TRIVARIATE MONOMIAL COMPLETE INTERSECTIONS AND PLANE PARTITIONS

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ABSTRACT. We consider the homogeneous components U_r of the map on $R = \Bbbk[x, y, z]/(x^A, y^B, z^C)$ that multiplies by $x + y + z$. We prove a relationship between the Smith normal forms of submatrices of an arbitrary Toeplitz matrix using Schur polynomials, and use this to give a relationship between Smith normal form entries of U_r. We also give a bijective proof of an identity proven by J. Li and F. Zanello equating the determinant of the middle homogeneous component U_r when $(A, B, C) = (a + b, a + c, b + c)$ to the number of plane partitions in an $a \times b \times c$ box. Finally, we prove that, for certain vector subspaces of R, similar identities hold relating determinants to symmetry classes of plane partitions, in particular classes 3, 6, and 8.

1. INTRODUCTION

For a commutative ring $\Bbbk$ and positive integers A, B, C, consider the trivariate monomial complete intersection $R = \Bbbk[x, y, z]/(x^A, y^B, z^C)$. This carries a standard grading in which x, y, z each have degree one, and decomposes as a direct sum $R = \bigoplus_{r=0}^{e} R_r$ where $e := A + B + C - 3$, and each homogeneous component $R_r \cong \Bbbk^{h(r)}$, where $h(r)$ denotes the size of the set B_r consisting of all monomials of total degree r in x, y, z which are nonzero in R. It is easily seen that $(h(0), h(1), \dots, h(e))$ is a symmetric unimodal sequence. Furthermore, it is known that the maps

$$U_r: \ R_r \xrightarrow{\ \cdot(x+y+z)\ } R_{r+1}$$

have $U_{e-r}^t = U_r$, and that U_r is injective for $0 \leq r \leq \lfloor \frac{e-1}{2} \rfloor$ when working with $\Bbbk = \mathbb{Z}$ or $\mathbb{Q}$. The question of how these maps U_r behave for other rings or fields $\Bbbk$ is partly addressed in recent work of Li and Zanello [9] (see also [3]).

Our first main result attempts to address this more fully, by considering the Smith normal form of U_r when working over $\Bbbk = \mathbb{Z}$. We say that a matrix U in $\mathbb{Z}^{m \times n}$, $m \geq n$, has $\mathrm{SNF}(U) = (a_1, a_2, \dots, a_n)$ if there exist matrices $P, Q \in \mathrm{GL}_m(\mathbb{Z}), \mathrm{GL}_n(\mathbb{Z})$ such that PUQ takes the diagonal form

$$\begin{pmatrix}
a_1 & 0 & \cdots & 0 & 0 \\
0 & a_2 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & a_{n-1} & 0 \\
0 & 0 & \cdots & 0 & a_n \\
\hline
0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0
\end{pmatrix}$$

and a_i divides a_{i+1}.

Theorem 1.1. *Assume $A \geq B \geq C \geq 1$.*

(i) *For $0 \le r \le A - 2$,*
$$\mathrm{SNF}(U_r) = \underbrace{(1, 1, \ldots, 1)}_{h(r)}.$$

In particular, these maps U_r are injective when $\Bbbk$ is any field.

(ii) *For $A - 1 \le r \le \lfloor \frac{e-1}{2} \rfloor$, the non-unit entries in $\mathrm{SNF}(U_r)$ are the same as the non-unit entries of $\mathrm{SNF}(M_r(A, B, C))$ where*
$$M_r(A, B, C) := \left(\binom{A}{r - B + i - j + 2} \right)_{\substack{i=1,\ldots,B+C-r-2 \\ j=1,\ldots,r-A+2}}.$$

In particular, there are at most $r - A + 2$ such non-unit entries, so that for any field $\Bbbk$, the map U_r has nullity at most $r - A + 2$.

(iii) *Let $m := \lfloor \frac{e-1}{2} \rfloor$, so that $U_m : R_m \to R_{m+1}$ is a map closest to the "middle" of R. If $\mathrm{SNF}(U_m) = (a_1, a_2, \ldots, a_{h_m})$, then for $0 \le s \le m$ one has*
$$\mathrm{SNF}(U_{m-s}) = (\underbrace{1, 1, \ldots, 1}_{s-h(m)+h(m-s)}, a_1, a_2, \ldots, a_{h(m)-s}).$$

On the way to proving this theorem, we prove two lemmas. The first lemma provides a relationship between the Smith normal forms of submatrices of an arbitrary Toeplitz matrix using Schur polynomials (Lemma 2.6), and the second lemma proves an "inverse" Littlewood-Richardson rule (Lemma 3.6). Both of these lemmas might be of independent interest.

Our other main results relate to the middle map U_m when $A + B + C$ is even (and without loss of generality, $A \le B + C$ so that $r = m$ does not fall in the trivial case (i) of Theorem 1.1 above). In this case, one can check that $h(m) = h(m + 1)$, so that U_m is square. Li and Zanello proved the striking result that $\det(U_m)$, up to sign, counts the number of plane partitions that fit in an $a \times b \times c$ box where $a := \frac{A+B-C}{2}$, $b := \frac{A-B+C}{2}$, $c := \frac{-A+B+C}{2}$ so that $A = a + b$, $B = a + c$, $C = b + c$. Their proof proceeded by evaluating
$$\det(U_m) = \det(M_m(A, B, C))$$
directly, and comparing the answer to known formulae for such plane partitions. We respond to their call for a more direct, combinatorial explanation (see [9]) with the following:

Theorem 1.2. *Expressed in the monomial $\mathbb{Z}$-basis for $R = \mathbb{Z}[x, y, z]/(x^{a+b}, y^{a+c}, z^{b+c})$ the map $U_m : R_m \to R_{m+1}$ has its determinant $\det(U_m)$ equal, up to sign, to its permanent $\mathrm{perm}(U_m)$, and each nonzero term in its permanent corresponds naturally to a plane partition in an $a \times b \times c$ box.*

The same ideas then allow us to express the counts for other symmetry classes of plane partitions, namely those which are cyclically symmetric (class 3) or transpose complementary (class 6) or cyclically symmetric and transpose complementary (class 8), in terms of the determinant of U_m when restricted to certain natural $\mathbb{Z}$-submodules of R.

Theorem 1.3. *Assuming $a = b = c$, let $C_3 = \mathbb{Z}/3\mathbb{Z} = \{1, \rho, \rho^2\}$ act on $R = \mathbb{Z}[x, y, z]/(x^{2a}, y^{2a}, z^{2a})$ by cycling the variables $x \xrightarrow{\rho} y \xrightarrow{\rho} z \xrightarrow{\rho} x$. Then the map $U_m|_{R^{C_3}}$ restricted to the m-th homogeneous component of the C_3-invariant subring R^{C_3} has $\det(U_m|_{R^{C_3}})$ equal, up to sign, to the number of cyclically symmetric plane partitions in an $a \times a \times a$ box.*

Theorem 1.4. *Assuming $a = b$ and the product abc is even, let $C_2 = \mathbb{Z}/2\mathbb{Z} = \{1, \tau\kappa\}$ act on $R = \mathbb{Z}[x, y, z]/(x^{2a}, y^{a+c}, z^{a+c})$ by swapping $y \leftrightarrow z$. Then the map $U_m|_{R^{C_2,-}}$ restricted to the m-th homogeneous component of the anti-invariant submodule $R^{C_2,-} := \{f \in R : \tau\kappa(f) = -f\}$ has $\det(U_m|_{R^{C_2,-}})$ equal, up to sign, to the number of transpose complementary plane partitions in an $a \times a \times c$ box.*

Theorem 1.5. *Assuming $a = b = c$ are all even, let C_2, C_3 act on $R = \mathbb{Z}[x, y, z]/(x^{2a}, y^{2a}, z^{2a})$ as before. Then the map $U_m|_{R^{C_3} \cap R^{C_2,-}}$ restricted to the m-th homogeneous component of the intersection $R^{C_3} \cap R^{C_2,-}$ has $\det(U_m|_{R^{C_3} \cap R^{C_2,-}})$ equal, up to sign, to the number of cyclically symmetric transpose complementary plane partitions in an $a \times a \times a$ box.*

2. Proof of Theorem 1.1

Proof of Theorem 1.1, part (i). Recall the statement of Theorem 1.1, part (i):

Theorem. *For $0 \le r < A - 1$, $\mathrm{SNF}(U_r) = \{\underbrace{1, \ldots, 1}_{h(r) \ ones}\}$. In other words, the cokernel of U_r is free of rank $h(r + 1) - h(r)$.*

Proof. Let B_s denote the monomial basis of R_s. We represent U_r by a matrix whose columns and rows are indexed by elements of B_r and B_{r+1}, respectively. We will prove our claim by showing that, for some ordering of the rows and columns of the matrix of U_r, there exists a lower unitriangular maximal submatrix. To do this, it suffices to show that there exists an ordering of the columns such that, for each column j, there exists a row i such that the (i, j) entry is 1 and the entries to the right are 0. We arrange the columns lexicographically, so that monomials with higher x power are on the left, and if x powers are equal, then we use the power of y to break ties, and then the power of z to break remaining ties. For example, for $A = B = C = 4$ and $r = 4$, the monomials would be ordered

$$x^3y \quad x^3z \quad x^2y^2 \quad x^2yz \quad x^2z^2 \quad xy^3 \quad xy^2z \quad xyz^2 \quad xz^3 \quad y^3z \quad y^2z^2 \quad yz^3.$$

For any given monomial $x^i y^j z^k$ indexing a column, the monomial $x^{i+1} y^j z^k$ is nonzero in the quotient ring $\mathbb{k}[x, y, z]/(x^A, y^B, z^C)$ since $i + 1 \le i + j + k + 1 < A$. Therefore $x^{i+1} y^j z^k$ indexes a row. The entry of this row in column $x^i y^j z^k$ is 1, and any other column with a 1 in this row must be indexed by either $x^{i+1} y^{j-1} z^k$ or $x^{i+1} y^j z^{k-1}$, both of which lie to the left of $x^i y^j z^k$. $\square$

Remark 2.1. From the previous proposition, it follows that all the maps U_r of $\mathbb{k}[x, y]/(x^A, y^B)$ (setting $C = 1$) and $\mathbb{k}[x]/(x^A)$ (setting $B = C = 1$) have free cokernel. In the other extreme (with an arbitrary number of variables and bounded powers), Wilson completely determined the cokernel of the maps U_r for $\mathbb{k}[x_1, \ldots, x_n]/(x_i^2)$ [12]. Moreover, Hara and Watanabe provided an elementary proof of Wilson's result and used it to show the Strong Lefschetz Property for $\mathbb{k}[x_1, \ldots, x_n]/(x_i^2)$ [5].

Things get interesting when $A - 1 \le r \le \frac{e}{2}$. The up maps are no longer necessarily injective over arbitrary fields. However, we can still prove a least upper bound on the number of non-unit elements in $\mathrm{SNF}(U_r)$. In fact, we have the following

Proof of Theorem 1.1, part (ii). Recall the statement of Theorem 1.1, part (ii):

Theorem. *For $A - 1 \le r \le \lfloor \frac{e-1}{2} \rfloor$, the non-unit entries in $\mathrm{SNF}(U_r)$ are the same as the non-unit entries of $\mathrm{SNF}(M_r(A, B, C))$ where*

$$(1) \qquad M_r(A, B, C) := \left(\binom{A}{r - B + i - j + 2} \right)_{\substack{i=1,\ldots,B+C-r-2 \\ j=1,\ldots,r-A+2}}.$$

In particular, there are at most $r - A + 2$ such non-unit entries, so that for any field $\mathbb{k}$, the map U_r has nullity at most $r - A + 2$.

Proof. We construct a matrix $U_r(y, z)$ as follows. First, represent U_r as a matrix in the monomial basis with a lexicographic ordering on its rows and columns, as in the previous proof. For each 1 in U_r, if the row index divided by the column index equals y, replace the 1 with y, and similarly for z. Then move all the columns indexed by monomials with x^{A-1} to the right, maintaining the ordering

amongst these columns. There are $r - A + 2$ such columns: for $x^{A-1}y^j z^k$, with $j + k = r - A + 1$, note that since $r \leq \frac{A+B+C-3}{2}$ implies $r - A + 1 \leq \frac{B+C-A-1}{2} \leq \frac{C-1}{2} < B-1, C-1$, j can take on every value from 0 to $r - A + 1$. Now we claim that there are $B + C - r - 2$ monomials of rank $r + 1$ with no powers of x. First, note that $r \leq \frac{A+B+C-3}{2}$ implies $B + C - r - 2 \geq r - (A - 1) \geq 0$. Now, the monomials of rank $r + 1$ with no powers of x are precisely $y^{C-1}z^{r+2-C}, y^{C-2}z^{r+3-C}, \ldots, y^{r+2-B}z^{B-1}$.

We now have a matrix of the form

$$
U_r(x, y) = \left(\begin{array}{cc|c}
1 & 0 & \\
& \ddots & * \\
* & 1 & \\
\hline
& * & 0
\end{array} \right)
$$

where the lower right block of zeroes has $B + C - r - 2$ rows and $r - A + 2$ columns. Now we perform the following algorithm: initialize $M := U_r(y, z)$. While

$$
M = \left(\begin{array}{c|c}
1 & Y \\
\hline
X & M'
\end{array} \right)
$$

and M has more than $r - A + 2$ rows, set $M := M' - XY$. Each step of the algorithm is using the 1 in the first row and first column as a pivot and performing $\mathbb{Z}$-invertible row and column operations on M to eliminate the other entries in the same row and same column. Then we focus on the remaining submatrix M' and repeat.

Lemma 2.2. *At each step of the algorithm, the following holds: fix a monomial α of total degree $r - A + 1$ with no powers of x, fix a nonnegative integer γ and a monomial β of total degree $r + 1 - \gamma$ with no powers of x, and let μ be the entry of M whose column is indexed by $x^{A-1}\alpha$ and whose row is indexed by $x^\gamma \beta$, and suppose $\beta/\alpha = y^j z^k$. If no entry to the left of μ indexed by a monomial whose power of x is less than $A - 1$ is a y or z, then $\mu = (-1)^{A-\gamma-1}\binom{j+k}{j}y^j z^k$.*

Proof. This statement is true at the beginning of the algorithm. We now proceed inductively. Suppose μ is as in the statement of the Lemma. Suppose yz divides β. For some previous step of the algorithm, there existed an entry μ_y whose column was indexed by $x^{A-1}\alpha$ and whose row was indexed by $x^{\gamma+1}\beta/y$, and no entry to the left of μ_y indexed by something with x power less than $A - 1$ was a y or z. We have the following setup:

$$
\begin{array}{c}
\phantom{x^{\gamma+1}\beta/y} \\
x^{\gamma+1}\beta/y \\
\vdots \\
x^\gamma \beta
\end{array}
\begin{array}{c}
\begin{array}{cccc}
x^\gamma \beta/y & \cdots & x^{A-1}\alpha & \cdots
\end{array} \\
\left(\begin{array}{cccc}
1 & \cdots & \mu_y & \cdots \\
\vdots & \ddots & \vdots & \\
y & \cdots & &
\end{array} \right).
\end{array}
$$

Similarly, for some previous step of the algorithm, there existed an entry μ_z whose column was indexed by $x^{A-1}\alpha$ and whose row was indexed by $x^{\gamma+1}\beta/z$, and no entry to the left of μ_z indexed by something with x power less than $A - 1$ was a y or z. We have an analogous setup:

$$
\begin{array}{c}
\phantom{x^{\gamma+1}\beta/z} \\
x^{\gamma+1}\beta/z \\
\vdots \\
x^\gamma \beta
\end{array}
\begin{array}{c}
\begin{array}{cccc}
x^\gamma \beta/z & \cdots & x^{A-1}\alpha & \cdots
\end{array} \\
\left(\begin{array}{cccc}
1 & \cdots & \mu_z & \cdots \\
\vdots & \ddots & \vdots & \\
z & \cdots & &
\end{array} \right).
\end{array}
$$

By the algorithm, after both of these steps, we end up with $\mu = -y\mu_y - z\mu_z$. But by induction,

$$
\begin{aligned}
-y\mu_y - z\mu_z &= -y(-1)^{A-\gamma-2}\binom{j+k-1}{j-1}y^{j-1}z^k - (-1)^{A-\gamma-2}\binom{j+k-1}{j}y^j z^{k-1} \\
&= (-1)^{A-\gamma-1}\left[\binom{j+k-1}{j-1} + \binom{j+k-1}{j}\right]y^j z^k \\
&= (-1)^{A-\gamma-1}\binom{j+k}{j}y^j z^k.
\end{aligned}
$$

In the case where yz does not divide β, at least one of these previous steps did not exist, but our argument still applies and the result still follows. $\qquad\square$

In particular, the Lemma implies that, at the end of the algorithm, the matrix M has $(B+C-r-2)$ rows and $(r-A+2)$ columns and takes the form

$$
\left((-1)^{A-1}\binom{A}{r-B+i-j+2}y^{A+B-r-i+j-2}z^{r-B+i-j+2}\right)_{i,j}.
$$

Recall that at each step of the algorithm, we are simply performing a sequence of $\mathbb{Z}$-invertible row and column operations. Therefore $U_r(y,z)$ and $(-1)^{A-1}M$ have the same non-unit Smith normal form entries. Substituting $y = z = 1$ gives the result. $\qquad\square$

Remark 2.3. Note that after substituting $A = a+b$, $B = a+c$, $C = b+c$, we return to the context of the plane partitions. In this case, our matrix $M_r(A, B, C)$ becomes an $(r-a-b+2)\times(a+b+2c-r-2)$ matrix with entries

$$
(2) \qquad\qquad M_r(a+b, a+c, b+c) = \left(\binom{a+b}{r-a-c+i-j+2}\right)_{i,j}.
$$

Focusing on the middle rank $r = a + b + c - 2$ yields a $c \times c$ matrix

$$
M_r(a+b, a+c, b+c) = \left(\binom{a+b}{b+i-j}\right)_{i,j}.
$$

The matrix $M_{a+b+c-2}(a+b, a+c, b+c)$ is called a *Carlitz matrix* [7], and occurs in [9, Lemma 2.2] and [3, Theorem 4.3]. The Smith normal form of these matrices is not known, but Kuperberg conjectures that there is a potential combinatorial connection between plane partitions and Smith forms of Carlitz matrices [8]. Even for small numbers, however, it is subtle to understand (and even to compute!) the Smith forms. We give some small examples:

When $c = 1$, the only Smith entry is $\binom{a+b}{b}$. When $c = 2$ and $a = b$, the explicit row and column operations to turn $M_{a+b+c-2}(a + b, a + c, b + c)$ into Smith normal form are

$$
\begin{pmatrix} 1 & -1 \\ -1-3a & 2+3a \end{pmatrix} M_{a+b+c-2}(a+b, a+c, b+c)\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \binom{2a}{a} & 0 \\ 0 & \binom{2a+1}{a+1} \end{pmatrix}.
$$

Even though the first entry is the a-th Catalan number, a combinatorial explanation of the Smith entries eludes the authors.

Directly calculating the Smith entries by computing GCDs of various binomial coefficients seems intractable and does not seem to generalize for bigger c. In particular, the direct method seems hard to understand even when $c = 2$ and a, b are arbitrary. In that case, the Smith entries are

$s_1 = \gcd\left(\binom{a+b}{b-1}, \binom{a+b}{b}, \binom{a+b}{b+1}\right)$ and $s_2 = \frac{\binom{a+b}{b}^2 - \binom{a+b}{b-1}\binom{a+b}{b+1}}{s_1}$.

Remark 2.4. If $R = \Bbbk[x_1, \ldots, x_n]/(x_1^{A_1}, \ldots, x_n^{A_n})$, then we can easily generalize the proof above to show that the non-unit Smith entries of the map U_r (which is now defined by multiplication by $x_1 + \cdots + x_n$) for $A_1 - 1 \leq r \leq \frac{A_1 + \cdots + A_n - n}{2}$ are the same as those of the matrix with the following entries, assuming $A_1 \geq A_2 \geq \cdots \geq A_n$: if the column is indexed by $x_1^{A_1 - 1}\alpha$ and the row is indexed by β where x divides neither α nor β, and $\beta/\alpha = x_2^{i_2} \cdots x_n^{i_n}$, then the entry is the multinomial coefficient

$$\binom{A_1}{i_2, i_3, \ldots, i_n} = \frac{A_1!}{i_2! i_3! \cdots i_n!}.$$

Computer evidence suggests that the non-unit Smith entries of these matrices behave nicely for $n = 4$ (as in $\mathrm{SNF}(U_r)$ is a submultiset of $\mathrm{SNF}(U_{r+1})$), but the analogous result is unfortunately not true for $n = 5$: taking $A_1 = A_2 = A_3 = A_4 = A_5 = 4$, the Smith entry 70 occurs in $\mathrm{SNF}(U_6)$ but not in $\mathrm{SNF}(U_7)$.

Letting $s(r)$ denote the number of non-unit Smith normal form entries of U_r and $m = a+b+c-2$, Theorem 1.1(ii) implies that $s(m-i) \leq c - i$ for all $i \leq c$. In fact, something stronger holds, which is Theorem 1.1(iii).

Proof of Theorem 1.1, part (iii). Recall the statement of Theorem 1.1, part (iii):

Theorem. *Let* $m := \lfloor \frac{e-1}{2} \rfloor$, *so that* $U_m : R_m \to R_{m+1}$ *is a map closest to the "middle" of R. If* $\mathrm{SNF}(U_m) = (a_1, a_2, \ldots, a_{h(m)})$, *then for* $0 \leq s \leq m$ *one has*

$$\mathrm{SNF}(U_{m-s}) = (\underbrace{1, 1, \ldots, 1}_{s-h(m)+h(m-s)}, a_1, a_2, \ldots, a_{h(m-s)}).$$

Immediately from part (iii) of Theorem 1.1, we re-derive [10, Propositions 2.1 and 2.2]:

Corollary 2.5. *The maps U_r for $r \leq m$ are injective if and only if U_m is injective.*

Recall the matrices $M_r(A, B, C)$ given by (1). The first one, $M_{A-1}(A, B, C)$, is a matrix with 1 column and $-A+B+C-1$ rows, and in general, the i-th matrix has i columns and $-A+B+C-i$ rows. We observe that in fact these are all submatrices of the $(-A+B+C-1) \times (-A+B+C-1)$ lower triangular Toeplitz matrix with entries

$$\left(\binom{A}{A - B + i - j + 1} \right)_{i,j}$$

for $i \geq j$, and 0 otherwise, which, written out, looks like

$$\begin{pmatrix} \binom{A}{A-B+1} & 0 & \cdots & 0 & 0 \\ \binom{A}{A-B+2} & \binom{A}{A-B+1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \binom{A}{C-1} & \binom{A}{C-2} & \cdots & \binom{A}{A-B+2} & \binom{A}{A-B+1} \end{pmatrix}.$$

For $1 \leq i \leq \frac{-A+B+C}{2}$, the matrix $M_{A-2+i}(A, B, C)$ is simply the submatrix of the Toeplitz matrix created by choosing the first i columns and the last $-A + B + C - i$ rows. Surprisingly, there is nothing special about the entries of the large Toeplitz matrix! We have the following more general result, which immediately implies part (iii) of Theorem 1.1:

Lemma 2.6. *Let A be an arbitrary $n \times n$ Toeplitz matrix,*

$$A = \begin{pmatrix} h_n & & & & & & \\ h_{n-1} & h_n & & & & & \\ h_{n-2} & h_{n-1} & h_n & & & & \\ \vdots & \vdots & \vdots & \ddots & & & \\ h_3 & h_4 & h_5 & \cdots & h_n & & \\ h_2 & h_3 & h_4 & \cdots & h_{n-1} & h_n & \\ h_1 & h_2 & h_3 & \cdots & h_{n-2} & h_{n-1} & h_n \end{pmatrix}$$

with entries in a principal ideal domain, and for $1 \le c \le n$, let A_c denote the $(n-c+1) \times c$ submatrix of A formed by columns $1, \ldots, c$ and rows $c, \ldots, n$. Then, for $1 \le k \le \frac{n}{2}$, the k-th Smith normal form entry of A_c is the same for all $k \le c \le \frac{n}{2}$.

Our proof of Lemma 2.6 requires a bit of algebraic machinery, so we defer it to its own section.

3. Proof of Lemma 2.6

Suppose M is a matrix over a PID with Smith normal form entries $a_1 \le a_2 \le \cdots \le a_r$. Then it is known (see [11]) that

$$a_k = \frac{\gcd(k \times k \text{ minors of } M)}{\gcd((k-1) \times (k-1) \text{ minors of } M)}.$$

Therefore, to prove Lemma 2.6, it suffices to show that, for $1 \le k \le c \le \frac{n}{2}$, the ideal generated by the $k \times k$ minors of A_c is equal to the ideal generated by the $k \times k$ minors of A_k.

Example 3.1. Suppose $n = 7$. The matrix A is

$$A = \begin{pmatrix} h_7 & & & & & & \\ h_6 & h_7 & & & & & \\ h_5 & h_6 & h_7 & & & & \\ h_4 & h_5 & h_6 & h_7 & & & \\ h_3 & h_4 & h_5 & h_6 & h_7 & & \\ h_2 & h_3 & h_4 & h_5 & h_6 & h_7 & \\ h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 \end{pmatrix}$$

and

$$A_1 = \begin{pmatrix} h_7 \\ h_6 \\ h_5 \\ h_4 \\ h_3 \\ h_2 \\ h_1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} h_6 & h_7 \\ h_5 & h_6 \\ h_4 & h_5 \\ h_3 & h_4 \\ h_2 & h_3 \\ h_1 & h_2 \end{pmatrix}, \quad A_3 = \begin{pmatrix} h_5 & h_6 & h_7 \\ h_4 & h_5 & h_6 \\ h_3 & h_4 & h_5 \\ h_2 & h_3 & h_4 \\ h_1 & h_2 & h_3 \end{pmatrix}, \quad A_4 = \begin{pmatrix} h_4 & h_5 & h_6 & h_7 \\ h_3 & h_4 & h_5 & h_6 \\ h_2 & h_3 & h_4 & h_5 \\ h_1 & h_2 & h_3 & h_4 \end{pmatrix}.$$

The case $k = 1$ is trivial because the 1×1 minors of the matrices are just the entries themselves. The $k = 2$ case is slightly more complicated. For example, the minor

$$\begin{vmatrix} h_6 & h_7 \\ h_1 & h_2 \end{vmatrix}$$

can be found in A_2 but not A_3; however, we can write

$$\begin{vmatrix} h_6 & h_7 \\ h_1 & h_2 \end{vmatrix} = \begin{vmatrix} h_5 & h_7 \\ h_1 & h_3 \end{vmatrix} - \begin{vmatrix} h_5 & h_6 \\ h_2 & h_3 \end{vmatrix},$$

where the minors on the right hand side can be found in A_3. One can see that trying to do this systematically for $k \times k$ minors with $k \geq 3$ gets rather difficult.

Since minors are rather difficult to work with directly, we instead use Schur polynomials. The *complete homogeneous symmetric polynomials* in n variables $x_1, \ldots, x_n$ are the polynomials

$$
\begin{aligned}
h_0(x_1, \ldots, x_n) &= 1 \\
h_1(x_1, \ldots, x_n) &= \sum_{1 \leq i \leq n} x_i \\
h_2(x_1, \ldots, x_n) &= \sum_{1 \leq i \leq j \leq n} x_i x_j \\
h_3(x_1, \ldots, x_n) &= \sum_{1 \leq i \leq j \leq k \leq n} x_i x_j x_k \\
&\vdots
\end{aligned}
$$

where $h_d(x_1, \ldots, x_n)$ is the sum of all monomials of total degree d. What these polynomials actually are is not so important to our proof; the importance lies in the fact that $h_1, \ldots, h_n$ are algebraically independent. An *integer partition* $\lambda = (\lambda_1, \lambda_2, \ldots)$ of n is a non-increasing sequence of nonnegative integers λ_i such that $\sum_i \lambda_i = n$. If k is the largest number such that $\lambda_k > 0$, then we say λ *has k parts*. If $\lambda_i \leq \ell$, we say that the i-th part of λ is at most ℓ. If λ has k parts and $\lambda_1 \leq \ell$, then we say that λ *fits in a $k \times \ell$ box*. We associate to each integer partition λ a Young diagram (see Figure 1).
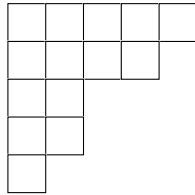
FIGURE 1. Young diagram representing $(5, 4, 2, 2, 1)$

If λ and μ are two partitions and $\mu_i \leq \lambda_i$ for all i, then we say μ is *contained in* λ. If so, then we associate a skew diagram to λ *mod* μ, written λ/μ (see Figure 2).
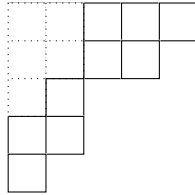
FIGURE 2. Skew diagram representing $(5, 4, 2, 2, 1)/(2, 2, 1, 0, 0)$

To any such λ/μ, we associate the *Schur polynomial*

$$
(3) \qquad\qquad S_{\lambda/\mu} = \det \left(h_{\lambda_i - \mu_j - i + j} \right).
$$

where we take $\mu_k = 0$ if λ has k parts. Schur polynomials correspond to minors found in Toeplitz matrices, which is precisely what we need. For a more comprehensive treatment of Schur polynomials

and Young diagram, we refer the reader to [4]. A Schur polynomial $S_{\lambda/\mu}$ is *non-skew* if $\mu = 0$, and *skew* otherwise.

Our primary weapon of attack will be the *Littlewood-Richardson rule*, which tells us how to decompose a skew Schur polynomial into a linear combination of non-skew Schur polynomials (with positive coefficients, even!). Suppose $S_{\lambda/\mu}$ is a skew Schur polynomial. Draw the diagram associated with λ/μ. A *labeling* of λ/μ, where λ has k parts, is defined as a labeling of the squares of the diagram with integers $1, \ldots, k$ such that:

- the numbers are weakly increasing from left to right in each row;
- the numbers are strictly increasing from top to bottom in each column;
- when reading the string of the numbers from right to left in each row, top row to bottom row, each initial substring must have at least as many i's as $i + 1$'s, for all i.

The first two properties say that the labeling forms a skew semistandard Young tableau. A partition π *arises from a labeling of* λ if, for each i, the i-th part of π has size equal to the number of i's in the labeling of λ. The Littlewood-Richardson rule states that

$$S_{\lambda/\mu} = \sum_\pi S_\pi$$

where the sum is over all partitions π that arise from a labeling of λ/μ (see Figure 3 for an illustration of this rule).
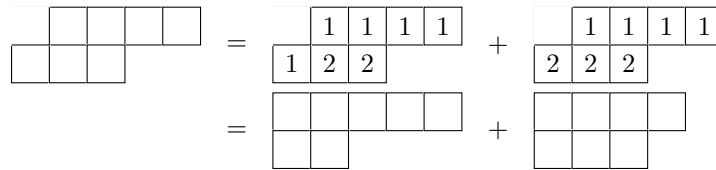
FIGURE 3. The Littlewood-Richardson rule showing $S_{(5,3)/(1,0)} = S_{(5,2)} + S_{(4,3)}$.

Since the Schur polynomials $h_1, \ldots, h_n$ in n variables are algebraically independent, we may treat the Littlewood-Richardson rule as an algebraic identity for $S_{\lambda/\mu}$ of the form (3), where the h_i are formal variables.

Recall our definition of the matrix A and its submatrices A_c in the statement of the Lemma. We now introduce a new term:

Definition 3.2. A Schur polynomial $S_{\lambda/\mu}$ is (k, c)-*legal* if it is equal to a $k \times k$ minor in A_c. We also say that λ/μ is (k, c)-legal, identifying the polynomial with the diagram.

By this definition, our task amounts to proving that the ideal generated by (k, k)-legal diagrams is equal to the ideal generated by (k, c)-legal diagrams, for all $1 \leq k \leq \frac{n}{2}$ and for all $k \leq c \leq \frac{n}{2}$. To make our job easier, we have the following characterization:

Proposition 3.3. *A skew diagram λ/μ with k parts is (k, c)-legal if and only if the following hold:*

- $\lambda_1 \leq n - k + 1$;
- $\lambda_k \geq k$;
- $\mu_1 \leq c - k$;
- $\lambda_i - \mu_i \geq k$ *for all i;*
- $\lambda_1 - \lambda_k \leq n - c - k + 1$.

Proof. Suppose λ/μ is (k,c)-legal. Then it corresponds to some $k \times k$ minor

$$
(4) \qquad
\begin{vmatrix}
h_{i+t} & \cdots & h_{i+t+r} \\
\vdots & \ddots & \vdots \\
h_i & \cdots & h_{i+r}
\end{vmatrix}
$$

of A_c, where the rows and columns represented by dots are not necessarily adjacent in A. Recall that

$$
A_c =
\begin{pmatrix}
h_{n-c+1} & \cdots & h_n \\
\vdots & \ddots & \vdots \\
h_1 & \cdots & h_c
\end{pmatrix}.
$$

This implies that

$$
\begin{array}{rcl}
1 \leq & i & \leq n-k-c+2, \\
k-1 \leq & r & \leq c-1, \\
k-1 \leq & t & \leq n-c, \\
k \leq & i+t & \leq n-c+1, \\
k \leq & i+r & \leq c, \\
2k-1 \leq & i+t+r & \leq n.
\end{array}
$$

From Eq. (3), we find that

$$
\begin{array}{rcl}
\lambda & = & (i+t+r-k+1,\ldots,i+r) \\
\mu & = & (r-k+1,\ldots,0).
\end{array}
$$

Therefore

$$
\begin{array}{rcl}
\lambda_1 & = & i+t+r-k+1 \leq n-k+1, \\
\lambda_k & = & i+r \geq k, \\
\mu_1 & = & r-k+1 \leq c-k, \\
\lambda_1 - \lambda_k & = & t-k+1 \leq n-c-k+1.
\end{array}
$$

The remaining inequality, $\lambda_i - \mu_i \geq k$, follows from the fact that the diagonal terms of the submatrix must be at least k, and are equal to the $\lambda_i - \mu_i$.

Now suppose λ/μ satisfies the above inequalities. It is straightforward to verify that the $k \times k$ minor with entries

$$
\begin{vmatrix}
h_{\lambda_1-\mu_1} & * & \cdots & * & h_{\lambda_1+k-1} \\
* & h_{\lambda_2-\mu_2} & \cdots & * & h_{\lambda_2+k-2} \\
& & \ddots & & \vdots \\
* & * & \cdots & h_{\lambda_{k-1}-\mu_{k-1}} & h_{\lambda_{k-1}+1} \\
* & * & \cdots & * & h_{\lambda_k}
\end{vmatrix}
$$

(where the $*$ denote entries that are determined by the choice of rows and columns) corresponds to λ/μ and can be found in A_c. $\qquad\square$

Corollary 3.4. *The (k,k)-legal diagrams are precisely non-skew λ such that $\lambda_1 \leq n-k+1$ and $\lambda_k \geq k$.*

Before we begin proving Lemma 2.6, we make the following handy definition:

Definition 3.5. If λ/μ has k parts, then the *spread* of λ/μ is the integer partition

$$(\lambda_1 - \lambda_k, \lambda_2 - \lambda_k, \ldots, \lambda_{k-1} - \lambda_k).$$

We put a lexicographical well-ordering on spreads. That is, if $\delta = (\delta_1, \ldots, \delta_{k-1})$ and $\epsilon = (\epsilon_1, \ldots, \epsilon_{k-1})$ are spreads, then $\delta < \epsilon$ if and only if the leftmost nonzero entry of $\epsilon - \delta$ is positive.

With this definition in mind, we prove our main lemma, which can be seen as an "inverse" Littlewood-Richardson rule:

Lemma 3.6. *Any (k,k)-legal diagram is a linear combination of (k,c)-legal diagrams.*

Proof. Suppose ν is a (k,k)-legal diagram. If $\nu_1 - \nu_k \le n - c - k + 1$, then by Proposition 3.3, ν is (k,c)-legal. If $\nu_1 - \nu_k > n - c - k + 1$, then its spread exceeds $(n - c - k + 1, n - c - k + 1, \ldots, n - c - k + 1)$. We construct the following skew diagram λ/μ (see Figure 4):
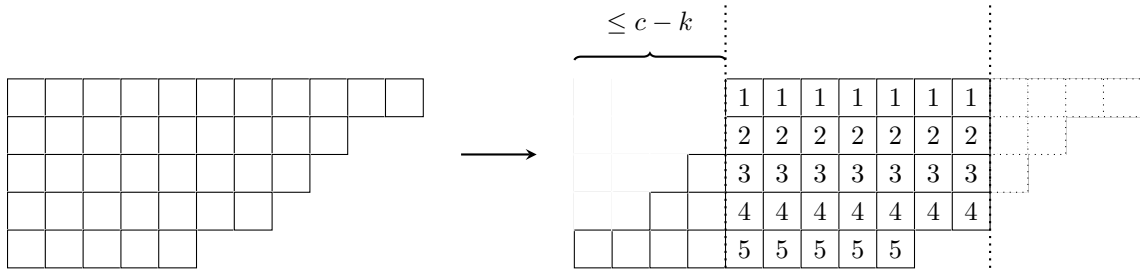
FIGURE 4. An example of constructing a λ/μ from a given non-skew ν.

Remove the portion of the spread of ν that exceeds $(n - c - k + 1, \ldots, n - c - k + 1)$, rotate it by 180°, and append it to the bottom left of ν. Algebraically, we are defining λ/μ by $\mu_i = \min\{\nu_1 - \nu_i, \nu_1 - \nu_k + c + k - n - 1\}$ and $\lambda_i - \mu_i = \min\{\nu_k + n - c - k + 1, \nu_i\}$. Suppose we label λ/μ and consider the non-skew diagram π that arises from it. We can maximize the spread of π by first maximizing the number of 1s in our labeling, then maximizing the number of 2s, and so on. This labeling is achieved by numbering the top of each column a 1, the second square of each column a 2, and so on. But this labeling gives rise to ν, by construction of λ/μ. Any other π has spread strictly less than the spread of ν. Therefore, we have written ν as a linear combination of a (k,c)-legal diagram and (k,k)-legal diagrams with strictly smaller spread. The result follows by induction. $\square$

Finally, we can prove Lemma 3.6:

Proof of Lemma 3.6. Let $1 \le k \le c \le \frac{n}{2}$. We first show that any (k,c)-legal diagram is a linear combination of (k,k)-legal diagrams. Suppose λ/μ is (k,c)-legal. By Proposition 3.3, $\lambda_1 \le n - k + 1$ and $\lambda_k \ge k$. If λ/μ is non-skew, i.e. $\mu = 0$, then by Corollary 3.4, λ/μ is (k,k)-legal. If λ/μ is skew, then we use the Littlewood-Richardson rule to write it as a sum of non-skew diagrams. Any labeling of λ/μ must have at most $n - k + 1$ copies of 1, since there are at most $n - k + 1$ columns and each column can have at most one 1. Thus $\nu_1 \le n - k + 1$ for all ν arising from a labeling. Similarly, a labeling must have at least $\lambda_k - \mu_1$ copies of k, since this is equal to the number of columns with length k. But if λ/μ is associated to the minor in Eq. (4), then $\lambda_k - \mu_1 = i + r - (r - k + 1) = i + k - 1 \ge k$. Therefore $\nu_k \ge k$ for all ν arising from a labeling. By Corollary 3.4, λ/μ is a sum of (k,k)-legal diagrams.

The other inclusion follows from Lemma 3.6. $\square$

4. Proof of Theorem 1.2

A *plane partition* of a positive integer n is a finite two-dimensional array of positive integers, weakly decreasing from left to right and from top to bottom, with sum n. For example, a plane partition of 43 could be

$$
\begin{array}{ccccc}
5 & 4 & 4 & 3 & 2 \\
4 & 4 & 3 & 3 & 2 \\
2 & 2 & 1 & 1 & \\
1 & 1 & & & \\
1 & & & &
\end{array}
$$

For more information on plane partitions, see [2] for a more comprehensive treatment.

We say that a plane partition *fits inside an* $a \times b \times c$ *box* if there are at most a rows, at most b columns, and each entry is at most c. For instance, the example above fits in an $a \times b \times c$ box for $a, b, c \leq 5$. This terminology stems from the fact that there is a natural way to view a plane partition as cubes stacked against the corner of the first octant in $\mathbb{R}^3$.
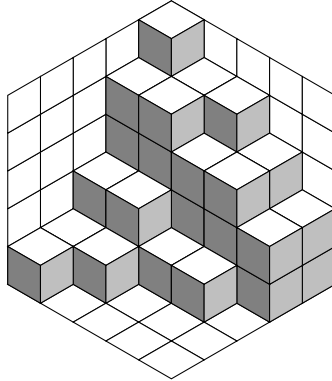
FIGURE 5. The plane partition $\{(5, 4, 4, 3, 2), (4, 4, 3, 3, 2), (2, 2, 1, 1), (1, 1), (1)\}$ in a $5 \times 5 \times 5$ box

Let $\mathrm{PP}(a, b, c)$ denote the number of plane partitions that fit in an $a \times b \times c$ box. MacMahon found the following elegant formula:

$$
(5) \qquad \mathrm{PP}(a, b, c) = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{i + j + k - 1}{i + j + k - 2}.
$$

A proof of this formula can be found in [2]. Li and Zanello essentially prove Theorem 1.2 in [9] by directly verifying that $|\det U_r|$ equals the right hand side of (5). In this section, we give a bijective proof of this result. Recall the statement of Theorem 1.2:

Theorem. *Expressed in the monomial $\mathbb{Z}$-basis for $R = \mathbb{Z}[x, y, z]/(x^{a+b}, y^{a+c}, z^{b+c})$ the map $U_m :$ $R_m \to R_{m+1}$ has its determinant $\det(U_m)$ equal, up to sign, to its permanent $\mathrm{perm}(U_m)$, and each nonzero term in its permanent corresponds naturally to a plane partition in an $a \times b \times c$ box.*

Proof. Our proof consists of two steps. First, we show that the number of plane partitions in an $a \times b \times c$ box is equal to the *permanent* of U_r. For ease of notation, let $U = U_r$. Let B_s denote the monomial basis of R_s as before. Recall that the permanent of U equals

$$
\sum_{\varphi} \prod_{\lambda \in B_r} U_{\lambda, \varphi(\lambda)}
$$

where the sum is over all bijections $\varphi : B_r \to B_{r+1}$ and $U_{\lambda,\mu}$ denotes the entry in row λ, column μ of the matrix representing U. Viewing the elements of $B_r \cup B_{r+1}$ as vertices of a bipartite graph, with B_r as one part and B_{r+1} as the other, and each 1 in the matrix of U corresponding to an edge, we see that each such φ gives rise to a perfect matching if $U_{\lambda,\varphi(\lambda)} = 1$ for all $i \in B_r$. Since U is a 0-1 matrix, the permanent of U counts the number of perfect matchings in this bipartite graph.

Step 1: Perfect matchings to plane partitions. Draw an equilateral triangle with side length $a+b+c$, and insert the monomials of $\Bbbk[x, y, z]$ of degree $r + 1 = a + b + c - 1$; Figure 6 illustrates the case of $r = 3$.
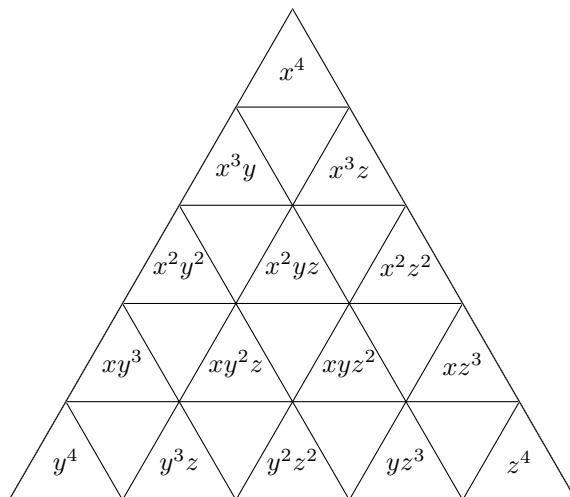
FIGURE 6. Monomials in B_4

Fill the remaining spaces with the monomials of degree $r = a + b + c - 2$ as in Figure 7; notice that two monomials in the diagram are adjacent if and only if one divides the other.

Now truncate the equilateral triangle by removing a triangle of length c from the top corner, a triangle of length b from the lower left corner, and a triangle of length a from the lower right corner. We are left with an $a \times b \times c$ hexagon containing the monomials in $B_r \cup B_{r+1}$, the vertices of the bipartite graph mentioned in the previous paragraph. Figure 8 shows the hexagon for $(a, b, c) = (2, 2, 1)$. Figure 9 shows this hexagons with monomials in B_r as black dots and monomials in B_{r+1} as white dots. A perfect matching between B_r and B_{r+1} is a bijection such that for any $\lambda \in B_r$, λ and its image are adjacent in the hexagon, i.e., they lie in adjacent equilateral triangles. We can thus represent a perfect matching pictorially as in Figure 9. It is thus evident that there is a bijection between perfect matchings and rhombus tilings of this $a \times b \times c$ hexagon. But the bijection between such rhombus tilings and plane partitions in an $a \times b \times c$ box is well-known; we view the tiling as a three-dimensional picture of the plane partition (rotated by 90°). Hence, the number of perfect matchings is equal to the number of such plane partitions, as desired.

Step 2: Permanent equals determinant. Consider the bijection between perfect matchings and plane partitions given above. To get from one plane partition to another, we perform the operations of adding or removing a block. For example, Figure 10 shows the plane partition with the right-most block removed (or added, depending on perspective, but we fix perspective at the beginning).
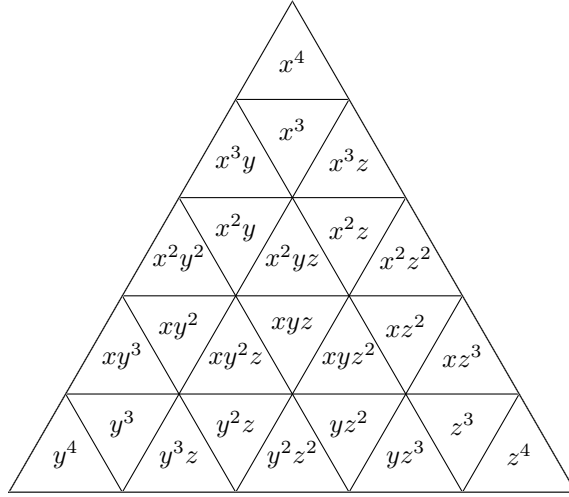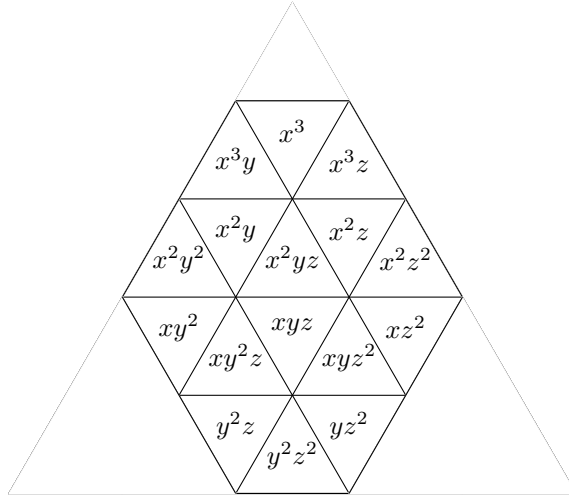
FIGURE 7. Adding monomials in B_3

FIGURE 8. Truncating equilateral triangle

As Figure 10 shows, if φ is the perfect matching associated with a plane partition, the associated perfect matching after such an operation is $\sigma\varphi$, where σ is a 3-cycle. Note that σ is an even permutation; hence, we can get from any perfect matching to another using even permutations. This shows that all perfect matchings have the same sign, completing the proof. $\square$

5. PROOFS OF THEOREMS 1.3, 1.4, AND 1.5

In [2], Bressoud describes the ten symmetry classes of plane partitions that are of interest. Each class consists of plane partitions that are invariant under certain actions. Each of these actions is some composition $\rho^i \tau^j \kappa^k$ where $(i, j, k) \in \{0, 1\}^3$, ρ is cyclically permuting the three coordinate axes, τ is reflection across the $y = z$ plane, and κ is complementation. Viewing plane partitions as
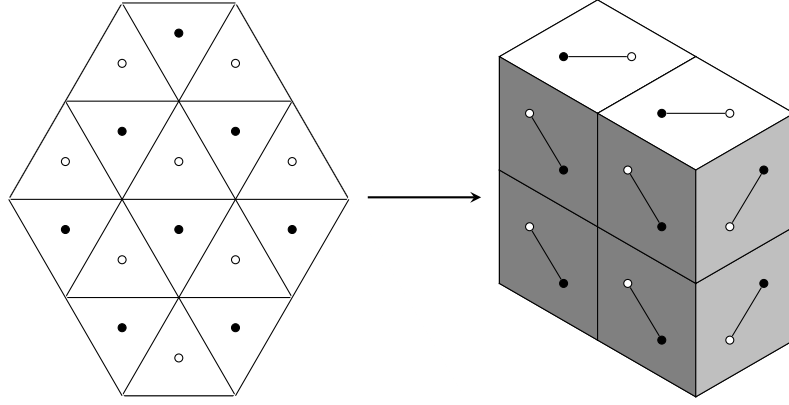
FIGURE 9. Left: Simplified diagram. Right: A perfect matching viewed as a plane partition
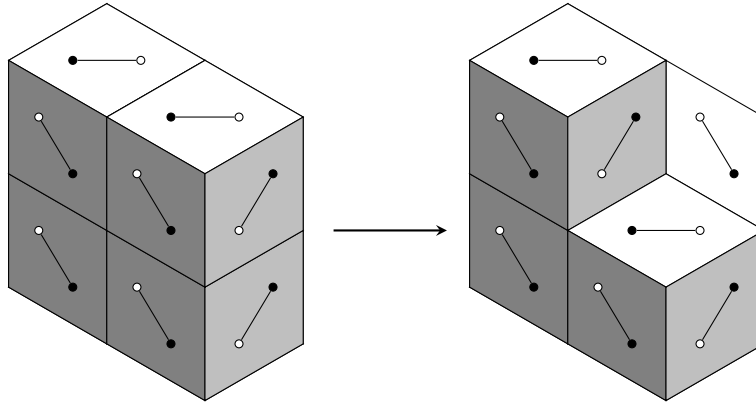
FIGURE 10. Removing a block

rhombus tilings in the plane, it is straightforward to check that ρ acts by counterclockwise rotation by 120°, τ acts by reflection about the vertical axis of symmetry, and κ acts by rotation by 180°. Note that $\langle \rho, \tau, \kappa \rangle = D_{12}$, the dihedral group of order 12, which is the group of symmetries on a regular hexagon. Given the above bijection, each symmetry class of plane partitions can be viewed as a subset of perfect matchings of the aforementioned bipartite graphs which are invariant under some subgroup of D_{12}. The ten symmetry classes are enumerated in Figure 11 in the same order as in [2].

We have shown that, for $R = \Bbbk[x, y, z]/(x^{a+b}, y^{a+c}, z^{b+c})$ and $r = a+b+c-2$, we have $|\det U_r| = \mathrm{PP}(a, b, c)$. In this section, we show that, for certain natural submodules of R over various invariant subrings (for appropriate group actions in each case), the determinant of the restriction of U_r to the submodule is equal (up to sign) to the number of a symmetry class of plane partitions that fit in an $a \times b \times c$ box.

Number	Class	Abbreviation	Subgroup
1	Plane partitions	PP	$\langle e \rangle$
2	Symmetric plane partitions	SPP	$\langle \tau \rangle$
3	Cyclically symmetric plane partitions	CSPP	$\langle \rho \rangle$
4	Totally symmetric plane partitions	TSPP	$\langle \rho, \tau \rangle$
5	Self-complementary plane partitions	SCPP	$\langle \kappa \rangle$
6	Transpose complement plane partitions	TCPP	$\langle \tau\kappa \rangle$
7	Symmetric self-complementary plane partitions	SSCPP	$\langle \tau, \kappa \rangle$
8	Cyclically symmetric transpose complement plane partitions	CSTCPP	$\langle \rho, \tau\kappa \rangle$
9	Cyclically symmetric self-complementary plane partitions	CSSCPP	$\langle \rho, \kappa \rangle$
10	Totally symmetric self-complementary plane partitions	TSSCPP	$\langle \rho, \tau, \kappa \rangle = D_{12}$

FIGURE 11. The symmetry classes of plane partitions

Proof of Theorem 1.3. Recall the statement of Theorem 1.3:

Theorem. *Assuming $a = b = c$, let $C_3 = \mathbb{Z}/3\mathbb{Z} = \{1, \rho, \rho^2\}$ act on $R = \mathbb{Z}[x, y, z]/(x^{2a}, y^{2a}, z^{2a})$ by cycling the variables $x \xrightarrow{\rho} y \xrightarrow{\rho} z \xrightarrow{\rho} x$. Then the map $U_m|_{R^{C_3}}$ restricted to the m-th homogeneous component of the C_3-invariant subring R^{C_3} has $\det(U_m|_{R^{C_3}})$ equal, up to sign, to the number of cyclically symmetric plane partitions in an $a \times a \times a$ box.*

Proof. The proof is in the same spirit as our proof for vanilla plane partitions: we show that the permanent of U'_r is equal to the cyclically invariant perfect matchings (which correspond to cyclically symmetric plane partitions), and then show that the permanent equals the determinant up to sign.

The elements $x^i y^j z^k + y^i z^j x^k + z^i x^j y^k$ form a basis for R^ρ. Hence, U'_r can be realized as a matrix with columns indexed by the basis elements of rank r of R^ρ, and rows indexed by the basis elements of rank $r + 1$ of R^ρ.

The element ρ acts on the hexagonal bipartite graph G by rotation by 120 degrees, so we can consider the quotient graph $G/\langle\rho\rangle$, where the vertices and edges are $\langle\rho\rangle$-orbits of vertices and edges in G, respectively. Figure 12 shows an example of the honeycomb graph and its corresponding quotient graph when $a = b = c = 2$.

Since $r = 3a - 2 \equiv 1 \pmod 3$ and $r + 1 \equiv 2 \pmod 3$, no monomial in either rank is fixed under ρ. It follows that each vertex and edge orbit correspond to three vertices and edges in G, respectively. Hence, cyclically symmetric perfect matchings on G correspond to perfect matchings on $G/\langle\rho\rangle$.

Moreover, there is a natural correspondence between the vertices in the quotient graph and the basis elements of the two middle ranks. Depending on the context, we will abuse notation and write $[i, j, k]$ to mean both a vertex orbit $[x^i y^j z^k]$ or a basis element $x^i y^j z^k + y^i z^j x^k + z^i x^j y^k$.

In the quotient graph, every vertex orbit $[i, j, k]$ is connected to $[i+1, j, k]$, $[i, j+1, k]$ or $[i, j, k+1]$ (if those orbits exist), and similarly, U'_r sends any basis element $[i, j, k]$ to a sum of basis elements $[i + 1, j, k] + [i, j + 1, k] + [i, j, k + 1]$ (if those elements exist.)

Note that the orbits $[a, a, a - 1]$ and $[a, a - 1, a - 1]$ share two edges in the quotient graph. The only way for two orbits to share two edges is if all elements of both orbits entirely make up the vertices of a hexagon in G. Hence, only the middle hexagon gives rise to this double edge in the quotient graph. Likewise, as basis elements, $U'_r[a, a - 1, a - 1] = [a + 1, a - 1, a - 1] + 2 \cdot [a, a, a - 1]$, and $[a, a - 1, a - 1]$ is the only element that gets sent to two times another basis element. Hence, U'_r is the adjacency matrix of $G/\langle\rho\rangle$. The number of perfect matchings on $G/\langle\rho\rangle$ is equal to the permanent of U'_r (by treating the sole 2 in U'_r as the two possible ways to have $[a, a - 1, a - 1]$ and
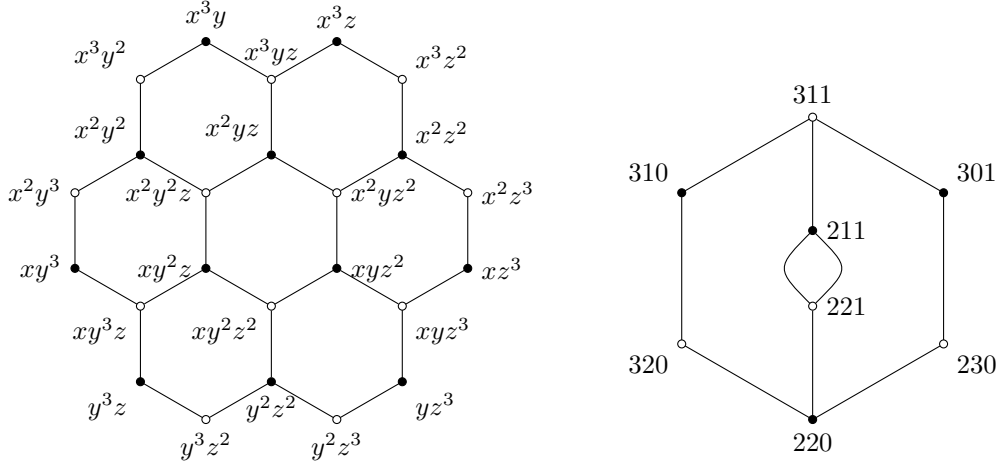
FIGURE 12. G (left) and $G/\langle\rho\rangle$ (right) for $a = b = c = 2$

$[a, a, a - 1]$ connected in a perfect matching), and it follows that the permanent of U'_r equals the number of cyclically symmetric plane partitions.

Next, we show that the permanent equals the determinant up to sign by showing all the matchings of $G/\langle\rho\rangle$ have the same sign. To get from a cyclically symmetric partition to one with fewer blocks, we can remove three blocks in the same orbit, or remove the block in the center of the partition. If φ is the perfect matching in $G/\langle\rho\rangle$ that corresponds to the partition, the partition after removing three blocks in the same orbit corresponds to $\sigma\varphi$, where σ is a 3-cycle, so this operation doesn't change the sign of the matching. Removing the central block corresponds to switching the edge between $[a, a - 1, a - 1]$ and $[a, a, a - 1]$ to the other edge, and thus does not change the sign. Figure 13 gives an example of these removals in action.
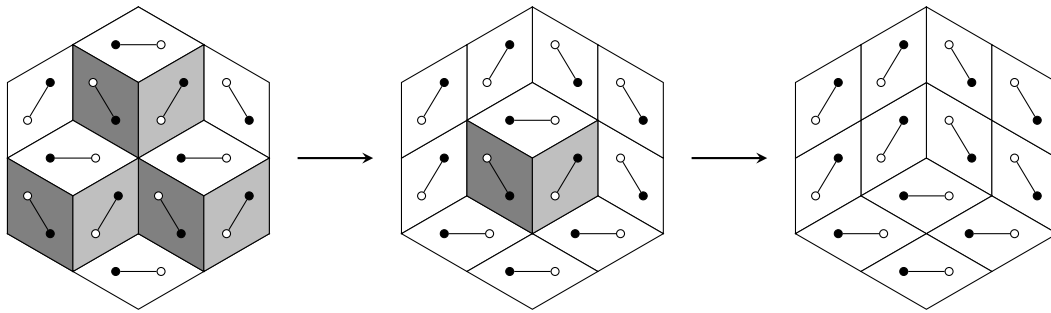

FIGURE 13. Removing three blocks in the same orbit, then removing the central block

The signs of all the matchings in $G/\langle\rho\rangle$ are the same, so the permanent is equal to the determinant up to sign, and the result follows.

$\square$

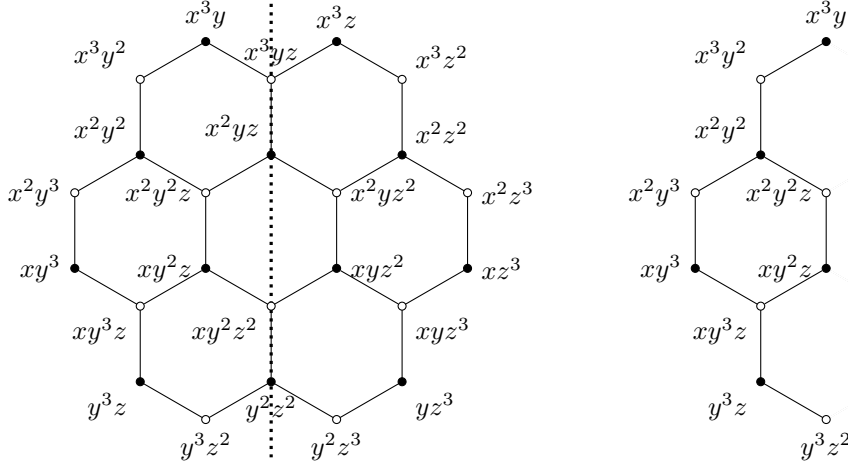**Proof of Theorem 1.4.** Recall the statement of Theorem 1.4:

FIGURE 14. The graph G, with axis of symmetry shown, and the quotient G' for $a = b = c = 2$

Theorem. *Assuming $a = b$ and the product abc is even, let $C_2 = \mathbb{Z}/2\mathbb{Z} = \{1, \tau\kappa\}$ act on $R = \mathbb{Z}[x, y, z]/(x^{2a}, y^{a+c}, z^{a+c})$ by swapping $y \leftrightarrow z$. Then the map $U_m|_{R^{C_2, -}}$ restricted to the m-th homogeneous component of the anti-invariant submodule $R^{C_3, -} := \{f \in R : \tau\kappa(f) = -f\}$ has $\det(U_m|_{R^{C_2, -}})$ equal, up to sign, to the number of transpose complementary plane partitions in an $a \times a \times c$ box.*

Proof. This time, the transpose complementary plane partitions correspond to perfect matchings on G that are symmetric about the vertical axis of symmetry (flip symmetric). Note that all the vertices on the vertical axis are of the form $x^i y^j z^j$, and in any symmetric perfect matching, $x^i y^j z^j$ must be matched with $x^{i+1} y^j z^j$ to preserve flip symmetry.

On $G/\langle\tau\kappa\rangle$, erase all vertices $[i, j, j]$ and all edges coming out of them. Call this new graph G'. From flip symmetry and our previous observation, flip symmetric matchings on G correspond directly to matchings on G'. Figure 14 gives an example of G' when $a = b = c = 2$.

Moreover, every vertex $[x^i y^j z^k]$ of G' corresponds to two elements $x^i y^j z^k$ and $x^i z^j y^k$. These vertices are in bijection with $x^i y^j z^k - x^i z^j y^k$ ($j > k$), the basis elements of $R^{\tau\kappa}$. Again, we abuse notation and let $[i, j, k]$ represent either a vertex of G' or a basis element depending on context. In G', every vertex orbit $[i, j, k]$ is connected to $[i + 1, j, k]$, $[i, j + 1, k]$ or $[i, j, k + 1]$ (if those orbits exist), and similarly, U_r sends any basis element $[i, j, k]$ ($j > k$) to a linear combination of basis elements $[i + 1, j, k] + [i, j + 1, k] + [i, j, k + 1]$ (if those elements exist.) Notice if $k + 1 = j$, then $[i, j, k + 1] = 0$.

Hence, U'_r, realized as a matrix indexed by the basis elements of rank r and $r + 1$, is the adjacency matrix for G', and has entries in $\{0, 1\}$. Therefore the permanent of U'_r counts the number of flip symmetric perfect matchings in G.

To show permanent equals determinant, we show matchings in G' have the same sign. To get from one transpose complementary plane partition to another, we move a visible block (meaning three sides of the block are visible) to its transpose complementary location (which is empty by symmetry.) This corresponds to hitting the perfect matching on G' with a three cycle, so this preserves sign. All the transpose complementary plane partitions are connected in this way because we can transform
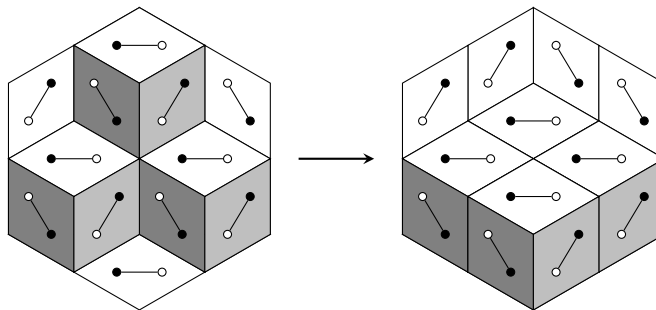
FIGURE 15. Moving visible block to its transpose complementary location

them into the basic transpose complementary plane partition where the top layer of blocks is level (see the right plane partition in Figure 15.) Hence, every matching on G' has the same sign, making the permanent equal the determinant.

$\square$

Proof of Theorem 1.5. Recall the statement of Theorem 1.5:

Theorem. *Assuming $a = b = c$ are all even, let C_2, C_3 act on $R = \mathbb{Z}[x, y, z]/(x^{2a}, y^{2a}, z^{2a})$ as before. Then the map $U_m|_{R^{C_3} \cap R^{C_2, -}}$ restricted to the m-th homogeneous component of the intersection $R^{C_3} \cap R^{C_2, -}$ has $\det(U_m|_{R^{C_3} \cap R^{C_2, -}})$ equal, up to sign, to the number of cyclically symmetric transpose complementary plane partitions in an $a \times a \times a$ box.*

Proof. This is a hybrid of the two previous proofs: the basis elements are now $x^i y^j z^k - x^i z^k y^j + y^i z^j x^k - z^i y^j x^k + z^i x^j y^k - y^i x^j z^k$, and now we move three visible blocks in the same ρ orbit to their transpose complementary locations to get from one CSTCPP to another. $\square$

6. QUESTIONS

Question 6.1. Is there some way to determine the Smith entries for the middle map $U_{\lfloor \frac{e-1}{2} \rfloor}$ for R, and is there a combinatorial explanation for the Smith entries in the context of plane partitions when $A = a + b$, $B = a + c$, and $C = b + c$?

Question 6.2. Why does the extension of Theorem 1.1 seem to work for four variables but not for five variables?

Question 6.3. Is there some way to achieve the counts for the other symmetry classes of plane partitions as determinants of maps like U_r restricted to natural submodules of R?

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