}$$

where the second equality follows from rearranging the columns.

Notice that in the new matrix, all the variables on the left side are distinct from all the variables on the right side. Renaming the variables on the left side by x_i's and the variables on the right side by y_i's, we find the above determinant is just a special case of the determinant of the following kind of square matrices of size $m+n$

$$\left[\begin{array}{ccccc|ccccc} x_n & \cdots & x_{m+n-3} & x_{m+n-2} & x_{m+n-1} & y_{m+1} & \cdots & y_{m+n-2} & y_{m+n-1} & y_{m+n} \\ x_{n-1} & & x_{m+n-4} & x_{m+n-3} & x_{m+n-2} & y_m & & y_{m+n-3} & y_{m+n-2} & y_{m+n-1} \\ \vdots & & \ddots & & \vdots & \vdots & & \ddots & & \vdots \\ 0 & & a & x_1 & x_2 & 0 & & y_1 & y_2 & y_3 \\ 0 & & 0 & a & x_1 & 0 & & 0 & y_1 & y_2 \\ 0 & \cdots & 0 & 0 & a & 0 & \cdots & 0 & 0 & y_1 \end{array}\right]$$

where the left side has m columns and the right side has n columns, and constant $a \neq 0$.

Denote the probability that such a matrix M is singular to be $p(m,n)$. We claim that for all $m \geq 0, n > 0$, we have $p(m,n) = 1/q$. We prove this claim by induction on m.

Base case. If $m = 0$, then the matrix degenerates to an upper triangular matrix with n many x_1's on its main diagonal. The determinant is x_1^n and the probability it is zero is just $1/q$. Hence $p(0,n) = 1/q$ for all $n > 0$.

Induction step. Assume for all $0 < k < m$, for all $n > 0$, $p(k,n) = 1/q$. Then consider $p(m,n)$. We have two possible cases: $n \leq m$ and $n > m$.

First notice that if $n \leq m$, then for each $1 \leq i \leq n$, we can subtract the i^{th} column (counted from the right) in the right side by y_1/a times the i^{th} column (counted from the right) in the left side, and obtain a determinant in the form

$$\left|\begin{array}{cccc|ccc} x_n & \cdots & x_{m+n-2} & x_{m+n-1} & \cdots & y_{m+n-1} - \frac{y_1 x_{m+n-2}}{a} & y_{m+n} - \frac{y_1 x_{m+n-1}}{a} \\ x_{n-1} & & x_{m+n-3} & x_{m+n-2} & & y_{m+n-2} - \frac{y_1 x_{m+n-3}}{a} & y_{m+n-1} - \frac{y_1 x_{m+n-2}}{a} \\ \vdots & \ddots & & \vdots & \ddots & & \vdots \\ 0 & & x_1 & x_2 & & y_2 - \frac{y_1 x_1}{a} & y_3 - \frac{y_1 x_2}{a} \\ 0 & & a & x_1 & & 0 & y_2 - \frac{y_1 x_1}{a} \\ 0 & \cdots & 0 & a & \cdots & 0 & 0 \end{array}\right|$$

$$= (-1)^n a \cdot \left|\begin{array}{ccc|ccc} x_n & \cdots & x_{m+n-2} & \cdots & y'_{m+n-2} & y'_{m+n-1} \\ x_{n-1} & & x_{m+n-3} & & y'_{m+n-3} & y'_{m+n-2} \\ \vdots & \ddots & & \ddots & & \vdots \\ 0 & & x_1 & & y'_1 & y'_2 \\ 0 & & a & & 0 & y'_1 \end{array}\right|$$

by expanding across the last row and renaming the variables. We are allowed to rename the y_i's as their values are independent of each other. In this way we obtain a smaller matrix of the same kind with $m-1$ columns on the left side and n columns on the right side. So for $m \geq n$, $p(m,n) = p(m-1,n) = 1/q$ by the induction hypothesis.

We are left with the case when $n > m$.

If $y_1 = 0$, then the last row of M will just be $(0, \cdots, a, 0, \cdots, 0)$ with a in the m^{th} entry and 0 in the remaining entries. Expand across the last row and we obtain $(-1)^n a$ times a smaller determinant of this kind with $m-1$ columns on the left side and n columns on the right side. Since a is nonzero, we have $P(\det M = 0 | y_1 = 0) = p(m-1,n) = 1/q$ by the induction hypothesis.

If $y_1 \neq 0$, then let $b = y_1$. Using a similar argument as in the case $n \leq m$, interchanging the role of m and n and letting b play the role of a, we obtain that the conditional probability that M is singular is given by $P(\det M = 0 | y_1 \neq 0) = p(m, n-1) = \cdots = p(m, m)$. Then it is reduced to the case when $n \leq m$ and $P(\det M = 0 | y_1 \neq 0) = 1/q$.

So in the case when $n > m$, we have $p(m, n) = P(\det M = 0 | y_1 = 0)P(y_1 = 0) + P(\det M = 0 | y_1 \neq 0)P(y_1 \neq 0) = 1/q \cdot 1/q + (q-1)/q \cdot 1/q = 1/q$.

Since in both cases we have $p(m, n) = 1/q$, we conclude that $p(m, n) = 1/q$ for all m, n.

In particular, we have $P(s_\lambda \mapsto 0) = p(\lfloor \frac{k}{2} \rfloor, \lceil \frac{k}{2} \rceil) = 1/q$. □

5. Classification of $1/q$

In this section we prove that the partition shapes with probability $1/q$ are exactly hooks, rectangles, and staircases.

For that purpose, we first state and prove a series of lemmas concerning when a reduced general Schur matrix has probability $P(\det M \mapsto 0) > 1/q$ as we assign the variables to numbers in $\mathbb{F}_q$ randomly. The key idea here is to view the probability $P(\det M \mapsto 0)$ as an average of different conditional probabilities coming from partial assignments of variables. By Theorem 3.1, each of these conditional probability is at least $1/q$, so our task is reduced to finding a particular conditional probability that is strictly larger than $1/q$.

Lemma 5.1. *Let the matrix M be a reduced general Schur matrix of size n with m free variables $x_1, \cdots, x_m$. If either of the upper left or lower right entries is zero or the upper left and lower right entries have different labels, then $P(\det M \mapsto 0) > 1/q$.*

Proof. First notice by the definition of general Schur matrix, if an entry is 0, then all the entries below it or to the right of it must be 0 as well. In particular, if either of the upper left or lower right entries is zero, then it means either the first column or the last row is the zero vector and we automatically have $P(\det M = 0) = 1 > 1/q$. So we can assume that neither of these two entries is zero.

Without loss of generality let the label of the upper left entry be the smaller of the two, and denote it by k. We have $P(\det M \mapsto 0) = \frac{1}{q^k} \sum_{(a_1, \cdots, a_k) \in \mathbb{F}_q^k} P(\det M \mapsto 0 | x_1 = a_1, \cdots, x_k = a_k)$. Notice for any $a_1, \cdots, a_k$, the matrix $M' = \varphi(M; x_1 = a_1, \cdots, x_k = a_k)$ is nonempty and thus a reduced general Schur matrix, since the labels of the entries in the last column of M are all strictly greater than k. Hence $P(\det M \mapsto 0 | x_1 = a_1, \cdots, x_k = a_k) = P(\det M' \mapsto 0)$, which is at least $1/q$ by Theorem 3.1. Therefore, to achieve $P(\det M \mapsto 0) > 1/q$, we only need one conditional probability to be strictly larger than $1/q$.

Now we can choose values $b_1, \cdots, b_k$ for $x_1, \cdots, x_k$ in order such that under this assignment all the entries in the first column become 0. Then the first column is just the zero column and the conditional probability $P(\det M \mapsto 0 | x_1 = b_1, \cdots, x_k = b_k) = 1$, as desired. □

Lemma 5.2. *We can strengthen the condition in the previous lemma to be "if the labels of the entries in first column from bottom to top are not exactly the same as the labels of the entries in the last row from left to right," then $P(\det M \mapsto 0) > 1/q$.*

Proof. We may assume the the upper left entry and the lower right entry are both nonzero and have the same label l, otherwise from the previous lemma we immediately have the desired result. Look at the labels of the nonzero entries in the first column from bottom to top and nonzero entries in the last row from left to right. Compare them in order and find the first one that is different. Without loss of generality let the entry from the first column have the smaller label and denote it by $k < n$. We have $P(\det M \mapsto 0) = \frac{1}{q^k} \sum_{(a_1, \cdots, a_k) \in \mathbb{F}_q^k} P(\det M \mapsto$

$0|x_1 = a_1, \cdots, x_k = a_k)$. For any $a_1, \cdots, a_k$, the matrix $M' = \varphi(M; x_1 = a_1, \cdots, x_k = a_k)$ is nonempty and thus a reduced general Schur matrix, since the labels of the entries in the last column of M are all strictly greater than k. Again, we only need some of the conditional probabilities to be strictly larger than $1/q$.

Now assign $x_1, \cdots x_{k-1}$ by values $b_1, \cdots, b_{k-1}$ in order so that all the entries in the first column below the one with label k are zero. If in this process any entry in the last row becomes a nonzero constant, then because of how ψ works, the last row of M is deleted in the precess of calculating $M'_1 = \varphi(M; x_1 = a_1, \cdots, x_{k-1} = a_{k-1})$. Hence the lower right entry in M'_1 comes from some entry in the last column of M that lies strictly above the lower right entry, and thus have label strictly larger than l. In comparison, the upper left entry of M'_1 still have label l since we never delete the first column. Since the labels of the upper left and lower right entries no longer match, applying the previous lemma we know $P(\det M'_1 \mapsto 0) > 1/q$, which gives $\sum_{b_k \in \mathbb{F}_q} P(\det M \mapsto 0|x_1 = b_1, \cdots, x_k = b_k) = P(\det M \mapsto 0|x_1 = b_1, \cdots, x_{k-1} = b_{k-1}) = P(\det M'_1 \mapsto 0) > 1/q$.

If instead there is no nonzero constant in the first row, then we just assign some value b_k to x_k to make the entry in the first column with label k to be nonzero. Then look at $M'_2 = \varphi(M; x_1 = a_1, \cdots, x_{k-1} = a_{k-1})$. Now the upper left entry in M'_2 has some label strictly larger than l while the lower right entry of M'_2 still have label l. Applying the previous lemma we know $P(\det M \mapsto 0|x_1 = b_1, \cdots, x_k = b_k) = P(\det M'_2 \mapsto 0) > 1/q$. □

Lemma 5.3. *Let the square matrix M of size n be a reduced general Schur matrix of size n with m free variables $x_1, \cdots, x_m$. Let all the entries on the main diagonal be nonzero and let the label of the upper left and the lower right entry be the same and be no greater than the label of any other entries on the main diagonal. Then if some diagonal entry actually has a strictly larger label, we have $P(\det M \mapsto 0) > 1/q$.*

Proof. Denote the label of the the upper left entry by k. For any assignment of $x_1 = a_1, \cdots, x_{k-1} = a_{k-1}$, the matrix $M' = \varphi(M; x_1 = a_1, \cdots, x_{k-1} = a_{k-1})$ is nonempty and thus a reduced general Schur matrix, since the labels of the entries in the last column of M are all strictly greater than $k-1$. Hence $P(\det M \mapsto 0|x_1 = a_1, \cdots, x_{k-1} = a_{k-1}) = P(\det M' \mapsto 0) \geq 1/q$ by Theorem 3.1. Again, we only need one conditional probability to be larger than $1/q$.

Assign values $b_1, \cdots, b_{k-1}$ to $x_1, \cdots, x_{k-1}$ in order so that all entries in the first column expect the first one becomes zero. If in this process we have some nonzero constant in the last row, then by the proof of the previous lemma we already have $P(\det M \mapsto 0) > 1/q$. Hence we can assume without loss of generality that all except the last entry in the last row also become zero. Let $M' = \varphi(M; x_1 = a_1, \cdots, x_{k-1} = a_{k-1})$. Notice in M' the labels of the upper left and lower right entries are still k while labels of the other diagonal entries get larger or stay the same compared to M. So the new matrix M' still satisfy the assumption in the lemma.

Then consider the upper left entry of M' and assign x_k. If x_k equals to some value such that the upper left entry becomes zero (this has probability $1/q$), then we has a zero column and the determinant is automatically zero. If the upper left entry is not zero (this has probability $(q-1)/q$), then we apply ψ to obtain a reduced general Schur matrix M'' of smaller size. Notice that since there are some entries on the diagonal with label strictly larger than k, the number of nonzero constants in M' must be strictly smaller than the its size, so M'' is nonempty. By Theorem 3.1, $P(\det M'' \mapsto 0) \geq 1/q$. Combining, we get the conditional probability is $P(\det M \mapsto 0|x_1 = b_1, \cdots, x_k = b_k) = P(\det M' \mapsto 0) = 1/q \cdot 1 + (q-1)/q \cdot P(\det M'' \mapsto 0) \geq 1/q + (q-1)/q \cdot 1/q > 1/q$. $\square$

Next we turn our attention to special Schur matrices and state and prove some necessary conditions for a special Schur matrix M to have $P(\det M \mapsto 0) = 1/q$. The idea is to use proof of contradiction and find one conditional probability strictly larger than $1/q$.

Lemma 5.4. *For a special Schur matrix M of size n, if we have $P(\det M \mapsto 0) = 1/q$, then the entries on the main diagonal all have the same label.*

Proof. If not, let the largest among all the labels of main diagonal entries be k. Assign 0 to $x_1, \cdots, x_{k-1}$. Then every entry below the diagonal is 0. And since the main diagonal entries have different labels, some of them have become 0 as well. So under this assignment the conditional probability that the determinant is zero is 1. Hence we have $P(\det M \mapsto 0) > 1/q$, a contradiction. $\square$

Lemma 5.5. *For a special Schur matrix M of size n, if we have $P(\det M \mapsto 0) = 1/q$, then for each diagonal below the main diagonal, the label of the entries in the first column and the last column are the smallest.*

Proof. If not, let $i < n$ be the smallest integer such that on the i^{th} diagonal, the label of the two entries $M_{n-i+1,1}$ and $M_{n,i}$ is not the smallest. (These two entries have the same label by previous lemmas.) Let k be the smallest of all the labels on this diagonal. Then entries above the i^{th} diagonal, $M_{n-i+1,1}$ and $M_{n,i}$ all have labels strictly larger than k. Hence the number of entries in M with label k is at most $i - 2 \leq n - 3$. Let $M' = \varphi(M; x_1 = \cdots = x_{k-1} = 0, x_k = 1)$. Notice in the application of φ rows and columns are only deleted after x_k is assigned. Since in M the number of entries with label k is at most $n - 3$, the size of M' is at least 3. The previous lemma gives that all the entries on the main diagonal of M have the same label. Now the upper left entry and the lower right entry of M' still have the same label as they come from the corresponding entries of M, but some other entries on the main diagonal have strictly larger labels as some columns are shifted to the left. By Lemma 5.3, we have $P(\det M \mapsto 0) > 1/q$, a contradiction. $\square$

Lemma 5.6. *For a special Schur matrix M of size n, if $P(\det M \mapsto 0) = 1/q$, then all the entries on the $(n-1)^{th}$ diagonal have the same label.*

Proof. Assume not. Denote the largest label among the entries on the $(n-1)^{\text{th}}$ diagonal by k. Since $m_{2,1}$ and $m_{n,n-1}$ have the smallest label on the $(n-1)^{\text{th}}$ diagonal and the labels of the main diagonal entries are all equal and strictly larger than k, there are at most $n - 3$ entries in M with label k. Let $M' = \varphi(M; x_1 = \cdots = x_{k-1} = 0, x_k = 1)$. Notice the size M' is at least 3 and in M' the labels of upper left and lower right entries are equal as in M, while the labels of some other main diagonal entries are increased. So on the main diagonal, the labels of the upper left and the lower right entries are the smallest and some other entries have strictly larger labels. By Lemma 5.3, we have $P(\det M \mapsto 0) > 1/q$, a contraction. $\square$

Corollary 5.7. *Let M be a special Schur matrix of size n. If we have $P(\det M \mapsto 0) = 1/q$, then on each diagonal, the labels of the entries are equal. Further, the labels of each diagonal form an arithmetic progression.*

Proof. Use Lemma 5.4 and 5.6 and induct from the main diagonal down to the the 1^{st} diagonal and up to the $(2n-1)^{\text{th}}$ diagonal. Use the fact that by definition, for any 2×2

submatrix of a special Schur matrix, the sum of labels of the two entries on the diagonal equals the sum of labels of the two entries on the antidiagonal. □

With these lemmas at hand, we can narrow our attention down to Jacobi-Trudi matrices. For the rest of this section, let M be the Jacobi-Trudi matrix of a partition shape $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$. We introduce the following main theorem which characterizes all the possible shapes λ that have probability $1/q$.

Theorem 5.8 (Characterization of $1/q$)**.** *The shapes where we have the probability of* $1/q$ *are exactly hooks, rectangles and staircases.*

We divide the proof of this theorem into several cases.

Case 1: M does not contain any constants. This implies that every entries in M is a variable. Therefore M is a special Schur matrix by the properties of Jacobi-Trudi matrix. Then by Corollary 5.7, the labels of the entries on the main diagonal are equal, which implies that $\lambda_1 = \lambda_2 = \cdots = \lambda_n$. Then λ is a rectangle shape.

Case 2: M contains some constants. In this case we first state some special cases of M, and then generalize all the possible shapes that has probability $P(\det M \mapsto 0)$ exactly $1/q$.

Lemma 5.9. *If* M *contains some constants,* $\psi(M)$ *is at least* λ_n*-by-*λ_n*.*

Proof. Since M contains some constants, the last row must have some constants. Hence it must have at least λ_n columns. Since the last λ_n columns does not contain any constants, they will not be canceled by the operation ψ. Hence $\psi(M)$ is at least λ_n-by-λ_n. □

Remark 5.10. $\psi(M)$ is exactly λ_n-by-λ_n if and only if every other columns of M contains nonzero constant, i.e., every other columns will be canceled by ψ.

Lemma 5.11. *If* $\lambda_n \geq 2$ *and* $P(\det M \mapsto 0) = 1/q$*, then we have* $\lambda = (\lambda_1^m, \lambda_n^{n-m})$ *where* $1 \leq m \leq n$*.*

Proof. Let $M' = \psi(M)$, and denote the size of M' as m. By Corollary 5.7, the labels of each diagonal of M' form an arithmetic progression. We denote the common difference as k. Since $\lambda_n \geq 2$, we have $M_{nn} = h_{\lambda_n}$, $M_{(n-1)n} = h_{\lambda_n - 1}$, and they are both non-constant. Therefore, the entries $M'_{(m-1)m}$ and M'_{mm} come from the last two columns of M, and the difference between their labels is 1. We hence have $k = 1$, which means in each row of M', the labels form a consecutive sequence. Hence we must have that the 1's in the original matrix M are in the leftmost $n - m$ rows, otherwise the difference between some adjacent entries in the same row of M' is at least 2. And by definition we already know the 1's are in the bottom $n - m$ rows of M. We can thus divide M into four blocks and write it as

$$\left[\begin{array}{cccc|ccc}
h_{\lambda_1} & h_{\lambda_1+1} & \cdots & & h_a & h_{a+1} & \cdots \\
\vdots & h_{\lambda_2} & \cdots & & h_{a-1} & h_a & \cdots \\
 & & \ddots & & \vdots & & \ddots \\
\hline
1 & h_1 & h_2 & \cdots & \ddots & & \\
0 & 1 & h_1 & \cdots & & & \\
\vdots & & \ddots & & & \vdots & \vdots \\
0 & \cdots & 0 & 1 & \cdots & h_{\lambda_n - 1} & h_{\lambda_n}
\end{array}\right]$$

Notice that all the 1's in M must appear consecutively along the diagonal in the lower left block of M, and M' comes from the block $M[[m], \{n-m+1, \cdots, n\}]$ on the upper right corner of M. Since $n-m$ is the number of 1's, we have $\lambda_1 = \lambda_2 = \cdots = \lambda_m$ and $\lambda_{m+1} = \lambda_{m+2} = \ldots = \lambda_n$, which gives the desired result. $\square$

What the previous lemma tells us is that for a partition λ with the last part at least 2, if we have $P(s_\lambda \mapsto 0) = 1/q$, then λ must be a rectangle or a fattened hook. The next lemma further narrows the possibility and shows λ can only be a rectangle.

Lemma 5.12. *Let M be the Jacobi-Trudi matrix corresponding to a partition shape $\lambda = (a^p, b^m)$ where $p, m \in \mathbb{Z}^+$ and $a > b \geq 2$. Then $P(\det M \mapsto 0) > 1/q$.*

Proof. By Corollary 3.2, we only need to find one partial assignment such that the conditional probability is larger than $1/q$.

Let $k = a - b \geq 1$. We can draw the partition shape λ as follows.

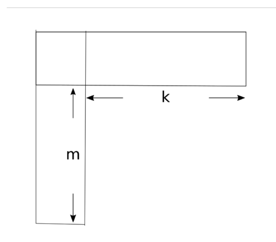

FIGURE 1. Partition Shape of λ

We can see from the figure that if $k < m$, we can take the transpose of λ and calculate its Jacobi-Trudi matrix N. By Corollary 4.2, $P(\det M \mapsto 0) = P(\det N \mapsto 0)$. Hence we can assume $k \geq m$ without loss of generality.

Now we have

$$M = \left[\begin{array}{ccccc|cccc} h_a & \cdots & & & & & & & \\ \vdots & & & & \ddots & & \vdots & & \\ h_{k+2} & \cdots & & h_{m+k} & h_{m+k+1} & h_{m+k+2} & \cdots & & \\ h_{k+1} & h_{k+2} & & \cdots & h_{m+k} & h_{m+k+1} & h_{m+k+2} & \cdots & \\ \hline 1 & h_1 & h_2 & \cdots & h_{m-1} & h_m & h_{m+1} & h_{m+2} & \cdots \\ 0 & 1 & h_1 & \cdots & h_{m-2} & h_{m-1} & h_m & h_{m+1} & \cdots \\ \vdots & & \ddots & & \vdots & \vdots & \vdots & & \\ 0 & & \cdots & 0 & 1 & h_1 & h_2 & \cdots & h_b \end{array}\right] \tag{5}$$

This gives

$$\psi(M) = \begin{bmatrix} & \vdots & & \\ h_{m+k+2} - s_{m+k+1} & & \cdot^{\cdot^{\cdot}} & \\ h_{m+k+1} - r_{m+k} & & h_{m+k+2} - t_{m+k+1} & \cdots \end{bmatrix}$$

where r is a polynomial of h_1 through h_{m+k} with no constant term while s, t are polynomials of h_1 through h_{m+k+1} with no constant terms.

We assign $h_1 = h_2 = \cdots = h_m = h_{m+k+1} = 0$ and $h_{m+1} = h_{k+1} = 1$. Since $k+1 > m$, this assignment is legal, which means no single variables is assigned two different values. Under this assignment, we have

$$M = \left[\begin{array}{ccccc|cccc}
h_a & \cdots & & & & & & & \\
\vdots & & & & \ddots & & \vdots & & \\
h_{k+2} & \cdots & & h_{m+k} & 0 & h_{m+k+2} & \cdots & & \\
1 & h_{k+2} & & \cdots & h_{m+k} & 0 & h_{m+k+2} & \cdots & \\
\hline
1 & 0 & 0 & \cdots & 0 & 0 & 1 & h_{m+2} & \cdots \\
0 & 1 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots \\
\vdots & & \ddots & & \vdots & \vdots & \vdots & & \\
0 & & \cdots & 0 & 1 & 0 & 0 & \cdots & h_b
\end{array}\right]$$

We can now see that in $\psi(M)$, we have $h_{m+k+1} - r_{m+k} = 0$, $s_{m+k+1} = 0$ while $t_{m+k+1} = h_{m+1}h_{k+1} = 1 \neq 0$. Therefore by Lemma 5.2 we have $P(\det M \mapsto 0) > 1/q$. $\square$

Proof.[Proof of Theorem 5.8] By Corollary 3.2, for any shape except hooks, rectangles and staircases, we only need to show that there exists one partial assignment such that the conditional probability is larger than $1/q$.

We perform ψ on M and denote $M' = \psi(M)$. By Remark 2.10, M' is a special Schur matrix. By Corollary 5.7, the labels of each diagonal of M' form an arithmetic progression, otherwise $P(\det M \mapsto 0) > 1/q$.

We denote the common difference of the labels of two consecutive entries to be k.

Subcase 1: $k = 0$. This means that $\psi(M)$ is a 1-by-1 matrix and does not contain any common differences. By Lemma 5.9, $\lambda_n = 1$. By Remark 5.10, every other columns of M contains 1. This implies that $\lambda_2 = \lambda_3 = \cdots = \lambda_n = 1$. Hence λ is a hook.

Subcase 2: $k = 1$. Then in order to maintain the probability 1/q, we claim that $\psi(M)$ must be λ_n-by-λ_n. This holds because if $\psi(M)$ contains another column, in M this column must be at least one column away from the last λ_n columns, since the last $(\lambda_n + 1)$th column contains 1. In this case, the labels in $\psi(M)$ cannot form an arithmetic progression. By Corollary 5.7, $P(\det M \mapsto 0) > 1/q$. Since $\psi(M)$ is at least 2-by-2, $\lambda_n \geq 2$. By Lemma 5.11 and 5.12 we have $P(\det M \mapsto 0) > 1/q$ unless λ is a rectangle.

Subcase 3: $k \geq 2$.

In Subcase 2 and proof of Lemma 5.11 we have shown that $k = 1$ if and only if $\lambda_n \geq 2$. Hence in this subcase $\lambda_n = 1$, which implies $M_{n(n-1)} = 1$. By the fact that k is the common difference of the labels of two consecutive entries in M, there must be $k-1$ columns between any two remaining columns in M. This means that between any two columns with no nonzero constants, there are $k-1$ columns that has nonzero constants. By properties of the partition shape the constant 1's must appear consecutively starting from the bottom to the top along some diagonal. Therefore, let $\psi(M)$ has size b, we can write the lower right corner of M in the following form:

$$
M_{lr} = \begin{vmatrix}
0 & 1 & e_1 & \cdots & e_{k-2} & e_{k-1} & \cdots & & & \\
0 & 0 & 1 & & \cdots & e_{k-2} & \cdots & & & \\
& & & \ddots & & & & & & \\
& & & & 1 & e_1 & \cdots & & & \\
& & & & & 0 & 1 & \cdots & & \\
& & & & & & & \ddots & & \\
& & & & & & & & 1 & e_1
\end{vmatrix}
$$

where M_{lr} is an (kb)-by-$(k-1)b$ matrix. Since we have included all the columns that will not be removed by the operation ψ in M_{lr}, the rest columns must have constant 1, and must appear on the left of M. Assume there are p such columns, hence similar to equation 5 in Lemma 5.12 we can write M in the following form:

$$
M = \left[\begin{array}{cccccccccc}
\vdots & & \ddots & & & & & & & \\
e_{r+k+1} & & \cdots & e_{r+p+k} & e_{r+p+k+1} & \cdots & & & & \\
e_{r+1} & \cdots & & e_{r+p} & e_{r+p+1} & & \cdots & & e_{r+p+k+1} & \cdots \\
1 & e_1 & \cdots & e_{p-1} & e_p & e_{p+1} & \cdots & & e_{p+k} & \cdots \\
0 & 1 & e_1 & \vdots & e_{p-1} & e_p & \cdots & \vdots & & \\
\vdots & & \ddots & & \vdots & \vdots & & & & \\
0 & \cdots & 0 & 1 & e_1 & e_2 & \cdots & e_k & e_{k+1} & \cdots \\
\hline
\multicolumn{4}{c|}{0} & \multicolumn{6}{c}{M_{lr}}
\end{array}\right]
$$

where r is the difference between the two rows. We can assume $p \geq 1$ because if $p = 0$, we let $p = k-1$ and still preserve the shape of M.

In terms of the partition shape, in M_{lr} we notice that there are $k-1$ parts of length 1, since $\lambda_n = 1$ and there are $k-1$ consecutive 1s on the diagonal left to the main diagonal. Since there is only one column with no nonzero constants between two sets of consecutive 1s, in λ we have $k-1$ 2s, 3s, etc. Also notice that starting from the row with e_{r+1} to the top, the difference in labels between two consecutive rows is exactly k, the difference between λ_i and λ_{i+1} is $k-1$. Then we can draw the partition shape λ as shown in Figure 5.

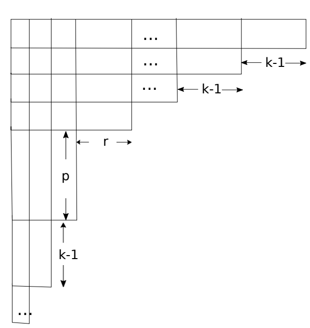

FIGURE 2. Partition Shape of λ

From the figure, we can see that if $r < p$, we can take the transpose of λ and use Corollary 4.2 to get the same probability. Therefore we assume without loss of generality that $r \geq p$, which means we can assume $r \geq 1$.

Then $\psi(M)$=

$$\left| \begin{array}{ccc} \vdots & & \\ & & \\ e_{r+p+k} - g_{r+p+k-1} & \iddots & \\ e_{r+p} - f_{r+p-1} & e_{r+p+k} - h_{r+p+k-1} & \cdots \end{array} \right|$$

where f_i, g_i, h_i are polynomials of e_1 through e_i.

If $p \neq k-1$, $p+1 \neq k$, so $p+1$ must be included in $[1, p+k] \backslash \{k\}$ since $k \geq 2$. Also, $r+k \neq r+p+1$ and $r+k > r+1 \geq p+1$ by the assumption. Then we assign $e_1 = e_2 = \cdots = e_p = e_{r+p+1} = 0$, while assign $e_{p+1} = e_{r+k} = 1$. The inequalities $1 < 2 < \cdots < p < p+1 \leq r+k$ and $p \neq k-1$ ensures that no two different values are assigned in a single entry. Under this assignment, in $\psi(M)$, $g_{r+p+k-1} = 0$, while $h_{r+p+k-1} \geq 1$ since it contains the term $c \cdot e_{p+1} e_{r+k-1}$, where c is a nonzero constant. Therefore by Lemma 5.6 we have $P(\det(M) \mapsto 0) > 1/q$.

If $p = k-1$, we assign $x_1 = x_2 = \cdots = x_p = x_{r+p+1} = 0$, while assign $x_{k+1} = x_{r+p} = 1$. We know that $p < r+p < r+p+1$, so we only need to consider if $k+1$ is the same as some entries that have been assigned 0. Since $p = k-1$, $k-p = 1 \leq r$, hence $p < k+1 \leq r+p+1$ and the equality is achieved if and only if $r = 1$. If $r = 1$, since we have $r \geq p$ we have $p = 1$, $k = 2$, so λ is a staircase, which has $P(\det(M) \mapsto 0) = 1/q$. If $r \geq 2$, by previous argument no two different values are assigned in a single entry. By similar argument we show that in $\psi(M)$, $g_{r+p+k-1} = 0$, while $h_{r+p+k-1} \geq 1$. Combining two cases gives us the desired result. $\square$

6. Independence Results for families with $P = 1/q$

In Section 4.1, we saw that Schur functions of hook shapes are as independent as possible. In this section, we investigate the independence of vanishing of the other two shapes with probability of vanishing $\frac{1}{q}$, namely rectangles and staircases.

6.1. **Rectangles.**

Theorem 6.1. *Let $c \in \mathbb{N}$ be arbitrary. Then the events $\{s_{k^\ell} \mapsto 0 \mid k - \ell = c\}$ are set-wise independent.*

Proof. We show the result in the case $c = 0$; for other values of c, the result follows in the same way.

We first reduce to showing the following result: Let $k \in \mathbb{N}$ be arbitrary and C a collection of conditions $\{C_i\}_{i=1,\ldots,k-1}$, where C_i is either $s_{i^i} \mapsto 0$ or $s_{i^i} \not\mapsto 0$ for each i. Then

$$P(s_{k^k} \mapsto 0 \mid C) = \frac{1}{q}.$$

Assuming this result, we show that $P(s_{a^a} \mapsto 0 \mid s_{b^b} \mapsto 0) = \frac{1}{q}$ for $a \neq b$. By Corollary 4.2, we may assume WLOG that $b < a$. We have

$$P(s_{a^a} \mapsto 0 \mid s_{b^b} \mapsto 0) = \frac{P(s_{a^a} \mapsto 0 \;\&\; s_{b^b} \mapsto 0)}{P(s_{b^b} \mapsto 0)} = q \cdot P(s_{a^a} \mapsto 0 \;\&\; s_{b^b} \mapsto 0),$$

so it suffices to show that $P(s_{a^a} \mapsto 0 \;\&\; s_{b^b} \mapsto 0) = \frac{1}{q^2}$. We have

$$\begin{aligned}
&P(s_{a^a} \mapsto 0 \;\&\; s_{b^b} \mapsto 0) \\
&\quad = P(s_{a^a} \mapsto 0 \;\&\; s_{b^b} \mapsto 0 \;\&\; s_{(a-1)^{a-1}} \mapsto 0) + P(s_{a^a} \mapsto 0 \;\&\; s_{b^b} \mapsto 0 \;\&\; s_{(a-1)^{a-1}} \not\mapsto 0) \\
&\quad = \cdots \\
&\quad = \sum_{C \in \mathcal{C}} P(s_{a^a} \mapsto 0 \;\&\; s_{b^b} \mapsto 0 \;\&\; C),
\end{aligned}$$

where $\mathcal{C}$ is the collection of conditions $\bigcup_{\substack{1 \le i \le b \\ i \neq a,b}} C_i$. Therefore we reduce to showing that each summand is $\frac{1}{q^a}$.

Indeed, we can write

$$P(s_{a^a} \mapsto 0 \;\&\; s_{b^b} \mapsto 0 \;\&\; C) = \frac{P(s_{a^a} \mapsto 0 \mid s_{b^b} \mapsto 0 \;\&\; C)}{P(s_{b^b} \mapsto 0 \;\&\; C)} = \frac{1}{qP(s_{b^b} \mapsto 0 \;\&\; C)}$$

by our assumed result. Therefore, induction on a gives that $P(s_{b^b} \mapsto 0 \;\&\; C) = \frac{1}{q^{a-1}}$, which gives our desired result.

We are now left with showing the result claimed at the beginning. We claim this follows immediately from Corollary 3.2. Indeed, write

$$P(s_{k^k} \mapsto 0) = P_1 P(s_{k^k} \mapsto 0 \mid s_{1^1} \mapsto 0) + (1 - P_1) P(s_{k^k} \mapsto 0 \mid s_{1^1} \not\mapsto 0).$$

Expanding each term similarly, we see inductively that we can write

$$P(s_{k^k} \mapsto 0) = \sum_{C = \{C_i\}_{i=1,\ldots k-1}} a_C P(s_{k^k} \mapsto 0 \mid C),$$

where $\sum_C a_C = 1$. Because LHS is $\frac{1}{q}$ and each summand probability is $\geq \frac{1}{q}$ by the proof of Corollary 3.2, each must be precisely $\frac{1}{q}$. This completes the proof. $\square$

Using the same argument on the collection $\{s_{k^\ell} \mid k + \ell = c\}$ gives the following.

Theorem 6.2. *Let $c \in \mathbb{N}$ be arbitrary. Then the events $\{s_{k^\ell} \mapsto 0 \mid k + \ell = c\}$ are set-wise independent.*

Remark 6.3. The events $s_\lambda \mapsto 0$ are not generally even independent, however, as the following example shows.

Example 6.4. Let $\lambda = 2^2$ and $\mu = 3^2$. We show that $P(s_\lambda \mapsto 0 \mid s_\mu \mapsto 0) \neq \frac{1}{q}$. For this, it suffices to show that $P(s_\lambda \mapsto 0 \;\&\; s_\mu \mapsto 0) \neq \frac{1}{q^2}$.

We have

$$s_\lambda = \begin{vmatrix} h_2 & h_3 \\ h_1 & h_2 \end{vmatrix}$$

$$s_\mu = \begin{vmatrix} h_3 & h_4 \\ h_2 & h_3 \end{vmatrix}$$

We do casework on whether $h_2 = 0$ and/or $h_1 = 0$.

<u>Case 1</u>: $h_2 = h_1 = 0$. Then we must have $h_3 = 0$. This gives q cases.

<u>Case 2</u>: $h_2 = 0$ and $h_1 \neq 0$. Then we must have $h_3 = 0$, giving $q(q-1)$ possibilities.

Case 3: $h_2 \neq 0$. Then $h_3 = \frac{h_2^2}{h_1}$ and $h_4 = \frac{s_2^3}{s_1^2}$, giving $q(q-1)$ cases.
Altogether, we have $2q^2 - q$ instances our of q^4 where $s_\lambda \mapsto 0$ and $s_\mu \mapsto 0$. Hence the two events are not independent.

6.2. **Staircases.** We now turn our attention to staircases. Throughout this section, let $\lambda_k = (k, k-1, \dots, 1)$ be the k-staircase partition.

Theorem 6.5. *For all $k \in \mathbb{N}$ with $k \geq 3$, we have*

$$P(s_{\lambda_k} \mapsto 0 \mid s_{\lambda_{k-2}} \mapsto 0) = \frac{1}{q}.$$

Proof. By Jacobi-Trudi, we have

$$s_{\lambda_k} = \begin{vmatrix} h_k & \cdots & h_{2k-4} & h_{2k-3} & h_{2k-2} & h_{2k-1} \\ \vdots & & & \ddots & & \vdots \\ 0 & \cdots & h_2 & h_3 & h_4 & h_5 \\ 0 & \cdots & 1 & h_1 & h_2 & h_3 \\ 0 & \cdots & 0 & 0 & 1 & h_1 \end{vmatrix}.$$

Note that $s_{\lambda_{k-2}}$ can be obtained from this matrix by removing the first and last rows, along with the last two columns. Hence expanding the determinant about the first row and then last row, we see the determinant will contain a term of the form

$$h_{2k-1} s_{\lambda_{k-2}},$$

from the $(1,n)$- and then $(n-1, n-1)$-cofactor. Further, this is the only term in which h_{2k-1} appears. So

$$P(s_{\lambda_k} \mapsto 0 \mid s_{\lambda_{k-2}} \not\mapsto 0) = \frac{1}{q}.$$

From Theorem 4.9, we have

$$\begin{aligned} \frac{1}{q} &= P(s_{\lambda_k} \mapsto 0) = P(s_{\lambda_{k-2}} \mapsto 0) P(s_{\lambda_k} \mapsto 0 \mid s_{\lambda_{k-2}} \mapsto 0) \\ &\quad + P(s_{\lambda_{k-2}} \not\mapsto 0) P(s_{\lambda_k} \mapsto 0 \mid s_{\lambda_{k-2}} \not\mapsto 0) \\ &= \frac{1}{q} \cdot \frac{1}{q} + \frac{q-1}{q} P(s_{\lambda_k} \mapsto 0 \mid s_{\lambda_{k-2}} \mapsto 0). \end{aligned}$$

Solving this, we obtain the desired implication. ☐

Corollary 6.6. *For all $k \in \mathbb{N}$, the results $s_{\lambda_k} \mapsto 0$ is independent of $s_{\lambda_{k+2}} \mapsto 0$.*

Proof. We have

$$P(s_{\lambda_k} \mapsto 0 \ \&\ s_{\lambda_{k-2}} \mapsto 0) = P(s_{\lambda_{k-2}} \mapsto 0) P(s_{\lambda_k} \mapsto 0 \mid s_{\lambda_{k-2}} \mapsto 0) = \frac{1}{q^2}$$

by Theorem 6.5 and Theorem 4.9. Swapping the roles of k and $k-2$ above and again using Theorem 4.9, we obtain the desired result. ☐

Corollary 6.7. *For all $k \in \mathbb{N}$, the events $s_{\lambda_k} \mapsto 0$ is independent of $s_{\lambda_1} = s_1 = h_1 = e_1 \mapsto 0$.*

Proof. The proof of Theorem 4.9 showed that $P(s_{\lambda_k} \mapsto 0 \mid h_1 \mapsto 0) = P(s_{\lambda_k} \mapsto 0 \mid h_1) = \frac{1}{q}$. Hence $s_{\lambda_k} \mapsto 0$ is independent of $h_1 \mapsto 0$. The reverse follows along the same lines as Corollary 6.6. ☐

Conjecture 6.8. The families $\{s_{\lambda_k} \mid k \text{ odd}\}$ and $\{s_{\lambda_k} \mid k \text{ even}\}$ are set-wise independent.

7. Distribution of Other Values

We now broaden our focus to consider not only $P(s_\lambda \mapsto 0)$, but $P(s_\lambda \mapsto a)$ for any a in $\mathbb{F}_q$, particularly in the case where λ is a rectangle.

7.1. General Results.

Proposition 7.1. *Let λ be a partition with $|\lambda| = n$. Then $P(s_\lambda \to a) = P(s_\lambda \to x^n a)$ for any $a, x \in \mathbb{F}_q$ with $x \neq 0$*

Proof. s_λ is a homogeneous polynomial of degree n, and each e_i is a homogeneous polynomial of degree i, so if some assignment $e_1 = a_1, ..., e_n = a_n$ gives $s_\lambda = a$, then the assignment $e_1 = xa_1, e_2 = x^2 a_2, ..., e_n = x^n a_n$ gives $s_\lambda = x^n a$. Since x is nonzero, this creates a bijection between assignments of the e_i such that $s_\lambda = a$ and assignments of the e_i such that $s_\lambda = x^n a$. $\square$

Corollary 7.2. *Let λ be a partition with $|\lambda| = n$, and let q be a prime power with* $\gcd(n, q-1) = 1$. *Then $P(s_\lambda \to a) = P(s_\lambda \to b)$ for any nonzero $a, b \in \mathbb{F}_q$.*

Proof. Since $\gcd(n, q-1) = 1$, $\{x^n | x \in \mathbb{F}_q^\times\} = \mathbb{F}_q^\times$, so the result follows directly from Proposition 7.1 . $\square$

7.2. Rectangles.

Lemma 7.3. *Let b be a nonzero element of $\mathbb{F}_q$ and let $a \geq n$, then*

$$P(s_{a^n} \mapsto b) = \sum_{(c_1, c_2, ..., c_k) \in C(n)} \frac{(q-1)^{k-1}}{q^n} g_b(\gcd(c_1, c_2, ..., c_k, q-1))$$

where $C(n)$ is the set of all compositions (i.e. ordered partitions) of n, and

$$g_b(d) = \begin{cases} 0 & d \nmid \frac{q-1}{ord(b)} \\ d & d | \frac{q-1}{ord(b)} \end{cases}$$

Proof. Recall the map φ, based on ψ defined in section 2, which takes a general Schur matrix and a set of assignments to a reduced general Schur matrix. For this proof we use a modified version of both φ and ψ, that we will denote $\tilde{\varphi}$ and $\tilde{\psi}$. While φ and ψ reduce the size of the matrix, $\tilde{\varphi}$ and $\tilde{\psi}$ will keep the size constant.

Definition 7.4. Let M be a general Schur matrix with m free variables. Define an operation $\tilde{\psi}$ that takes general Schur matrices with m free variables to matrices over $\mathbb{F}_q[x_1, x_2, ..., x_k]$:

(a) If M has no nonzero constants as entries, then $\psi(M) = M$.
(b) If M has $k \geq 1$ many nonzero constant entries, then from top to bottom, for each of these k entries we use it as a pivot to turn all the other entries in its column into zero by subtracting multiple of the row it is in from each of the rows above. Then we further use these nonzero constants to turn all the other entries in the their rows into zero by column operations, giving a new matrix M'. Unlike ψ, we do not delete these rows and columns. Define $\tilde{\psi}(M) = M'$ in this case.

$\tilde{\psi}$ only performs determinant-preserving row and column operations on the matrix, so $\det(M) = \det \tilde{\psi}(M)$

Example 7.5. An example application of $\tilde{\psi}$

$$M = \begin{bmatrix} 0 & 2x_2 & x_4 & x_5 \\ 0 & 1 & 4x_3 & x_4 \\ 0 & 0 & x_1 & x_3 - x_2 \\ 0 & 0 & 0 & x_2 \end{bmatrix}$$

$$\xrightarrow{\text{use nonzero constants to do row and column operations}} \begin{bmatrix} 0 & 0 & x_4 - 8x_2x_3 & x_5 - 2x_2x_4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & x_1 & x_3 - x_2 \\ 0 & 0 & 0 & x_2 \end{bmatrix} = \tilde{\psi}(M)$$

We define $\tilde{\varphi}$ analogously to φ, replacing ψ in the definition with $\tilde{\psi}$.

Definition 7.6. Let M be a reduced general Schur matrix of size n with m variables. Define an operation $\tilde{\varphi}$ recursively:
(a) $\tilde{\varphi}(\emptyset) = \emptyset$.
(b) $\tilde{\varphi}(M; x_1 = a_1) = \tilde{\psi}(M(x_1 = a_1))$, where $M(x_1 = a)$ denotes the matrix obtained from M by assigning value a_1 to x_1.
(c) $\tilde{\varphi}(M; x_1 = a_1, \cdots, x_i = a_i) = \tilde{\varphi}(\tilde{\varphi}(M; x_1 = a_1, \cdots, x_{i-1} = a_{i-1}); x_i = a_i)$.

Lemma 7.7. *Let $\lambda = (a^n)$ be a rectangular partition, and let $A = (x_{j-i+n})_{1\leq i,j\leq n}$ be the Jacobi-Trudi matrix corresponding s_λ. Then $\tilde{\varphi}(A; x_1, x_2, ..., x_r)$ is a block anti-diagonal matrix, where the top right block is $\varphi(A; x_1, x_2, ..., x_r)$ and all others are either scalar multiples of the identity or the zero matrix.*

Example 7.8.

$$A = \begin{bmatrix} x_4 & x_5 & x_6 & x_7 \\ x_3 & x_4 & x_5 & x_6 \\ x_2 & x_3 & x_4 & x_5 \\ x_1 & x_2 & x_3 & x_4 \end{bmatrix}$$

$$\tilde{\varphi}(A; x_1 = 0, x_2 = 0, x_3 = 2) = \begin{bmatrix} 0 & 0 & x_6 - \frac{x_3x_5 - x_4^2}{2} & x_7 - x_4x_5 \\ 0 & 0 & x_5 - x_3x_4 & x_6 - \frac{x_3x_5 - x_4^2}{2} \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}$$

$$\varphi(A; x_1 = 0, x_2 = 0, x_3 = 2) = \begin{bmatrix} x_6 - \frac{x_3x_5 - x_4^2}{2} & x_7 - x_4x_5 \\ x_5 - x_3x_4 & x_6 - \frac{x_3x_5 - x_4^2}{2} \end{bmatrix}$$

Proof. We proceed by induction on r.

Base Case. $\tilde{\varphi}(A; x_1 = a_1)$ is either equal to $\varphi(A; x_1 = a_1)$ or can be decomposed into a $1x1$ nonzero block and a block equal to $\varphi(A; x_1 = a_1)$.

Inductive Step. Assume $\tilde{\varphi}(A; x_1 = a_1, x_2 = a_2, ..., x_{r-1} = a_{r-1})$ is of the desired form. Then $\tilde{\varphi}(A; x_1 = a_1, ..., x_r = a_r) = \tilde{\varphi}(\tilde{\varphi}(A; x_1 = a_1, ...x_{r-1} = a_{r-1}); x_r = a_r)$ will only change the final block of $\tilde{\varphi}(A; x_1 = a_1, x_2 = a_2, ..., x_{r-1} = a_{r-1})$ that is equal to $\varphi(A; x_1 = a_1, x_2 =$

$a_2, ..., x_{r-1} = a_{r-1})$, since it is the only block in which new nonzero constants can appear. Thus it suffices to show that $\tilde{\varphi}(\varphi(A; x_1 = a_1, x_2 = a_2, ..., x_{r-1} = a_{r-1}); x_r = a_r)$ is of the desired form.

By Lemma 5.1, $\varphi(A; x_1 = a_1, x_2 = a_2, ..., x_{r-1} = a_{r-1})$ has all entries the same on its lowest nonzero diagonal. The new assignment $x_r = a_r$ can set only these entries to some nonzero constant, since only they have the smallest label. If $x_r = a_r$ does set these entries to some nonzero constant and these entries are below the main diagonal, then $\tilde{\varphi}(\varphi(A; x_1 = a_1, ..., x_{r-1} = a_{r-1}); x_r = a_r)$ is block-antidiagonal with two blocks, where the lower left block is a scalar multiple of the identity, and the upper right block is $\varphi(A; x_1 = a_1, ..., x_r = a_r)$. If these entries are set to some nonzero constant and are above the main diagonal, then $\tilde{\varphi}(\varphi(A; x_1 = a_1, ..., x_{r-1} = a_{r-1}); x_r = a_r)$ is block-antidiagonal with two blocks, a block of zeros in the bottom left and a scalar multiple of the identity in the top right. If these entries are not sent to 0, then $\tilde{\varphi}(\varphi(A; x_1 = a_1, ..., x_{r-1} = a_{r-1}); x_r = a_r) = \varphi(A; x_1 = a_1, ..., x_r = a_r)$. All possibilities are of the desired form. ☐

If the determinant of A under some particular assingment $x_1 = a_1, x_2 = a_2, ..., x_{2n-1} = a_{2n-1}$ is nonzero, then each block in $\tilde{\varphi}(A; x_1 = a_1, ..., x_{2n-1} = a_{2n-1})$ must have nonzero determinant. Thus each block must be a scalar multiple of the identity, since φ of A with a full assignment of x_1 through x_{2n-1} must either be empty or only contain 0s. The sizes of these blocks therefore must sum to n.

Definition 7.9. Let $A = (x_{j-i+n})_{1 \le i,j \le n}$ be a matrix corresponding to some rectangular Schur function, and let $x_1 = a_1, x_2 = a_2, ...x_{2n-1} = a_{2n-1}$ be an assignment of the x_i such that A is nonsingular. Define the *block structure of A under assignment* $(a_1, a_2, ..., a_{2n-1})$ to be the sequence of the sizes of blocks of $\tilde{\varphi}(A; x_1 = a_1, x_2 = a_2, ...x_{2n-1} = a_{2n-1})$. Denote this by $B(A; x_1 = a_1, x_2 = a_2, ...x_{2n-1} = a_{2n-1}) = (c_1, c_2, ..., c_k)$

We now consider the probability that for some fixed A and $(c_1, .., c_K)$, $B(A; x_1 = a_1, ..., x_{2n-1} = a_{2n-1}) = (c_1, ..., c_k)$. Within a block of size c_i, the $c_i - 1$ diagonals below the main diagonal had to be set to 0, and the main diagonal had to be set to something nonzero. The values of the diagonals above the main diagonal could be anything. Therefore,

$$P(B(A; x_1 = a_1, ..., x_{2n-1} = a_{2n-1}) = (c_1, ..., c_k)) = \left(\frac{1}{q}\right)^{n-k} \left(\frac{q-1}{q}\right)^k = \frac{(q-1)^k}{q^n}$$

The determinant of a matrix of this form is $\pm y_1^{c_2} y_2^{c_2} \cdots y_k^{c_k}$, where the y_i are the values on the diagonal of the i^{th} block. From the definition of $\tilde{\varphi}$ and the structure of A we can see that these y_i are all independent and uniformly distributed across the nonzero values of $\mathbb{F}_q$. Also note that if the determinant is $-y_1^{c_2} y_2^{c_2} \cdots y_k^{c_k}$, then some c_i must be odd, and replacing y_i with $-y_i$ lets us ignore the sign.

We now want to calculate the probability that $y_1^{c_1} y_2^{c_2} \cdots y_k^{c_k} = b$. If $c_1 = c_2 = ... = c_k$, then this is just the probability that for some nonzero $y \in \mathbb{F}_q$, $y^{c_1} = b$, so

$$P(y_1^{c_1} y_2^{c_2} \cdots y_k^{c_k} = b) = \frac{g_b(\gcd(q-1, c_1))}{q-1}$$

If some $c_i > c_j$, then $y_1^{c_1} y_2^{c_2} \cdots y_k^{c_k} = y_1^{c_1} y_2^{c_2} \cdots y_i^{c_i - c_j} \cdots y_{j-1}^{c_{j-1}} (y_i y_j)^{c_j} y_{j+1}^{c_{j+1}} \cdots y_k^{c_k}$

$y_i y_j$ is still uniformly distributed and independent of y_h when $h \neq j$, so

$$P(y_1^{c_1} y_2^{c_2} \cdots y_i^{c_i - c_j} \cdots y_{j-1}^{c_{j-1}} (y_i y_j)^{c_j} y_{j+1}^{c_{j+1}} \cdots y_k^{c_k} = b) = P(y_1^{c_1} y_2^{c_2} \cdots y_i^{c_i - c_j} \cdots y_{j-1}^{c_{j-1}} y_j^{c_j} y_{j+1}^{c_{j+1}} \cdots y_k^{c_k} = b)$$

We can then use the Euclidean algorithm to get that

$$P(y_1^{c_1} y_2^{c_2} \cdots y_k^{c_k} = b) = P(y_1^d y_2^d \cdots y_k^d = b) = \frac{g_b(\gcd(q-1, d))}{q-1}$$

where $d = \gcd(c_1, c_2, ..., c_k)$.

Therefore we have

$$P(S_{a^n} \mapsto b \quad \& \quad B(A; x_1 = a_1, ..., x_{2n-1} = a_{2n-1}) = (c_1, ..., c_k)) = \frac{(q-1)^{k-1}}{q^n} g_b(\gcd(q-1, c_1, ..., c_k))$$

Summing over all compositions gives the lemma. $\square$

Theorem 7.10. *Let b be a nonzero element of $\mathbb{F}_q$, and let $a \geq n$. Then*

$$P(S_{a^n} \mapsto b) = \sum_{d | \gcd(q-1,n)} \frac{f_b(d)}{q^{n(d-1)/d+1}}$$

where

$$f_b(d) = \sum_{e|d} \mu(e) g_b(\frac{d}{e})$$

is the Möbius inverse of g_b.

Proof. By the previous lemma, we have

$$P(S_{a^n} \mapsto b) = \sum_{(c_1, c_2, ..., c_k) \in C(n)} \frac{(q-1)^{k-1}}{q^n} g_b(\gcd(c_1, c_2, ..., c_k, q-1))$$

Note that the summand only depends on $\gcd(c_1, c_2, ..., c_k, q-1)$ and k, so we can rewrite the sum as

$$P(S_{a^n} \mapsto b) = \sum_{\substack{d | \gcd(n, q-1) \\ 1 \leq k \leq n/d}} N(d, k) \frac{(q-1)^{k-1}}{q^n} f_b(d)$$

where $N(d, k)$ counts the number of compositions $(c_1, ..., c_k)$ of n with k parts such that d divides $\gcd(c_1, c_2, ..., c_k, q-1)$ and

$$\sum_{e|d} f_b(e) = g_b(d)$$

or equivalently, by Möbius inversion,

$$f_b(d) = \sum_{e|d} \mu(e) g_b(\frac{d}{e})$$

This switch from g_b to f_b is necessary because some compositions are counted by multiple $N(d,k)$.

$N(d,k)$ is easy to compute, we can consider a k-part composition of n as a choice of $k-1$ break points along a line of length n. To satisfy the gcd requirement we only have $n/d-1$ places where these break points can be, thus $N(d,k) = \binom{a/d-1}{k-1}$.

We can then simplify the sum using the binomial formula:

$$P(S_{a^n} \mapsto b) = \sum_{\substack{d \mid \gcd(n,q-1) \\ 1 \le k \le n/d}} \binom{n/d-1}{k-1} \frac{(q-1)^{k-1}}{q^n} f_b(d) = \sum_{d \mid \gcd(n,q-1)} \frac{q^{n/d-1}}{q^n} f_b(d)$$

This completes the proof. $\square$

Corollary 7.11. *Let b be an element of $\mathbb{F}_q$ of multiplicative order $q-1$, and let $a \geq n$. Then*

$$P(S_{a^n} \mapsto b) = \sum_{d \mid \gcd(q-1,n)} \frac{\mu(d)}{q^{n(d-1)/d+1}}$$

and

$$P(S_{a^n} \mapsto 1) = \sum_{d \mid \gcd(q-1,n)} \frac{\varphi(d)}{q^{n(d-1)/d+1}}$$

where μ is the Möbius function and φ is Euler's totient function.

Proof. These are both immediate consequences of the preceding theorem, when g_b is particularly nice, and therefore f_b is easy to compute. $\square$

8. Miscellaneous Shapes

In this section, we look at miscellaneous shapes λ and calculate $P(s_\lambda \mapsto 0)$. In general, this probability is hard to compute for a random shape, so we mainly focus on generalizations of the special shapes that we have investigated in previous sections.

8.1. Shapes of probability $(q^2+q-1)/q^3$.

We first investigate shapes with probability $(q^2+q-1)/q^3$ which is the next simplest probability after $1/q$. Notice that $(q^2+q-1)/q^3$ is not the next smallest probability, though; the probability $(q^4+(q-1)(q^2-q))/q^5$ from Proposition 3.6 is strictly smaller.

For a partition $\lambda = (a,b)$ with only two parts where $a > b > 1$, it is not hard to see we have $P(s_\lambda \mapsto 0) = (q^2+q-1)/q^3$. Similarly, looking at partitions with three parts, we can also find shapes with this probability.

Proposition 8.1. *Let $\lambda = (a, a-1, a-2)$ for some $a \geq 5$. Then*

$$P(s_\lambda \mapsto 0) = \frac{q^2+q-1}{q^3}.$$

Proof. By Jacobi-Trudi, we have

$$s_\lambda = \begin{vmatrix} h_a & h_{a+1} & h_{a+2} \\ h_{a-2} & h_{a-1} & h_a \\ h_{a-4} & h_{a-3} & h_{a-2} \end{vmatrix}.$$

We count the number of maps $\{h_{a-4}, h_{a-3}, \ldots, h_{a+2}\} \to \mathbb{F}_q^7$ so that this matrix is invertible. From this point onward, we treat the h_i's as elements of $\mathbb{F}_q$.

We first pick the entries in the first column, where we distinguish whether h_{a-2} and h_a are 0. In both cases, we focus on making the third column linearly independent from the first. We have four cases.

<u>Case 1</u>: $h_{a-2}, h_a \neq 0$.
For any choice of h_a, there is precisely one $c \in \mathbb{F}_q^\times$ such that $c \cdot h_{a-2} = h_a$. Hence we may choose h_{a-1} and h_{a+2} in $q^2 - 1$ ways. Since there are then $q^3 - q^2$ ways to pick the second column, and we had $(q-1)^2$ ways of picking h_{a-2} and h_a, this choice accounts for $(q^2-1)(q-1)^2(q^3-q^2)$.

<u>Case 2</u>: $h_{a-2} = 0$ but $h_a \neq 0$.
In this case, the first and third columns are linearly independent regardless of our choice of h_{a+2} and h_{a-1}. Hence we see there are $1 \cdot (q-1)q^2(q^3-q^2) = q^2(q-1)(q^3-q^2)$ invertible matrices obtained this way.

<u>Case 3</u>: $h_{a-2} = h_a = 0$.
In this case, we must choose $h_{a-1}, h_{a+2} \neq 0$ to maintain linear independence of the first and third columns. Hence we obtain $1^2 \cdot (q-1)^2(q^3-q^2)$ invertible matrices in this way.

<u>Case 4</u>: $h_{a-2} \neq 0$, but $h_a = 0$.
In this case, we again automatically obtain linear independence of the first and third column. Hence there are $1 \cdot (q-1)q^2(q^3-q^2) = q^2(q-1)(q^3-q^2)$ invertible matrices obtained in this way.

Putting this all together and dividing by q^7 gives us $P(s_\lambda \not\mapsto 0)$. Subtraction from 1 gives the desired result. □

It turns out that two generalized hook shapes also have this probability.

Proposition 8.2. *Let $\lambda = (a, b, 1^m)$, where $b \geq 2$ and $a \neq b + m$. Then $P(s_\lambda \mapsto 0) = (q^2 + q - 1)/q^3$.*

Proof. By Jacobi-Trudi Identity, we have

$$s_\lambda = \begin{vmatrix} h_a & h_{a+1} & h_{a+2} & \cdots & h_{a+m+1} \\ h_{b-1} & h_b & h_{b+1} & \cdots & h_{b+m} \\ 0 & 1 & h_1 & \cdots & h_m \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & h_1 \end{vmatrix}.$$

Let n be the number of distinct h_i's appearing in the above determinant. Depending on h_a and h_{b-1}, we have three cases.

<u>Case 1</u>: $h_{b-1} \neq 0$. Noting that the cofactor matrix formed by removing the first row and last column is upper triangular with diagonal entries $h_{b-1}, 1, \ldots, 1$, we see that expanding along the last column gives

$$\pm h_{b-1}h_{a+m+1} + P(h_1, \ldots, h_{a+m}),$$

where P is a polynomial. Since $h_{b-1} \neq 0$, for s_λ to be zero the value of h_{a+m+1} is uniquely determined by the choices of the other h_i's, giving us a total of $q^{n-2}(q-1)$ choices.

Case 2: $h_{b-1} = h_a = 0$. Then $s_\lambda = 0$ and the remaining h_i's may be chosen arbitrarily, which gives q^{n-2} ways.

Case 3: $h_{b-1} = 0$ and $h_a \neq 0$. Expanding along the first column gives

$$s_\lambda = h_a \begin{vmatrix} h_b & h_{b+1} & \cdots & h_{b+m} \\ 1 & h_1 & \cdots & h_m \\ 0 & \ddots & \ddots & \vdots \\ 0 & 0 & 1 & h_1 \end{vmatrix}.$$

Noting that the cofactor matrix formed by removing the first row and last column is upper triangular with 1's on the diagonal, we see that expanding along the last column gives

$$s_\lambda = \pm h_a(h_{b+m} + P(h_1, \ldots, h_{b+m-1})).$$

Since $h_a \neq 0$ and $a \neq b+m$, the value of h_{b+m} is uniquely determined by the choices of the other h_i's, giving us a total of $q^{n-3}(q-1)$ choices.

Adding up the number of choices from each case and dividing by q^n gives the desired probability. $\square$

Notice that we cannot drop the condition $a \neq b+m$ in the previous proposition. For example, the shape $\lambda = (4,3,1)$, has probability $(q^3+q^2-2q+1)/q^4$.

Proposition 8.3. *For a more general type of hook $\lambda = (a^m, 1^n)$ where $a, m > 1$, we have $P(s_\lambda \mapsto 0) = (q^2+q-1)/q^3$.*

Proof.

Using Jacobi-Trudi, we have

$$M = \begin{vmatrix} x_a & x_{a+1} & \cdots & x_{a+m-1} & x_{a+m} & x_{a+m+1} & \cdots & x_{a+m+n-1} \\ x_{a-1} & x_a & \cdots & \vdots & & & & \\ \vdots & & \ddots & & \vdots & & & \\ x_{a-m+1} & \cdots & x_{a-1} & x_a & x_{a+1} & x_{a+2} & \cdots & x_{a+n} \\ 0 & \cdots & 0 & 1 & x_1 & x_2 & \cdots & x_n \\ 0 & 0 & \cdots & 0 & 1 & x_1 & \cdots & x_{n-1} \\ & \vdots & & \cdots & 0 & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & & & \cdots & 1 & x_1 \end{vmatrix}$$

Then we perform ψ on M, and get

$$\psi(M) = \begin{vmatrix} x_a & x_{a+1} & \cdots & x_{a+m-2} & x_{a+m+n-1} - f_k \\ x_{a-1} & x_a & \cdots & \vdots & \\ \vdots & & \ddots & & \vdots \\ x_{a-m+2} & x_{a-m+3} & \cdots & x_a & x_{a+n+1} - f_2 \\ x_{a-m+1} & x_{a-m+2} & \cdots & x_{a-1} & x_{a+n} - f_1 \end{vmatrix}$$

where $f_i (1 \leq i \leq k)$ are polynomials of x_1 through x_n.

Notice that this shape is similar to the rectangle shape we proved in Section 5. We consider the submatrix at the lower left corner of M, and we denote it by

$$M' = \begin{vmatrix} x_{a-1} & x_a & \cdots & x_{a+m-3} \\ x_{a-2} & x_{a-1} & \cdots & \vdots \\ \vdots & & \ddots & \\ x_{a-m+1} & x_{a-m+2} & \cdots & x_{a-1} \end{vmatrix}$$

which is a rectangle shape. By Corollary 4.8, $P(\det M' = 0) = 1/q$. Also, if $\det M' \neq 0$, by Lemma 4.4 and equation 1 we can write

$$M' = \begin{bmatrix} & & & c_w I_w \\ & & \iddots & \\ & c_2 I_2 & & \\ c_1 I_1 & & & \end{bmatrix}. \tag{6}$$

Therefore, $\det M = \pm \det \psi(M) = \pm c_1 c_2 \cdots c_w (x_{a+m+n-1} - f_k)$, and it is zero if and only if $x_{a+m+n-1} = f_k$, which has probability $1/q$.

If instead $\det M' = 0$, we use similar argument as in the proof of Lemma 4.4 to write

$$M = \begin{bmatrix} & & & & B \\ & & & c_p I_p & \\ & & \iddots & & \\ & c_2 I_2 & & & \\ c_1 I_1 & & & & \end{bmatrix} \tag{7}$$

where matrix B is a square matrix of size $m \geq 2$. Moreover, B is of the form

$$B = \begin{bmatrix} c_{p+1} & & \cdots & & \\ 0 & c_{p+1} & & & \\ \vdots & 0 & \ddots & & \vdots \\ & & 0 & c_{p+1} & \\ & & \cdots & 0 & t \end{bmatrix} \tag{8}$$

since M is of rectangle form except the last column, which implies that t does not necessarily equal c_{p+1}. Then, $\det M = 0$ if and only if $\det B = 0$, which is equivalent to either $c_{p+1} = 0$ or $t = 0$. Therefore,

$$P(\det B = 0) = 1 - (\frac{q-1}{q})^2 = \frac{2q-1}{q^2}.$$

To sum over two different cases,

$$\begin{aligned} P(\det M = 0) =& P(\det M' = 0)\, P(\det M = 0 | \det M' = 0) \\ &+ P(\det M' \neq 0)\, P(\det M = 0 | \det M' \neq 0) \\ &= \frac{1}{q} \cdot \frac{2q-1}{q^2} + \frac{q-1}{q} \cdot \frac{1}{q} \\ &= \frac{q^2+q-1}{q^3} \end{aligned}$$

□

Remark 8.4. This is not always true for a fattened hook $\lambda = (a^m, b^n)$ where $b \geq 2$. In fact, it seems the probability for a fattened hook can be very complicated, and Proposition 3.6 gives an example.

We also work on generalization of staircases. Based on numerical data for n up to 7, we have the following conjecture.

Conjecture 8.5. For a 2-staircase $\lambda = (2n, 2n-2, \cdots, 4, 2)$, we have $P(s_\lambda \mapsto 0) = (q^2+q-1)/q^3$.

8.2. **Relaxing the condition for the "far-apart" shapes.**

Proposition 3.9 computes the probability for shape where all the rows are far apart, in which case we have no constants or repeated variable in the Jacobi-Trudi matrix. If we relax the condition so that we have one repeated variable, then we obtain the following proposition.

Proposition 8.6. *Let $\lambda = (\lambda_1, \ldots, \lambda_k)$, where $\lambda_j - \lambda_{j+1} = k-2$ for some $j < k$, $\lambda_i - \lambda_{i+1} \geq k-1$ for all $i < k, i \neq j$ and $\lambda_k \geq k$. Then*

$$P(s_\lambda \mapsto 0) = \frac{1}{q^{k^2-2k+2}} \left(q^{k^2-2k+2} - (q^{2k-2} - q^{k-1} - q^{k-2} + 1) \prod_{i=0}^{k-3} (q^{k-2} - q^i) \right).$$

Proof. By Jacobi-Trudi, we have

$$s_\lambda = \begin{vmatrix} h_{\lambda_1} & h_{\lambda_1+1} & \cdots & h_{\lambda_i+k-1} \\ h_{\lambda_2-1} & h_{\lambda_2} & \cdots & h_{\lambda_2+k-2} \\ \vdots & & \ddots & \vdots \\ h_{\lambda_j-j+1} & h_{\lambda_j} & \cdots & h_{\lambda_j+k-j} \\ h_{\lambda_{j+1}-j} & h_{\lambda_{j+1}} & \cdots & h_{\lambda_{j+1}+k-(j+1)} \\ \vdots & & \ddots & \vdots \\ h_{\lambda_k-k+1} & & \cdots & h_{\lambda_k} \end{vmatrix}.$$

By the condition $\lambda_j - \lambda_{j+1} = k-2$, we have $\lambda_j - j + 1 = \lambda_{j+1} + k - (j+1)$, so the two entries h_{λ_j-j+1} and $h_{\lambda_{j+1}+k-(j+1)}$ equal. At the same time, the condition $\lambda_i - \lambda_{i+1} \geq k-1$ for all $i < k, i \neq j$ ensures that all the other h_l's are distinct and different from h_{λ_j-j+1}, and the condition $\lambda_k \geq k$ ensures that no entry in the above determinant is a constant. Hence the only repeated entry is h_{λ_j-j+1}, and we have k^2-1 different variables to choose.

We count the number of choices of the h_l's such that the above matrix is invertible. To start with, consider the first and the last columns and the choices that make them linearly independent. Denote the two columns by $c_1 = (x_1, x_2, \cdots, x_j, x_{j+1} \cdots, x_k)^t$ and $c_k = (y_1, y_2, \cdots, y_j, x_j, y_{j+2}, \cdots, y_k)^t$ respectively, where we use x_l's and y_l's instead of h_l's for simplicity. As we have shown, x_l's and y_l's are all distinct. We need c_1 and c_k to be linearly independent, and depending on whether x_j and x_{j+1} are zero or not, there are four cases:

Case 1: $x_j = 0$ and $x_{j+1} = 0$. Then the two columns become

$$c_1 = \begin{bmatrix} x_1 \\ \vdots \\ x_{j-1} \\ 0 \\ 0 \\ x_{j+2} \\ \vdots \\ x_k \end{bmatrix}, \quad c_k = \begin{bmatrix} y_1 \\ \vdots \\ y_{j-1} \\ y_j \\ 0 \\ y_{j+2} \\ \vdots \\ y_k \end{bmatrix}$$

For c_1 and c_k to be linearly independent, we just need c_1 to be a nonzero vector and c_k not a scalar multiple of it. We have $q^{k-2} - 1$ choices for c_1, and we have $q^{k-1} - q$ choices of c_k since there are q^{k-1} ways to choose c_k randomly and q many of them will be a multiple of c_1. Hence in total we have $(q^{k-2} - 1) \cdot (q^{k-1} - q)$ choices in this case.

Case 2: $x_j = 0$ and $x_{j+1} \neq 0$. Then the two columns become

$$c_1 = \begin{bmatrix} x_1 \\ \vdots \\ x_{j-1} \\ 0 \\ x_{j+1} \\ x_{j+2} \\ \vdots \\ x_k \end{bmatrix}, \quad c_k = \begin{bmatrix} y_1 \\ \vdots \\ y_{j-1} \\ y_j \\ 0 \\ y_{j+2} \\ \vdots \\ y_k \end{bmatrix}$$

We can easily see that c_k is not a multiple of c_k, and as long as c_k is not the zero vector c_1 is not a multiple of c_1. Hence in this case the two columns are linearly independent if and only if c_k is not the zero vector. Now we have $q - 1$ choices for x_{j+1} to be nonzero, q choices for each of the other $k - 2$ many x_l's, and $q^{k-1} - 1$ for c_k to not be the zero vector. Hence in total we obtain $(q - 1) \cdot q^{k-2} \cdot (q^{k-1} - 1)$ many choices in this case.

Case 3: $x_j \neq 0$ and $x_{j+1} = 0$. Then the two columns are just

$$c_1 = \begin{bmatrix} x_1 \\ \vdots \\ x_j \\ 0 \\ x_{j+2} \\ \vdots \\ x_k \end{bmatrix}, \quad c_k = \begin{bmatrix} y_1 \\ \vdots \\ y_j \\ x_j \\ y_{j+2} \\ \vdots \\ y_k \end{bmatrix}$$

Since $x_j \neq 0$, neither of the two columns is the zero vector and c_k i cannot be a multiples of c_1 as every multiple of c_1 has the $(j+1)^{\text{th}}$ entry to be zero. Hence all such c_1 and c_2 are linearly independent. We have $q-1$ choices for x_j to be nonzero and we have q choices for each of the remaining $2k-3$ many x_l's and y_l's. In total we have $(q-1)\cdot q^{2k-3}$ many choices in this case.

Case 4: $x_j \neq 0$ and $x_{j+1} \neq 0$.
Then neither of the two columns can be the zero vector, and we just need to make sure c_k is not a scalar multiple of c_1. We have $q-1$ choices of both x_j and x_{j+1}, and we have q choices for each of the remaining $k-2$ many x_l's, resulting in $(q-1)^2 \cdot q^{k-2}$ many choices for c_1. Once we have chosen c_1, the only way for c_k to be a multiple of c_1 is to be $\frac{x_j}{x_{j+1}}$ times of c_1, and all the other choices of c_k are linearly independent from c_1. Hence there are $(q^{k-1}-1)$ choices of c_k. In total we get $(q-1)^2 \cdot q^{k-2} \cdot (q^{k-1}-1)$ many choices in this case.

Summing over the four cases, we see that the total number of choices of independent c_1 and c_k are $q(q^{2k-2} - q^{k-1} - q^{k-2} + 1)$.

To choose the remaining $k-2$ many columns, notice that their entries are all distinct variables. For each $2 \leq i < k$, we can choose the i^{th} column so that it is not in the span of the previous $i-1$ columns and the last column, and there are $q^k - q^i$ many ways to do that.

Hence multiplying all these choices we have that there are $(q^{2k-2} - q^{k-1} - q^{k-2} + 1) \cdot \prod_{i=2}^{k-1}(q^k - q^i)$ many invertible matrices. We know there are q^{k^2-1} many possible matrices in total, so the probability that the matrix is singular (i.e., the determinant s_λ is zero) is

$$\begin{aligned}
P(s_\lambda \mapsto 0) &= 1 - \frac{q(q^{2k-2} - q^{k-1} - q^{k-2} + 1) \cdot \prod_{i=2}^{k-1}(q^k - q^i)}{q^{k^2-1}} \\
&= \frac{1}{q^{k^2-2}} \left(q^{k^2-2} - (q^{2k-2} - q^{k-1} - q^{k-2} + 1) \prod_{i=2}^{k-1}(q^k - q^i) \right) \\
&= \frac{1}{q^{k^2-2k+2}} \left(q^{k^2-2k+2} - (q^{2k-2} - q^{k-1} - q^{k-2} + 1) \prod_{i=0}^{k-3}(q^{k-2} - q^i) \right) \\
&= \frac{1}{q^{(k+2)(k-1)/2}} \left(q^{(k+2)(k-1)/2} - (q^{2k-2} - q^{k-1} - q^{k-2} + 1) \prod_{i=1}^{k-2}(q^i - 1) \right)
\end{aligned}$$

□

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