THE SPLITTING SUBSPACE CONJECTURE

ERIC CHEN AND DENNIS TSENG

Abstract. We answer a question by Niederreiter concerning the enumeration of a class of subspaces of finite dimensional vector spaces over finite fields by proving a conjecture by Ghorpade and Ram.

1. Introduction

We positively resolve the Splitting Subspace Conjecture, stemming from a question posed by Niederreiter (1995) [3, p. 11] and stated by Ghorpade and Ram [2]. We first define the notion of a σ-splitting subspace.

Definition. In the vector space $\mathbb{F}_{q^{mn}}$ over the finite field $\mathbb{F}_q$, given a $\sigma \in \mathbb{F}_{q^{mn}}$ such that $\mathbb{F}_{q^{mn}} = \mathbb{F}_q(\sigma)$, a ($m$-dimensional) subspace W of $\mathbb{F}_{q^{mn}}$ is a σ-*splitting subspace* if

$$W \oplus \sigma W \oplus \cdots \oplus \sigma^{n-1} W = \mathbb{F}_{q^{mn}}.$$

For example, $\{1, \sigma^m, \sigma^{2m}, \ldots, \sigma^{(n-1)m}\}$ spans a σ-splitting subspace. If $n = 1$, then $F_{q^{mn}}$ is the only σ-splitting subspace; if $m = 1$, then each 1-dimensional subspace of $F_{q^{mn}}$ is σ-splitting.

Conjecture 1 (Ghorpade-Ram)**.** The number of σ-splitting subspaces is

$$\frac{q^{mn} - 1}{q^m - 1} q^{m(m-1)(n-1)}.$$

This follows as Corollary 3.4 from our main result, Theorem 3.3. The next two sections are devoted to proving this theorem. We first construct a recursion that gives the cardinality of more general classes of subspaces, including the σ-splitting subspaces, and then solve this recurrence to obtain the result. Finally, we discuss some special cases of our more general result.

2. Recursion

For the remainder of this report, unless otherwise noted, consider more generally the vector space $\mathbb{F}_{q^N} (= \mathbb{F}_q^N)$ over the finite field $\mathbb{F}_q$, given a $\sigma \in \mathbb{F}_{q^N}$ such that $\mathbb{F}_{q^N} = \mathbb{F}_q(\sigma)$.

We begin by isolating the key property of the linear transformation $v \mapsto \sigma v$.

Proposition 2.1. *The linear endomorphisms of* $\mathbb{F}_{q^N}$ *that preserve no subspaces other than* $\{0\}$ *and all of* $\mathbb{F}_{q^N}$ *are exactly those which act as multiplication by a primitive element* σ *that generates the extension* $\mathbb{F}_q(\sigma) = \mathbb{F}_{q^N}$.

Proof. Operators defined as multiplication by a primitive element σ generating the extension $\mathbb{F}_q(\sigma) = \mathbb{F}_{q^N}$ cannot preserve any subspaces except $\{0\}$ and $\mathbb{F}_{q^N}$, for if W is such a subspace with nonzero $w \in W$, then $w \sum_{i=0}^{N-1} a_i \sigma^i \in W,\ a_i \in \mathbb{F}_q$, so $W = \mathbb{F}_{q^N}$. Conversely, note that any linear operator T together with the vector space $\mathbb{F}_q^N$ can be viewed as a finitely generated

$\mathbb{F}_q[x]$ module M, where x acts as T. Since $\mathbb{F}_q[x]$ is a principal ideal domain, we can use the primary decomposition of M to find $M \cong \bigoplus_{i=1}^{k} \mathbb{F}_q[x]/(p_i(x)^{r_i})$, where p_i is a polynomial for each i and r_i is a positive integer.

If T preserves no proper subspaces of $\mathbb{F}_{q^N}$, then $k = 1$. Also, $r_1 = 1$ unless $p_1(T)M$ is a proper submodule of M. Therefore, we have M is equal to $\mathbb{F}_q[x]/(p_1(x))$, where p_1 is an irreducible polynomial. This is exactly what it means for $x(= T)$ to act as the primitive element of the field extension $\mathbb{F}_q(\sigma) = \mathbb{F}_{q^N} = \mathbb{F}_q^N$ with minimal polynomial $p_1(x)$. □

We next define notation to describe the sets to be counted by the general recursion.

Definition. Suppose that $A_1, A_2, \dots, A_k$ are sets of subspaces of $\mathbb{F}_{q^N}$. Let $[A_1, A_2, \dots, A_k]$ be the set of all k-tuples $(W_1, W_2, \dots, W_k)$ such that

$$\begin{aligned} W_i &\in A_i \quad \text{for} \quad 1 \le i \le k, \\ W_i &\supseteq W_{i+1} + \sigma W_{i+1} \quad \text{for} \quad 1 \le i \le k-1. \end{aligned}$$

If A_i is the set of all subspaces of $\mathbb{F}_{q^N}$ with dimension d_i, then A_i is denoted within the brackets as d_i. For example, $[3, A_2]$ denotes all tuples (W_1, W_2) such that $\dim(W_1) = 3$, $W_2 \in A_2$ and $W_1 \supseteq W_2 + \sigma W_2$.

Definition. For nonnegative integers a, b with $N > a > b$ or $a = b = 0$

$$(a, b) := \left\{ W \subseteq \mathbb{F}_{q^N} : \dim(W) = a \text{ and } \dim(W \cap \sigma^{-1} W) = b \right\}.$$

For example, $(1, 0)$ is the set of all 1-dimensional subspaces and $(2, 1)$ is the set of all 2-dimensional subspaces W such that $\dim(W \cap \sigma^{-1} W) = 1$.

Definition. Given sets $[A_{1,1}, A_{1,2}], [A_{2,1}, A_{2,2}], \dots, [A_{r,1}, A_{r,2}]$ as defined above, let

$$\langle [A_{1,1}, A_{1,2}], [A_{2,1}, A_{2,2}], \dots, [A_{r,1}, A_{r,2}] \rangle$$

denote the set of $2r$-tuples of subspaces $(W_{1,1}, W_{1,2}, W_{2,1}, W_{2,2}, \dots W_{r,1}, W_{r,2})$ such that

$$\begin{aligned} (W_{i,1}, W_{i,2}) &\in [A_{i,1}, A_{i,2}] \quad \text{for} \quad 1 \le i \le r, \\ W_{i,2} &\supseteq W_{i+1,1} \quad \text{for} \quad 1 \le i \le r-1. \end{aligned}$$

For example, $\langle [3, 2], [2, 1] \rangle$ is the set of all 4-tuples of subspaces (W_1, W_2, W_3, W_4) such that

$$\begin{aligned} \dim(W_1) = 3,\ \dim(W_2) = 2, &\quad \dim(W_3) = 2,\ \dim(W_4) = 1, \\ W_1 \supseteq W_2 + \sigma W_2, &\quad W_3 \supseteq W_4 + \sigma W_4, \\ &\quad W_2 \supseteq W_3. \end{aligned}$$

We use the following proposition extensively in constructing the recursion.

Proposition 2.2. *For nonnegative integers $N > a > b$ or $a = b = 0$*

$$\begin{aligned} [a, b] &= \bigcup_{i=b}^{\max(a-1,0)} [(a, i), b] \\ &= \bigcup_{j=0}^{\max(b-1,0)} [a, (b, j)]. \end{aligned}$$

Proof. Follows from Proposition 2.1 and the definitions of [,], (,). □

We next define an ordering on the tuples labelling the sets of subspaces

$$[(a_{1,1}, a_{1,2}), (a_{2,1}, a_{2,2}), \ldots, (a_{r,1}, a_{r,2})].$$

The recursion in Lemma 2.3 will give the cardinality of sets of subspaces so labelled in terms of the cardinality of sets labelled by tuples before it in the ordering. The base case is $[(0, 0)]$, containing one element.

Definition. First, define an ordering on the ordered pairs of the form (a, b) such that $(a_1, b_1) \succ (a_2, b_2)$ if $a_1 > a_2$ or $a_1 = a_2$ and $b_1 < b_2$. Next, define an ordering on tuples of the form $[(a_{1,1}, a_{1,2}), (a_{2,1}, a_{2,2}), \ldots, (a_{r,1}, a_{r,2})]$ such that the order is lexicographic in terms of the ordered pairs $(a_{i,1}, a_{i,2})$ from left to right. Finally, define an ordering on the same tuples for $s \geq 0$ such that

$$[(a_{1,1}, a_{1,2}), (a_{2,1}, a_{2,2}), \ldots, (a_{r+s,1}, a_{r+s,2})] \succ [(a_{1,1}, a_{1,2}), (a_{2,1}, a_{2,2}), \ldots, (a_{r,1}, a_{r,2})].$$

For example, $(3, 1) \succ (3, 2) \succ (2, 0)$ and $[(6, 5), (4, 2)] \succ [(6, 5), (4, 3)] \succ [(5, 2), (2, 0)]$.

Lemma 2.3. *Suppose*

$$N > a_{1,1} > a_{1,2} \geq a_{2,1} > a_{2,2} \geq \ldots \geq a_{r,1} > a_{r,2} \geq 0 = a_{r+1,1} = a_{r+1,2} = \ldots = a_{r+s,1} = a_{r+s,2}$$

and after setting

$$a_{0,1} = a_{0,2} = N, \quad a_{r+1,1} = a_{r+1,2} = 0, \quad j_{r+1} = k_{r+1} = 0,$$

that (or else $[(a_{1,1}, a_{1,2}), (a_{2,1}, a_{2,2}), \ldots, (a_{r,1}, a_{r,2})]$ *is empty)*

$$a_{i-1,1} \geq 2a_{i,1} - a_{i,2} \quad \text{for} \quad 1 \leq i \leq r.$$

Let

$$C = \{(j_1, \ldots, j_r) : \max(a_{i+1,2}, 2a_{i,2} - a_{i,1}) \leq j_i \leq \max(a_{i,2} - 1, 0), 1 \leq i \leq r\},$$
$$D = \{(k_1, \ldots, k_r) : a_{i,2} \leq k_i \leq a_{i,1} - 1, 1 \leq i \leq r\}.$$

Then

$$\begin{aligned}
&|[(a_{1,1}, a_{1,2}), (a_{2,1}, a_{2,2}), \ldots, (a_{r+s,1}, a_{r+s,2})]| \\
&= |[(a_{1,1}, a_{1,2}), (a_{2,1}, a_{2,2}), \ldots, (a_{r,1}, a_{r,2})]| \\
&= \sum_{(j_1,\ldots,j_r)\in C} |[(a_{1,2}, j_1), (a_{2,2}, j_2), \ldots, (a_{r,2}, j_r)]| \prod_{i=1}^{r} \begin{bmatrix} a_{i-1,2} - (2a_{i,2} - j_i) \\ a_{i,1} - (2a_{i,2} - j_i) \end{bmatrix}_q \\
&- \sum_{(k_1,\ldots,k_r)\in D\setminus(a_{1,2},\ldots,a_{r,2})} |[(a_{1,1}, k_1), (a_{2,1}, k_2), \ldots, (a_{r,1}, k_r)]| \prod_{i=1}^{r} \begin{bmatrix} k_i - a_{i+1,1} \\ a_{i,2} - a_{i+1,1} \end{bmatrix}_q.
\end{aligned}$$

Proof. We give an example before the general case. Let $r = 2$; we compute $|[(3, 1), (1, 0)]|$ by counting $|\langle[3, 1], [1, 0]\rangle|$ in two different ways. Applying Proposition 2.2 to the terms on the left within the brackets gives

$$|\langle[3, 1], [1, 0]\rangle| = |\langle[(3, 2), 1], [(1, 0), 0]\rangle| + |\langle[(3, 1), 1], [(1, 0), 0]\rangle|.$$

Above, if $(W_1, W_2, W_3, W_4) \in \langle[(3, 2), 1], [(1, 0), 0]\rangle$ then $W_2 = W_3$ and $W_4 = \{0\}$. So

$$|\langle[(3, 2), 1], [(1, 0), 0]\rangle| = |[(3, 2), (1, 0)]|.$$

Likewise for $(W_1, W_2, W_3, W_4) \in \langle[(3, 1), 1], [(1, 0), 0]\rangle$ then $W_2 = W_3$ and $W_4 = \{0\}$. So

$$|\langle[(3, 1), 1], [(1, 0), 0]\rangle| = |[(3, 1), (1, 0)]|,$$

and

$$|\langle [3,1],[1,0]\rangle| = |[(3,2),(1,0)]| + |[(3,1),(1,0)]|.$$

Next, applying Proposition 2.2 to the terms on the right within the brackets gives

$$|\langle [3,1],[1,0]\rangle| = |\langle [3,(1,0)],[1,(0,0)]\rangle|.$$

If $(W_1, W_2, W_3, W_4) \in \langle [3,(1,0)],[1,(0,0)]\rangle$, then $W_3 = W_2$, $W_4 = \{0\}$ and thus W_1 is a 3-dimensional subspace containing the 2-dimensional space $W_2 + \sigma W_2$. So

$$|\langle [3,(1,0)],[1,(0,0)]\rangle| = |[(1,0),(0,0)]| \begin{bmatrix} N-2 \\ 1 \end{bmatrix}_q,$$

and therefore

$$|\langle [3,(1,0)],[1,(0,0)]\rangle| = |[(1,0),(0,0)]| \begin{bmatrix} N-2 \\ 1 \end{bmatrix}_q.$$

We then have, after rearranging, that

$$|[(3,1),(1,0)]| = |[(1,0),(0,0)]| \begin{bmatrix} N-2 \\ 1 \end{bmatrix}_q - |[(3,2),(1,0)]|.$$

Note that

$$[(3,2),(1,0)],\ \ [(1,0),(0,0)]$$

come before $[(3,1),(1,0)]$ in the ordering on tuples.

The proof of the Lemma is a generalization of this process. The first equality is clear. The size of $[(a_{1,1}, a_{1,2}), (a_{2,1}, a_{2,2}), \ldots, (a_{r,1}, a_{r,2})]$ is computed by applying Proposition 2.2

$$\begin{aligned} &|\langle [a_{1,1}, a_{1,2}], [a_{2,1}, a_{2,2}], \ldots, [a_{r,1}, a_{r,2}]\rangle| \\ &= \sum_{(k_1,\ldots,k_r)\in D} |\langle [(a_{1,1}, k_1), a_{1,2}], [(a_{2,1}, k_2), a_{2,2}], \ldots, [(a_{r,1}, k_r), a_{r,2}]\rangle| \\ \text{(R)}\qquad &= \sum_{(k_1,\ldots,k_r)\in D} |[(a_{1,1}, k_1), (a_{2,1}, k_2), \ldots, (a_{r,1}, k_r)]| \prod_{i=1}^{r} \begin{bmatrix} k_i - a_{i+1,1} \\ a_{i,2} - a_{i+1,1} \end{bmatrix}_q. \end{aligned}$$

Expanding in the other way, we get

$$\begin{aligned} &|\langle [a_{1,1}, a_{1,2}], [a_{2,1}, a_{2,2}], \ldots, [a_{r,1}, a_{r,2}]\rangle| \\ &= \sum_{(j_1,\ldots,j_r)\in C} |\langle [a_{1,1}, (a_{1,2}, j_1)], [a_{2,1}, (a_{2,2}, j_2)], \ldots, [a_{r,1}, (a_{r,2}, j_r)]\rangle| \\ \text{(L)}\qquad &= \sum_{(j_1,\ldots,j_r)\in C} |[(a_{1,2}, j_1), (a_{2,2}, j_2), \ldots, (a_{r,2}, j_r)]| \prod_{i=1}^{r} \begin{bmatrix} a_{i-1,2} - (2a_{i,2} - j_i) \\ a_{i,1} - (2a_{i,2} - j_i) \end{bmatrix}_q. \end{aligned}$$

Subtracting from (R) and (L) the quantity

$$|[(a_{1,1}, a_{1,2}), (a_{2,1}, a_{2,2}), \ldots, (a_{r,1}, a_{r,2})]|$$

produces the stated result of the Lemma. □

Finally, we relate sets of the form $[(a_{1,1}, a_{1,2}), (a_{2,1}, a_{2,2}), \ldots, (a_{r,1}, a_{r,2})]$ to σ-splitting subspaces.

Proposition 2.4. *Let $\mathbb{F}_{q^N} = \mathbb{F}_{q^{mn}}$. Then*

$$\begin{aligned}&[((n-1)m,(n-2)m),((n-2)m,(n-3)m),\ldots,(2m,m),(m,0)]\\ &=\left\{(\bigoplus_{i=0}^{n-2}\sigma^iW,\ \bigoplus_{i=0}^{n-3}\sigma^iW,\ \ldots,\ W\oplus\sigma W,\ W)\ :\ \bigoplus_{i=0}^{n-1}\sigma^iW=\mathbb{F}_{q^{mn}}\right\}.\end{aligned}$$

In particular $|[((n-1)m,(n-2)m),((n-2)m,(n-3)m),\ldots,(2m,m),(m,0)]|$ *is the number of σ-splitting subspaces.*

Proof. If W is a σ-splitting subspace, then

$$\begin{aligned}&(\bigoplus_{i=0}^{n-2}\sigma^iW,\bigoplus_{i=0}^{n-3}\sigma^iW,\ldots,W\oplus\sigma W,W)\\ &\qquad\in[((n-1)m,(n-2)m),((n-2)m,(n-3)m),\ldots,(2m,m),(m,0)].\end{aligned}$$

On the other hand, suppose that

$$(W_{n-1},\ldots,W_1)\in[((n-1)m,(n-2)m),((n-2)m,(n-3)m),\ldots,(2m,m),(m,0)]$$

Then for $1\le k\le n-2$

$$\begin{aligned}\dim(W_{k+1})&=(k+1)m\\ &=2km-(k-1)m\\ &=\dim(W_k+\sigma W_k).\end{aligned}$$

So $W_{k+1}=W_k+\sigma W_k$ for $1\le k\le n-2$. Also, $W_2=W_1\oplus\sigma W_1$ as $W_1\cap\sigma W_1=\{0\}$.

Suppose that $W_k=\bigoplus_{i=0}^{k-1}\sigma^iW_1$. Then, since $\dim(W_{k+1})=\dim(W_k+\sigma W_k)=(k+1)m$, we obtain

$$\begin{aligned}W_{k+1}&=W_k+\sigma W_k\\ &=W_1+\sigma^2W_1+\cdots+\sigma^kW_1\\ &=\bigoplus_{i=0}^{k}\sigma^iW_1.\end{aligned}$$

When $k=n-1$, we have that $W_{n-1}+\sigma W_{n-1}=\bigoplus_{i=0}^{n-1}\sigma^iW_1$, since $W_{n-1}+\sigma W_{n-1}=\mathbb{F}_{q^{mn}}$ is mn-dimensional. So W_1 is indeed a σ-splitting subspace. □

Corollary 2.5. *The number of σ-splitting subspaces in $\mathbb{F}_{q^{mn}}$ over F_q is independent of choice of primitive element σ.*

Proof. Neither the base case $|[(0,0]|$ nor Lemma 2.3 depends on the σ chosen. □

Remark. More generally, given an arbitrary invertible linear operator T on $\mathbb{F}_{q^{mn}}$ over $\mathbb{F}_q$, we might consider how many "T-splitting" subspaces exist; that is, the number of m-dimensional subspaces W such that

$$W\oplus TW\oplus\cdots\oplus T^{n-1}W=\mathbb{F}_{q^{mn}}.$$

We may then redefine $(\ ,\ ),[\ ,\ ],\langle\ ,\ \rangle$ by replacing the expressions $W+\sigma W$ with $W+TW$ and $W\cap\sigma^{-1}W$ with $W\cap T^{-1}W$.

Recall from Proposition 2.1 and Lemma 2.3 that when $T(v)=\sigma v$, the nonzero numbers $|[(a_{1,1},a_{1,2}),(a_{2,1},a_{2,2}),\ldots,(a_{r,1},a_{r,2})]|$ can be computed from the base case $|[0,0]|=1$.

But if T is any invertible linear operator, there may exist nonempty sets of the form $[(a_1, a_1), (a_2, a_2), \ldots, (a_r, a_r)]$ where $a_r \neq 0$. In fact, such sets cannot be computed recursively. For example

$$\begin{aligned} &|\langle [4,4],[2,2]\rangle| \\ &= |\langle [(4,4),4],[(2,2),2]\rangle| = |[(4,4),(2,2)]| \\ &= |\langle [4,(4,4)],[2,(2,2)]\rangle| = |[(4,4),(2,2)]|. \end{aligned}$$

We may still apply Lemma 2.3 in the case of general T, however, with the cardinalities of these sets as additional base cases.

Remark. One might try to apply the method in Lemma 2.3 to try to count *pointed* subspaces. Namely, given a fixed vector v, it is not difficult to show that the number of tuples in $[(a_{1,1}, a_{1,2}), (a_{2,1}, a_{2,2}), \ldots, (a_{r,1}, a_{r,2})]$ whose ℓ^{th} subspace contains v is equal to

$$\frac{q^{a_{\ell,1}}-1}{q^N-1}|[(a_{1,1}, a_{1,2}), (a_{2,1}, a_{2,2}), \ldots, (a_{r,1}, a_{r,2})]|,$$

and the number of tuples $(W_{1,1}, W_{1,2}, \ldots, W_{r,1}, W_{r,2})$ in $\langle [a_{1,1}, a_{1,2}], [a_{2,1}, a_{2,2}], \ldots, [a_{r,1}, a_{r,2}]\rangle$ such that $W_{\ell,1}$ contains v is equal to

$$\frac{q^{a_{\ell,1}}-1}{q^N-1}|\langle [a_{1,1}, a_{1,2}], [a_{2,1}, a_{2,2}], \ldots, [a_{r,1}, a_{r,2}]\rangle|.$$

Counting these tuples $(W_{1,1}, W_{1,2}, \ldots, W_{r,1}, W_{r,2})$ in $\langle [a_{1,1}, a_{1,2}], [a_{2,1}, a_{2,2}], \ldots, [a_{r,1}, a_{r,2}]\rangle$ by expanding at $a_{1,1}, a_{2,1}, \ldots, a_{r,1}$ as in (R), we get exactly

$$\frac{q^{a_{\ell,1}}-1}{q^N-1}\sum_{(k_1,\ldots,k_r)\in D}|[(a_{1,1}, k_1), (a_{2,1}, k_2), \ldots, (a_{r,1}, k_r)]|\prod_{i=1}^{r}\begin{bmatrix} k_i - a_{i+1,1} \\ a_{i,2} - a_{i+1,1}\end{bmatrix}_q,$$

which is just (R) scaled by $\frac{q^{a_{\ell,1}}-1}{q^N-1}$. It is less obvious that expanding at $a_{1,2}, \ldots, a_{r,2}$ will result in (L) scaled by $\frac{q^{a_{\ell,1}}-1}{q^N-1}$. Considering two cases, one where the subspace $W_{\ell,1} + \sigma W_{\ell,1}$ contains v and the other where the subspace $W_{\ell,1} + \sigma W_{\ell,1}$ does not contain v, yields the expansion

$$\sum_{(j_1,\ldots,j_r)\in C}\frac{\begin{bmatrix} 2a_{\ell,2}-j_\ell \\ 1\end{bmatrix}_q\begin{bmatrix} a_{\ell-1,2}-(2a_{\ell,2}-j_\ell) \\ a_{\ell,1}-(2a_{\ell,2}-j_\ell)\end{bmatrix}_q + \left(\begin{bmatrix} a_{\ell-1,2} \\ 1\end{bmatrix}_q - \begin{bmatrix} 2a_{\ell,2}-j_\ell \\ 1\end{bmatrix}_q\right)\begin{bmatrix} a_{\ell-1,2}-(2a_{\ell,2}-j_\ell)-1 \\ a_{\ell,1}-(2a_{\ell,2}-j_\ell)-1\end{bmatrix}_q}{\begin{bmatrix} N \\ 1\end{bmatrix}_q}$$
$$|[(a_{1,2}, j_1), (a_{2,2}, j_2), \ldots, (a_{r,2}, j_r)]|\prod_{i=1, i\neq \ell}^{r}\begin{bmatrix} a_{i-1,2} - (2a_{i,2} - j_i) \\ a_{i,1} - (2a_{i,2} - j_i)\end{bmatrix}_q.$$

It can be checked that

$$\frac{\begin{bmatrix} 2a_{\ell,2}-j_\ell \\ 1\end{bmatrix}_q\begin{bmatrix} a_{\ell-1,2}-(2a_{\ell,2}-j_\ell) \\ a_{\ell,1}-(2a_{\ell,2}-j_\ell)\end{bmatrix}_q + \left(\begin{bmatrix} a_{\ell-1,2} \\ 1\end{bmatrix}_q - \begin{bmatrix} 2a_{\ell,2}-j_\ell \\ 1\end{bmatrix}_q\right)\begin{bmatrix} a_{\ell-1,2}-(2a_{\ell,2}-j_\ell)-1 \\ a_{\ell,1}-(2a_{\ell,2}-j_\ell)-1\end{bmatrix}_q}{\begin{bmatrix} N \\ 1\end{bmatrix}_q} = \frac{\begin{bmatrix} a_{\ell,1} \\ 1\end{bmatrix}_q\begin{bmatrix} a_{\ell-1,2}-(2a_{\ell,2}-j_\ell) \\ a_{\ell,1}-(2a_{\ell,2}-j_\ell)\end{bmatrix}_q}{\begin{bmatrix} N \\ 1\end{bmatrix}_q},$$

which means that we again just get the expansion (L) scaled by $\frac{q^{a_{\ell,1}}-1}{q^N-1}$.

3. Solution to the Recursion

The next two lemmas are special cases of the following q-Chu-Vandermonde identity for N a nonnegative integer [1, p. 354]. Refer to Appendix A for proofs of the lemmas.

$$
{}_2\phi_1\begin{pmatrix} q^{-N}, & a; & cq^N/a \\ & c & \end{pmatrix} := \sum_{m=0}^{N} \frac{(q^{-N};q)_m (a;q)_m}{(q;q)_m (c;q)_m} \left(\frac{cq^N}{a}\right)^m = \frac{(c/a;q)_N}{(c;q)_N}.
$$

Lemma 3.1. *If $C \le B-1 \le D-1 \le A-1$ are non-negative integers, then*

$$
\sum_{s=C}^{B-1} \begin{bmatrix} A-B-1 \\ B-s-1 \end{bmatrix}_q \begin{bmatrix} B \\ s \end{bmatrix}_q \begin{bmatrix} s \\ C \end{bmatrix}_q \begin{bmatrix} A-(2B-s) \\ D-(2B-s) \end{bmatrix}_q q^{(B-s)(B-s-1)} = \frac{[B]_q}{[D-C]_q} \begin{bmatrix} B-1 \\ C \end{bmatrix}_q \begin{bmatrix} A-B-1 \\ D-B-1 \end{bmatrix}_q \begin{bmatrix} D-C \\ B-C \end{bmatrix}_q.
$$

Lemma 3.2. *If $C \le D \le B-1 \le A-1$ are non-negative integers, then*

$$
\sum_{s=D}^{B-1} \begin{bmatrix} A-B-1 \\ B-s-1 \end{bmatrix}_q \begin{bmatrix} B \\ s \end{bmatrix}_q \begin{bmatrix} s \\ C \end{bmatrix}_q \begin{bmatrix} s-C \\ D-C \end{bmatrix}_q q^{(B-s)(B-s-1)} = \frac{[B]_q}{[A-D]_q} \begin{bmatrix} B-1 \\ C \end{bmatrix}_q \begin{bmatrix} B-C-1 \\ D-C \end{bmatrix}_q \begin{bmatrix} A-D \\ B-D \end{bmatrix}_q.
$$

We now give the main theorem of this report.

Theorem 3.3. *Suppose that*

$$
N > a_{1,1} > a_{1,2} \ge a_{2,1} > a_{2,2} \ge \cdots \ge a_{r,1} > a_{r,2} \ge 0,
$$
$$
a_{0,1} = a_{0,2} = N, \quad a_{r+1,1} = a_{r+1,2} = 0.
$$

Then

$$
|[(a_{1,1}, a_{1,2}), \dots, (a_{r,1}, a_{r,2})]| = \frac{\begin{bmatrix} N \\ 1 \end{bmatrix}_q}{\begin{bmatrix} a_{1,1} \\ 1 \end{bmatrix}_q} \frac{\prod_{i=0}^{r-1} \begin{bmatrix} a_{i,1}-a_{i+1,1}-1 \\ a_{i+1,1}-a_{i+1,2}-1 \end{bmatrix}_q \begin{bmatrix} a_{i+1,1} \\ a_{i+1,2} \end{bmatrix}_q \begin{bmatrix} a_{i+1,2} \\ a_{i+2,1} \end{bmatrix}_q}{\prod_{i=1}^{r-1} \begin{bmatrix} a_{i,1}-1 \\ a_{i+1,1}-1 \end{bmatrix}_q} q^E, \tag{1}
$$

where

$$
E = \sum_{i=1}^{r} (a_{i,1}-a_{i,2})(a_{i,1}-a_{i,2}-1).
$$

Corollary 3.4 (Splitting Subspace Conjecture)**.** *We have, when $N \ge mn$, the equality*

$$
|[((n-1)m, (n-2)m), \dots, (2m, m), (m, 0)]| = \frac{\begin{bmatrix} N \\ 1 \end{bmatrix}_q}{\begin{bmatrix} m \\ 1 \end{bmatrix}_q} \begin{bmatrix} N-mn+m-1 \\ m-1 \end{bmatrix}_q q^{m(m-1)(n-1)}
$$

In particular, when $N = mn$,

$$
|[((n-1)m, (n-2)m), \dots, (2m, m), (m, 0)]| = \frac{\begin{bmatrix} mn \\ 1 \end{bmatrix}_q}{\begin{bmatrix} m \\ 1 \end{bmatrix}_q} q^{m(m-1)(n-1)}.
$$

Proof. From plugging into (1)

$$\begin{aligned}
&[((n-1)m,(n-2)m),((n-2)m,(n-3)m),\dots,(2m,m),(m,0)] \\
&= \frac{\begin{bmatrix}N\\1\end{bmatrix}_q}{\begin{bmatrix}(n-1)m\\1\end{bmatrix}_q} \frac{\left(\begin{bmatrix}N-(n-1)m-1\\m-1\end{bmatrix}_q \begin{bmatrix}m-1\\m-1\end{bmatrix}_q \cdots \begin{bmatrix}m-1\\m-1\end{bmatrix}_q\right)\left(\begin{bmatrix}(n-1)m\\(n-2)m\end{bmatrix}_q \cdots \begin{bmatrix}m\\0\end{bmatrix}_q\right)}{\begin{bmatrix}(n-1)m-1\\(n-2)m-1\end{bmatrix}_q \cdots \begin{bmatrix}m-1\\0\end{bmatrix}_q} q^{\sum_{i=1}^{n-1} m(m-1)} \\
&= \frac{\begin{bmatrix}N\\1\end{bmatrix}_q}{\begin{bmatrix}(n-1)m\\1\end{bmatrix}_q} \begin{bmatrix}N-(n-1)m-1\\m-1\end{bmatrix}_q \frac{\begin{bmatrix}(n-1)m\\(n-2)m\end{bmatrix}_q \cdots \begin{bmatrix}2m\\m\end{bmatrix}_q}{\begin{bmatrix}(n-1)m-1\\(n-2)m-1\end{bmatrix}_q \cdots \begin{bmatrix}m-1\\0\end{bmatrix}_q} q^{m(m-1)(n-1)}.
\end{aligned}$$

Since for each $1 \le k \le n-2$, $\frac{\begin{bmatrix}(n-k)m\\(n-k-1)m\end{bmatrix}_q}{\begin{bmatrix}(n-k)m-1\\(n-k-1)m-1\end{bmatrix}_q} = \frac{\frac{[(n-k)m]!_q}{[m]!_q[(n-k-1)m]!_q}}{\frac{[(n-k)m-1]!_q}{[m]!_q[(n-k-1)m-1]!_q}} = \frac{1-q^{(n-k)m}}{1-q^{(n-k-1)m}}$, this reduces to

$$\begin{aligned}
&\frac{\begin{bmatrix}N\\1\end{bmatrix}_q}{\begin{bmatrix}(n-1)m\\1\end{bmatrix}_q} \begin{bmatrix}N-(n-1)m-1\\m-1\end{bmatrix}_q \frac{1-q^{(n-1)m}}{1-q^m} q^{m(m-1)(n-1)} \\
&= \frac{\begin{bmatrix}N\\1\end{bmatrix}_q}{\begin{bmatrix}m\\1\end{bmatrix}_q} \begin{bmatrix}N-mn+m-1\\m-1\end{bmatrix}_q q^{m(m-1)(n-1)}.
\end{aligned}$$

□

Proof of Theorem 3.3. We verify that (1) satisfies the recursion in Lemma 2.3.

Recall that

$$L = \sum_{(j_1,\dots,j_r)\in C} |[(a_{1,2},j_1),(a_{2,2},j_2),\dots,(a_{r,2},j_r)]| \prod_{i=1}^{r} \begin{bmatrix}a_{i-1,2}-(2a_{i,2}-j_i)\\a_{i,1}-(2a_{i,2}-j_i)\end{bmatrix}_q,$$

$$R = \sum_{(k_1,\dots,k_r)\in D} |[(a_{1,1},k_1),(a_{2,1},k_2),\dots,(a_{r,1},k_r)]| \prod_{i=1}^{r} \begin{bmatrix}k_i-a_{i+1,1}\\a_{i,2}-a_{i+1,1}\end{bmatrix}_q.$$

We first check equality when $a_{r,2} \neq 0$ so that the expressions obtained for (L) and (R) using (1) do not contain negative q-binomials.

Substituting (1) and applying Lemma 3.1 to the resulting independent sums in (L) gives

$$\begin{aligned}
L &= \frac{\begin{bmatrix}N\\1\end{bmatrix}_q}{\begin{bmatrix}a_{1,2}\\1\end{bmatrix}_q} \frac{\prod_{i=1}^r \sum_{j_i} \begin{bmatrix}a_{i-1,2}-a_{i,2}-1\\a_{i,2}-j_i-1\end{bmatrix}_q \begin{bmatrix}a_{i,2}\\j_i\end{bmatrix}_q \begin{bmatrix}j_i\\a_{i+1,2}\end{bmatrix}_q \begin{bmatrix}a_{i-1,2}-(2a_{i,2}-j_i)\\a_{i,1}-(2a_{i,2}-j_i)\end{bmatrix}_q q^{(a_{i,2}-j_i)(a_{i,2}-j_i-1)}}{\prod_{i=1}^{r-1} \begin{bmatrix}a_{i,2}-1\\a_{i+1,2}-1\end{bmatrix}_q} \\
&= \frac{\begin{bmatrix}N\\1\end{bmatrix}_q}{\begin{bmatrix}a_{1,2}\\1\end{bmatrix}_q} \prod_{i=1}^{r} \frac{[a_{i,2}]_q}{[a_{i,1}-a_{i+1,2}]_q} \begin{bmatrix}a_{i,2}-1\\a_{i+1,2}\end{bmatrix}_q \begin{bmatrix}a_{i-1,2}-a_{i,2}-1\\a_{i,1}-a_{i,2}-1\end{bmatrix}_q \begin{bmatrix}a_{i,1}-a_{i+1,2}\\a_{i,2}-a_{i+1,2}\end{bmatrix}_q \prod_{i=1}^{r-1} \begin{bmatrix}a_{i,2}-1\\a_{i+1,2}-1\end{bmatrix}_q^{-1}.
\end{aligned}$$

Substituting (1) and applying Lemma 3.2 to the resulting independent sums in (R) gives

$$\begin{aligned}
R &= \frac{\begin{bmatrix}N\\1\end{bmatrix}_q}{\begin{bmatrix}a_{1,1}\\1\end{bmatrix}_q} \frac{\prod_{i=1}^r \sum_{k_i} \begin{bmatrix}a_{i-1,1}-a_{i,1}-1\\a_{i,1}-k_i-1\end{bmatrix}_q \begin{bmatrix}a_{i,1}\\k_i\end{bmatrix}_q \begin{bmatrix}k_i\\a_{i+1,1}\end{bmatrix}_q \begin{bmatrix}k_i-a_{i+1,1}\\a_{i,2}-a_{i+1,1}\end{bmatrix}_q q^{(a_{i,1}-k_i)(a_{i,1}-k_i-1)}}{\prod_{i=1}^{r-1} \begin{bmatrix}a_{i,2}-1\\a_{i+1,2}-1\end{bmatrix}_q} \\
&= \frac{\begin{bmatrix}N\\1\end{bmatrix}_q}{\begin{bmatrix}a_{1,1}\\1\end{bmatrix}_q} \prod_{i=1}^{r} \frac{[a_{i,1}]_q}{[a_{i-1,1}-a_{i,2}]_q} \begin{bmatrix}a_{i,1}-1\\a_{i+1,1}\end{bmatrix}_q \begin{bmatrix}a_{i,1}-a_{i+1,1}-1\\a_{i,2}-a_{i+1,1}\end{bmatrix}_q \begin{bmatrix}a_{i-1,1}-a_{i,2}\\a_{i,1}-a_{i,2}\end{bmatrix}_q \prod_{i=1}^{r-1} \begin{bmatrix}a_{i,1}-1\\a_{i+1,1}-1\end{bmatrix}_q^{-1}.
\end{aligned}$$

After simplification (see Appendix B)

$$L = R = \frac{(1-q^N)[N-a_{1,2}-1]!_q}{[N-a_{1,1}]!_q[a_{r,2}]!_q} \prod_{i=1}^{r} \frac{[a_{i,1}-a_{i+1,2}-1]!_q}{[a_{i,1}-a_{i,2}-1]!_q[a_{i,1}-a_{i,2}]!_q} \prod_{i=2}^{r} \frac{1}{[a_{i-1,2}-a_{i,1}]!_q}. \tag{2}$$

Finally, we deal with the case $a_{r,2} = 0$, when the expression obtained by directly applying (1) to (L) may contain negative q-binomials ((R) is unaffected). Suppose $r > 1$. By definition, we know that

$$|\langle [a_{1,1}, a_{1,2}], [a_{2,1}, a_{2,2}], \ldots, [a_{r,1}, 0]\rangle| = |\langle [a_{1,1}, a_{1,2}], [a_{2,1}, a_{2,2}], \ldots, [a_{r-1,1}, a_{r-1,2}]\rangle| \begin{bmatrix} a_{r-1,2} \\ a_{r,1} \end{bmatrix}_q.$$

This means that

$$\begin{aligned} L &= \sum_{(j_1,\ldots,j_r)\in C} |[(a_{1,2}, j_1), (a_{2,2}, j_2), \ldots, (a_{r,2}, j_r)]| \prod_{i=1}^{r} \begin{bmatrix} a_{i-1,2} - (2a_{i,2} - j_i) \\ a_{i,1} - (2a_{i,2} - j_i) \end{bmatrix}_q \\ &= |\langle [a_{1,1}, a_{1,2}], , \ldots, [a_{r-1,1}, a_{r-1,2}]\rangle| \begin{bmatrix} a_{r-1,2} \\ a_{r,1} \end{bmatrix}_q. \end{aligned}$$

Since $a_{r-1,2} \geq a_{r,1} > 0$, we may apply our previous result to obtain

$$\begin{aligned} &|\langle [a_{1,1}, a_{1,2}], \ldots, [a_{r-1,1}, a_{r-1,2}]\rangle| \begin{bmatrix} a_{r-1,2} \\ a_{r,1} \end{bmatrix}_q \\ &= \begin{bmatrix} a_{r-1,2} \\ a_{r,1} \end{bmatrix}_q \frac{(1-q^N)[N-a_{1,2}-1]!_q}{[N-a_{1,1}]!_q[a_{r-1,2}]!_q} \prod_{i=1}^{r-1} \frac{[a_{i,1}-a_{i+1,2}-1]!_q}{[a_{i,1}-a_{i,2}-1]!_q[a_{i,1}-a_{i,2}]!_q} \prod_{i=2}^{r-1} \frac{1}{[a_{i-1,2}-a_{i,1}]!_q}. \end{aligned}$$

We wish to show that this is equal to

$$\frac{(1-q^N)[N-a_{1,2}-1]!_q}{[N-a_{1,1}]!_q[a_{r,2}]!_q} \prod_{i=1}^{r} \frac{[a_{i,1}-a_{i+1,2}-1]!_q}{[a_{i,1}-a_{i,2}-1]!_q[a_{i,1}-a_{i,2}]!_q} \prod_{i=2}^{r} \frac{1}{[a_{i-1,2}-a_{i,1}]!_q}$$

when $a_{r,2} = 0$. Take the quotient to find

$$\begin{aligned} &\frac{\frac{(1-q^N)[N-a_{1,2}-1]!_q}{[N-a_{1,1}]!_q[a_{r,2}]!_q} \prod_{i=1}^{r} \frac{[a_{i,1}-a_{i+1,2}-1]!_q}{[a_{i,1}-a_{i,2}-1]!_q[a_{i,1}-a_{i,2}]!_q} \prod_{i=2}^{r} \frac{1}{[a_{i-1,2}-a_{i,1}]!_q}}{\begin{bmatrix} a_{r-1,2} \\ a_{r,1} \end{bmatrix}_q \frac{(1-q^N)[N-a_{1,2}-1]!_q}{[N-a_{1,1}]!_q[a_{r-1,2}]!_q} \prod_{i=1}^{r-1} \frac{[a_{i,1}-a_{i+1,2}-1]!_q}{[a_{i,1}-a_{i,2}-1]!_q[a_{i,1}-a_{i,2}]!_q} \prod_{i=2}^{r-1} \frac{1}{[a_{i-1,2}-a_{i,1}]!_q}} \\ &= \frac{[a_{r-1,2}]!_q \frac{[a_{r,1}-a_{r+1,2}-1]!_q}{[a_{r,1}-a_{r,2}-1]!_q[a_{r,1}-a_{r,2}]!_q} \frac{1}{[a_{r-1,2}-a_{r,1}]!_q}}{\begin{bmatrix} a_{r-1,2} \\ a_{r,1} \end{bmatrix}_q [a_{r,2}]!_q} \\ &= \frac{[a_{r-1,2}]!_q \frac{[a_{r,1}-1]!_q}{[a_{r,1}-1]!_q[a_{r,1}]!_q} \frac{1}{[a_{r-1,2}-a_{r,1}]!_q}}{\begin{bmatrix} a_{r-1,2} \\ a_{r,1} \end{bmatrix}_q} \\ &= \frac{[a_{r-1,2}]!_q \frac{1}{[a_{r,1}]!_q} \frac{1}{[a_{r-1,2}-a_{r,1}]!_q}}{\begin{bmatrix} a_{r-1,2} \\ a_{r,1} \end{bmatrix}_q} \\ &= 1, \end{aligned}$$

as desired. Therefore, when $a_{r,2} = 0$, the equality

$$\begin{aligned} L &= \sum_{(j_1,\dots,j_r)\in C} |[[a_{1,2}, j_1), [a_{2,2}, j_2), \dots, [a_{r,2}, j_r)]| \prod_{i=1}^{r} \begin{bmatrix} a_{i-1,2} - (2a_{i,2} - j_i) \\ a_{i,1} - (2a_{i,2} - j_i) \end{bmatrix}_q \\ &= \frac{(1-q^N)[N - a_{1,2} - 1]!_q}{[N - a_{1,1}]!_q [a_{r,2}]!_q} \prod_{i=1}^{r} \frac{[a_{i,1} - a_{i+1,2} - 1]!_q}{[a_{i,1} - a_{i,2} - 1]!_q [a_{i,1} - a_{i,2}]!_q} \prod_{i=2}^{r} \frac{1}{[a_{i-1,2} - a_{i,1}]!_q} \\ &= R \end{aligned}$$

still holds.

Finally, suppose $r = 1$. Then,

$$L = |[(0,0)]| \begin{bmatrix} N \\ a_{1,1} \end{bmatrix}_q = \begin{bmatrix} N \\ a_{1,1} \end{bmatrix}_q .$$

If we plug in $\langle [a_{1,1}, 0] \rangle$ into (2), then we get $\begin{bmatrix} N \\ a_{1,1} \end{bmatrix}_q$, as desired. ☐

Corollary 3.5. *The numbers*

$$| \langle [a_{1,1}, a_{1,2}], [a_{21}, a_{22}], \dots, [a_{r,1}, a_{r,2}] \rangle |$$

are given by

$$L = R = \frac{(1-q^N)[N - a_{1,2} - 1]!_q}{[N - a_{1,1}]!_q [a_{r,2}]!_q} \prod_{i=1}^{r} \frac{[a_{i,1} - a_{i+1,2} - 1]!_q}{[a_{i,1} - a_{i,2} - 1]!_q [a_{i,1} - a_{i,2}]!_q} \prod_{i=2}^{r} \frac{1}{[a_{i-1,2} - a_{i,1}]!_q}.$$

4. Special Case: (k,k-1)

Note that when $r = 1$ and $a_{1,1} = k, a_{1,2} = k - 1$, with $k \le N - 1$, the formula (1) gives

$$|(k, k-1)| = \begin{bmatrix} N \\ 1 \end{bmatrix}_q ,$$

a number independent of k.

Proposition 4.1. *There is a bijection between sets of the form* $(k_1, k_1 - 1)$ *and* $(k_2, k_2 - 1)$ *when* $k_1, k_2 \le N - 1$.

Proof. It suffices to show that there exists a bijection between $(k, k-1)$ and $(k-1, k-2)$ for $2 \le k \le N - 1$. Define

$$\begin{aligned} \phi : (k-1, k-2) &\to (k, k-1) \\ W &\mapsto W + \sigma W. \end{aligned}$$

The map ϕ is well defined:

$$\begin{aligned} \dim(W + \sigma W) &= k, \\ \dim((W + \sigma W) \cap (\sigma^{-1} W + W)) &= k - 1. \end{aligned}$$

The second equality follows from the fact that $(W + \sigma W) \cap (\sigma^{-1} W + W)$ contains W and has dimension *strictly* less than k by Proposition 2.1.

Next, ϕ is injective: if $W_1,\ W_2 \in (k-1, k-2)$ and $W_1 + \sigma W_1 = W_2 + \sigma W_2 = W' \in (k, k-1)$, then $W' \cap \sigma^{-1} W' = W_1 = W_2$.

Finally, ϕ is surjective: if $W' \in (k, k-1)$, then $W' \cap \sigma^{-1} W' \in (k-1, k-2)$ since $(W' \cap \sigma^{-1} W') + \sigma(W' \cap \sigma^{-1} W') \subseteq W'$; in fact $(W' \cap \sigma^{-1} W') + \sigma(W' \cap \sigma^{-1} W') = W'$. ☐

5. A $q = 1$ analogue

In light of our results, we might ask what (1) counts when we set $q = 1$. In this section we will see that the situation translates from enumerating subspaces of vector spaces to enumerating subsets of sets.

Instead of subspaces of $\mathbb{F}_{q^N}$, we consider subsets of $\{1, \ldots, N\}$. Rather than multiplying by the element σ, we let σ cyclically permute the elements of $\{1, \ldots, N\}$, so that σ preserves no proper subset, in analogy with Proposition 2.1. Conversely, it is easy to see that any permutation of the set $\{1, \ldots, N\}$ that preserves no proper subset is cyclic, and we can reorder the elements so that $\sigma = (12 \cdots N)$ in cycle notation. For example, we have $\sigma\{1, 3, 4\} = \{2, 4, 5\}$ for $N \geq 5$. When $N = mn$, the number of m-element subsets W of $\{1, \ldots, N\}$ such that $\bigcup_{i=0}^{n-1} \sigma^i W = \{1, \ldots, N\}$ is easily seen to be n. Proposition 5.1 will show this in a slightly more general setting. We retain the $[\ ,\ ], (\ ,\ ), <\ ,\ >$ notation as before with the definitions restated in the setting of subsets of $\{1, \ldots, N\}$ below.

Definition. Suppose $A_1, A_2, \ldots, A_k$ are sets of subsets of $\{1, \ldots, N\}$. Let $[A_1, A_2, \ldots, A_k]$ be the set of all k-tuples $(W_1, W_2, \ldots, W_k)$ such that

$$\begin{aligned} W_i \in A_i \quad &\text{for} \quad 1 \leq i \leq k, \\ W_i \supseteq W_{i+1} \cup \sigma W_{i+1} \quad &\text{for} \quad 1 \leq i \leq k-1. \end{aligned}$$

If A_i is the set of all subsets of $\{1, \ldots, N\}$ with cardinality d_i, then A_i is denoted within the brackets as d_i.

Definition. For nonnegative integers a, b with $N > a > b$ or $a = b = 0$

$$(a, b) := \left\{ W \subseteq F_{q^N} : |W| = a, |W \cap \sigma^{-1} W| = b \right\}.$$

Definition. Given sets $[A_{1,1}, A_{1,2}], [A_{2,1}, A_{2,2}], \ldots, [A_{r,1}, A_{r,2}]$ as defined above, let

$$\langle [A_{1,1}, A_{1,2}], [A_{2,1}, A_{2,2}], \ldots, [A_{r,1}, A_{r,2}] \rangle$$

denote the set of $2r$-tuples of subsets $(W_{1,1}, W_{1,2}, W_{2,1}, W_{2,2}, \ldots W_{r,1}, W_{r,2})$ such that

$$\begin{aligned} (W_{i,1}, W_{i,2}) \in [A_{i,1}, A_{i,2}] \quad &\text{for} \quad 1 \leq i \leq r, \\ W_{i,2} \supseteq W_{i+1,1} \quad &\text{for} \quad 1 \leq i \leq r-1. \end{aligned}$$

It can be checked that Lemma 2.3 is still valid in this $q = 1$ setting, so our formulas are still valid by just plugging in $q = 1$. This occurs since the q-binomials counting ways to extend subspaces become binomial terms counting ways to enlarge subsets. However, we can directly count some special cases and check that they agree with the general formula.

Proposition 5.1. *(Analogue of the Splitting Subspace Conjecture) For $q = 1$ and $N \geq mn$ we have*

$$\begin{aligned} &|[((n-1)m, (n-2)m), ((n-2)m, (n-3)m), \ldots, (2m, m), (m, 0)]| \\ &= \frac{N}{m} \binom{N - mn + m - 1}{m - 1}. \end{aligned}$$

In particular, if $N = mn$, then

$$[((n-1)m, (n-2)m), ((n-2)m, (n-3)m), \ldots, (2m, m), (m, 0)] = n.$$

Proof. The same argument used in Proposition 2.4 can be used to show that if

$$(W_{n-1},\dots,W_1)\in[((n-1)m,(n-2)m),((n-2)m,(n-3)m),\dots,(2m,m),(m,0)],$$

then $W_{i+1}=W_i\cup\sigma W_i$. Therefore, the problem of counting this set reduces to finding the number of m-element subsets W such that $W\cup\sigma W\cdots\cup\sigma^{n-1}W$ has cardinality mn.

We count the number of ordered pairs (W,k), where W is an m-element subset satisfying the desired property and k is an element of W. First, we fix k and count the number of sets W that contain k. Without loss of generality, suppose $k=N-n+1$. Since W cannot have any element between $N-2n+2$ and $N-n$ inclusive, choosing the rest of W amounts to choosing $m-1$ elements from $\{1,\dots,N-2n+1\}$ such that the elements are at least n apart.

By an elementary counting argument, there is a bijection between choosing $m-1$ elements from $\{1,\dots,N-2n+1\}$ such that the elements are at least n apart and choosing $m-1$ elements from an $N-2n+1-(m-2)(n-1)=N-mn+m-1$ element set. This is exactly $\binom{N-mn+m-1}{m-1}$. Therefore, the number of ordered pairs (W,k) is $N\binom{N-mn+m-1}{m-1}$, since there are N choices of k.

We can also count the number of ordered pairs (W,k) by choosing W first and then k. Since there are m possibilities once we fix W, each W appears in m ordered pairs. So as desired

$$\begin{aligned}&|[((n-1)m,(n-2)m),((n-2)m,(n-3)m),\dots,(2m,m),(m,0)]|\\&=\frac{N}{m}\binom{N-mn+m-1}{m-1}.\end{aligned}$$

In particular, if $N=mn$ then $\frac{mn}{m}\binom{mn-mn+m-1}{m-1}=n$. □

Remark. As $q\to 1$, the formula in Proposition 5.1 agrees with the formula in Corollary 3.4.

Proposition 5.2. *If $q=1$, then* $|(m,k)|=\frac{N}{N-m}\binom{N-m}{m-k}\binom{m-1}{k}=\frac{N}{m}\binom{N-m-1}{m-k-1}\binom{m}{k}$.

Proof. We count in two ways the ordered pairs (W,a) where W is an m-element set such that $|W\cap\sigma W|=k$ and $a\notin W$. First, fix a and count the number of possible W not containing a. Without loss of generality, suppose $a=N$.

Then, define a *block* of $W\subset\{1,\dots,N-1\}$ to be a subset $\{b,b+1,\dots,b+\ell-1\}$ of consecutive numbers contained in W such that $b-1,b+\ell\notin W$ (if $b=1$, then $b-1$ is understood to be N, which is already fixed to not be in W). We know that the sum of the sizes of the blocks of W is m, since the union of the blocks is m. Also, if $B_1,\dots,B_i$ are the blocks of W, then we know that $(|B_1|-1)+\cdots+(|B_i|-1)=k$, since the intersection of W with σW is precisely the disjoint union of all the blocks of W without the first element of each block. In particular, the number of blocks must be $m-k$.

Therefore, to count the number of possibilities of W, we count the number of ways to space out the blocks of W: the number of ways to choose $m-k$ elements out of $(N-1)-k$ consecutive numbers such that no two elements are adjacent. This is equivalent to the number of ways to choose $m-k$ elements out of an $N-1-k-(m-k-1)=N-m$ element set.

Now that we have fixed the spacing of the blocks, the number of ways to distributing the remaining k elements of W into the $m-k$ blocks is $\binom{(m-k-1)+k}{k}$. Therefore, the number of possible ordered pairs (W,a) is $N\binom{N-m}{m-k}\binom{m-1}{k}$, since there are N choices for the initial value of a.

We can also count the number of ordered pairs (W,a) by fixing W and then finding the number of possibilities for a. For a fixed W there are $N-m$ possible a. This means that $|(m,k)| = \frac{N}{N-m}\binom{N-m}{m-k}\binom{m-1}{k} = \frac{N}{m-k}\binom{N-m-1}{m-k-1}\binom{m-1}{k} = \frac{N}{m}\binom{N-m-1}{m-k-1}\binom{m}{k}$. □

Remark. For q a power of a prime, (1) shows that

$$|(m,k)| = \frac{\begin{bmatrix}N\\1\end{bmatrix}_q}{\begin{bmatrix}m\\1\end{bmatrix}_q}\begin{bmatrix}N-m-1\\m-k-1\end{bmatrix}_q\begin{bmatrix}m\\k\end{bmatrix}_q q^{(m-k)(m-k-1)}.$$

This gives the same answer as Proposition 5.2 when $q\to 1$.

Appendix A. Proof of Lemmas 3.1 and 3.2

Proof of Lemma 3.1. First substitute $B-1-s$ for s to find that the identity is equivalent to

$$\sum_{s=0}^{B-1-C}\begin{bmatrix}A-B-1\\s\end{bmatrix}_q\begin{bmatrix}B\\B-1-s\end{bmatrix}_q\begin{bmatrix}B-1-s\\C\end{bmatrix}_q\begin{bmatrix}A-B-1-s\\D-B-1-s\end{bmatrix}_q q^{s(s+1)}$$
$$=\frac{[B]_q}{[D-C]_q}\begin{bmatrix}B-1\\C\end{bmatrix}_q\begin{bmatrix}A-B-1\\D-B-1\end{bmatrix}_q\begin{bmatrix}D-C\\B-C\end{bmatrix}_q.$$

Then

$$\begin{aligned}&\begin{bmatrix}A-B-1\\s\end{bmatrix}_q\\&=\frac{(1-q^{A-B-1})\cdots(1-q^{A-B-s})}{(1-q)\cdots(1-q^s)}\\&=(-1)^s q^{(A-B-1)s-\binom{s}{2}}\frac{(q^{B+1-A};q)_s}{(q;q)_s},\end{aligned}$$

and

$$\begin{aligned}&\begin{bmatrix}B\\B-1-s\end{bmatrix}_q\begin{bmatrix}B-1-s\\C\end{bmatrix}_q\\&=\frac{(q;q)_B(q;q)_{B-1-s}}{(q;q)_{1+s}(q;q)_{B-1-s}(q;q)_C(q;q)_{B-C-1-s}}\\&=\frac{(q;q)_B}{(1-q)(q^2;q)_s(q;q)_C(q;q)_{B-C-1-s}}\\&=\frac{1}{(1-q)(q^2;q)_s}(1-q^B)\frac{(1-q^{B-1})\cdots(1-q^{B-C-s})}{(q;q)_C}\\&=\frac{1}{(1-q)(q^2;q)_s}(1-q^B)\begin{bmatrix}B-1\\C\end{bmatrix}_q(1-q^{B-C-1})\cdots(1-q^{B-C-s})\end{aligned}$$

$$= \frac{1}{(1-q)(q^2;q)_s}(1-q^B)\begin{bmatrix}B-1\\C\end{bmatrix}_q(-1)^s q^{(B-C-1)s-\binom{s}{2}}(q^{C+1-B};q)_s.$$

Finally

$$\begin{aligned}
&\begin{bmatrix}A-B-1-s\\D-B-1-s\end{bmatrix}_q\\
&=\begin{bmatrix}A-B-1\\D-B-1\end{bmatrix}_q\frac{\begin{bmatrix}A-B-1-s\\D-B-1-s\end{bmatrix}_q}{\begin{bmatrix}A-B-1\\D-B-1\end{bmatrix}_q}\\
&=\begin{bmatrix}A-B-1\\D-B-1\end{bmatrix}_q\frac{\frac{(q;q)_{A-B-1-s}}{(q;q)_{D-B-1-s}(q;q)_{A-D}}}{\frac{(q;q)_{A-B-1}}{(q;q)_{D-B-1}(q;q)_{A-D}}}\\
&=\begin{bmatrix}A-B-1\\D-B-1\end{bmatrix}_q\frac{(1-q^{D-B-1})\cdots(1-q^{D-B-s})}{(1-q^{A-B-1})\cdots(1-q^{A-B-s})}\\
&=\begin{bmatrix}A-B-1\\D-B-1\end{bmatrix}_q\frac{(-1)^s q^{(D-B-1)s-\binom{s}{2}}(q^{B+1-D};q)_s}{(-1)^s q^{(A-B-1)s-\binom{s}{2}}(q^{B+1-A};q)_s}.
\end{aligned}$$

Combining these terms

$$\begin{aligned}
&\begin{bmatrix}A-B-1\\s\end{bmatrix}_q\begin{bmatrix}B\\B-1-s\end{bmatrix}_q\begin{bmatrix}B-1-s\\C\end{bmatrix}_q\begin{bmatrix}A-B-1-s\\D-B-1-s\end{bmatrix}_q q^{s(s+1)}\\
&=\frac{1-q^B}{1-q}\begin{bmatrix}B-1\\C\end{bmatrix}_q\begin{bmatrix}A-B-1\\D-B-1\end{bmatrix}_q q^{(D-C)s}\frac{(q^{B+1-A};q)_s(q^{C+1-B};q)_s(q^{B+1-D};q)_s}{(q;q)_s(q^2;q)_s(q^{B+1-A};q)_s}\\
&=\frac{1-q^B}{1-q}\begin{bmatrix}B-1\\C\end{bmatrix}_q\begin{bmatrix}A-B-1\\D-B-1\end{bmatrix}_q q^{(D-C)s}\frac{(q^{C+1-B};q)_s(q^{B+1-D};q)_s}{(q;q)_s(q^2;q)_s}.
\end{aligned}$$

The power of q is $(D-C)s$ since

$$\begin{aligned}
&(A-B-1)s-\binom{s}{2}+(B-C-1)s-\binom{s}{2}\\
&+(D-B-1)s-\binom{s}{2}-((A-B-1)s-\binom{s}{2})+s(s+1)\\
&=(D-C)s.
\end{aligned}$$

This means that

$$\begin{aligned}
&\sum_{s=0}^{B-1-C}\begin{bmatrix}A-B-1\\s\end{bmatrix}_q\begin{bmatrix}B\\B-1-s\end{bmatrix}_q\begin{bmatrix}B-1-s\\C\end{bmatrix}_q\begin{bmatrix}A-B-1-s\\D-B-1-s\end{bmatrix}_q q^{s(s+1)}\\
&=\frac{1-q^B}{1-q}\begin{bmatrix}B-1\\C\end{bmatrix}_q\begin{bmatrix}A-B-1\\D-B-1\end{bmatrix}_q\sum_{s=0}^{B-1-C}\frac{(q^{C+1-B};q)_s(q^{B+1-D};q)_s}{(q;q)_s(q^2;q)_s}q^{(D-C)s}\\
&=\begin{bmatrix}B\\1\end{bmatrix}_q\begin{bmatrix}B-1\\C\end{bmatrix}_q\begin{bmatrix}A-B-1\\D-B-1\end{bmatrix}_q\,{}_2\phi_1\left(\begin{matrix}q^{C+1-B}, & q^{B+1-D}; & q^{D-C}\\ & q^2 & \end{matrix}\right)
\end{aligned}$$

$$= \begin{bmatrix} B \\ 1 \end{bmatrix}_q \begin{bmatrix} B-1 \\ C \end{bmatrix}_q \begin{bmatrix} A-B-1 \\ D-B-1 \end{bmatrix}_q \frac{(q^{D+1-B};q)_{B-C-1}}{(q^2;q)_{B-C-1}}$$
$$= \begin{bmatrix} B \\ 1 \end{bmatrix}_q \begin{bmatrix} B-1 \\ C \end{bmatrix}_q \begin{bmatrix} A-B-1 \\ D-B-1 \end{bmatrix}_q \frac{(1-q^{D+1-B})\cdots(1-q^{D-C-1})}{(1-q^2)\cdots(1-q^{B-C})}$$
$$= \begin{bmatrix} B \\ 1 \end{bmatrix}_q \begin{bmatrix} B-1 \\ C \end{bmatrix}_q \begin{bmatrix} A-B-1 \\ D-B-1 \end{bmatrix}_q \begin{bmatrix} D-C \\ B-C \end{bmatrix}_q \frac{1-q}{1-q^{D-C}}$$
$$= \frac{1-q^B}{1-q^{D-C}} \begin{bmatrix} B-1 \\ C \end{bmatrix}_q \begin{bmatrix} A-B-1 \\ D-B-1 \end{bmatrix}_q \begin{bmatrix} D-C \\ B-C \end{bmatrix}_q$$

as desired. □

Proof of Lemma 3.2. The proof is identical to that of Lemma 3.1. Substitute $B-1-s$ for s to find that the lemma is equivalent to

$$\sum_{s=0}^{B-1-D} \begin{bmatrix} A-B-1 \\ s \end{bmatrix}_q \begin{bmatrix} B \\ B-1-s \end{bmatrix}_q \begin{bmatrix} B-1-s \\ C \end{bmatrix}_q \begin{bmatrix} B-1-C-s \\ D-C \end{bmatrix}_q q^{s(s+1)}$$
$$= \frac{[B]_q}{[A-D]_q} \begin{bmatrix} B-1 \\ C \end{bmatrix}_q \begin{bmatrix} B-C-1 \\ D-C \end{bmatrix}_q \begin{bmatrix} A-D \\ B-D \end{bmatrix}_q.$$

We see that

$$\begin{bmatrix} B-1-C-s \\ D-C \end{bmatrix}_q$$
$$= \begin{bmatrix} B-1-C \\ D-C \end{bmatrix}_q \frac{\begin{bmatrix} B-1-C-s \\ D-C \end{bmatrix}_q}{\begin{bmatrix} B-1-C \\ D-C \end{bmatrix}_q}$$
$$= \frac{\frac{(q;q)_{B-1-C-s}}{(q;q)_{D-C}(q;q)_{B-1-D-s}}}{\frac{(q;q)_{B-1-C}}{(q;q)_{D-C}(q;q)_{B-D-1}}}$$
$$= \begin{bmatrix} B-1-C \\ D-C \end{bmatrix}_q \frac{(1-q^{B-D-1})\cdots(1-q^{B-D-s})}{(1-q^{B-C-1})\cdots(1-q^{B-C-s})}$$
$$= \begin{bmatrix} B-1-C \\ D-C \end{bmatrix}_q \frac{q^{(B-D-1)s-\binom{s}{2}}(q^{D+1-B};q)_s}{q^{(B-C-1)s-\binom{s}{2}}(q^{C+1-B};q)_s}.$$

Combining this with the expressions for $\begin{bmatrix} A-B-1 \\ s \end{bmatrix}_q$ and $\begin{bmatrix} B \\ B-1-s \end{bmatrix}_q \begin{bmatrix} B-1-s \\ C \end{bmatrix}_q$ from the proof of Lemma 3.1

$$\begin{bmatrix} A-B-1 \\ s \end{bmatrix}_q \begin{bmatrix} B \\ B-1-s \end{bmatrix}_q \begin{bmatrix} B-1-s \\ C \end{bmatrix}_q \begin{bmatrix} B-1-C-s \\ D-C \end{bmatrix}_q q^{s(s+1)}$$
$$= \frac{1-q^B}{1-q} \begin{bmatrix} B-1 \\ C \end{bmatrix}_q \begin{bmatrix} B-1-C \\ D-C \end{bmatrix}_q q^{(A-D)s} \frac{(q^{B+1-A};q)_s(q^{C+1-B};q)_s(q^{D+1-B};q)_s}{(q;q)_s(q^2;q)_s(q^{C+1-B};q)_s}$$
$$= \frac{1-q^B}{1-q} \begin{bmatrix} B-1 \\ C \end{bmatrix}_q \begin{bmatrix} B-1-C \\ D-C \end{bmatrix}_q q^{(A-D)s} \frac{(q^{B+1-A};q)_s(q^{D+1-B};q)_s}{(q;q)_s(q^2;q)_s}.$$

The power of q is $(A-D)s$ because

$$\begin{aligned}
&(A-B-1)s - \binom{s}{2} + (B-C-1)s - \binom{s}{2} \\
&+ (D+1-B)s - \binom{s}{2} - ((B-C-1)s - \binom{s}{2}) + s(s+1) \\
&= (A-D)s.
\end{aligned}$$

This means that

$$\begin{aligned}
&\sum_{s=0}^{B-1-D} \begin{bmatrix} A-B-1 \\ s \end{bmatrix}_q \begin{bmatrix} B \\ B-1-s \end{bmatrix}_q \begin{bmatrix} B-1-s \\ C \end{bmatrix}_q \begin{bmatrix} B-1-C-s \\ D-C \end{bmatrix}_q q^{s(s+1)} \\
&= \frac{1-q^B}{1-q} \begin{bmatrix} B-1 \\ C \end{bmatrix}_q \begin{bmatrix} B-1-C \\ D-C \end{bmatrix}_q \sum_{s=0}^{B-1-D} \frac{(q^{B+1-A};q)_s (q^{D+1-B};q)_s}{(q;q)_s (q^2;q)_s} q^{(A-D)s} \\
&= \frac{1-q^B}{1-q} \begin{bmatrix} B-1 \\ C \end{bmatrix}_q \begin{bmatrix} B-1-C \\ D-C \end{bmatrix}_q {}_2\phi_1 \left(\begin{matrix} q^{B+1-A}, \quad q^{D+1-B}; \quad q^{A-D} \\ q^2 \end{matrix} \right) \\
&= \frac{1-q^B}{1-q} \begin{bmatrix} B-1 \\ C \end{bmatrix}_q \begin{bmatrix} B-1-C \\ D-C \end{bmatrix}_q \frac{(q^{B+1-D};q)_{A-B-1}}{(q^2;q)_{A-B-1}} \\
&= \frac{1-q^B}{1-q} \begin{bmatrix} B-1 \\ C \end{bmatrix}_q \begin{bmatrix} B-1-C \\ D-C \end{bmatrix}_q \frac{(1-q^{B+1-D}) \cdots (1-q^{A-D-1})}{(1-q^2) \cdots (1-q^{A-B})} \\
&= \frac{1-q^B}{1-q} \begin{bmatrix} B-1 \\ C \end{bmatrix}_q \begin{bmatrix} B-1-C \\ D-C \end{bmatrix}_q \begin{bmatrix} A-D \\ A-B \end{bmatrix}_q \frac{1-q}{1-q^{A-D}} \\
&= \frac{1-q^B}{1-q^{A-D}} \begin{bmatrix} B-1 \\ C \end{bmatrix}_q \begin{bmatrix} B-1-C \\ D-C \end{bmatrix}_q \begin{bmatrix} A-D \\ B-D \end{bmatrix}_q
\end{aligned}$$

as desired.

□

Appendix B. Proof of Theorem 3.3, L=R

Proof.

$$L = \frac{\begin{bmatrix} N \\ 1 \end{bmatrix}_q}{\begin{bmatrix} a_{1,2} \\ 1 \end{bmatrix}_q} \prod_{i=1}^{r} \frac{[a_{i,2}]_q}{[a_{i,1}-a_{i+1,2}]_q} \begin{bmatrix} a_{i,2}-1 \\ a_{i+1,2} \end{bmatrix}_q \begin{bmatrix} a_{i-1,2}-a_{i,2}-1 \\ a_{i,1}-a_{i,2}-1 \end{bmatrix}_q \begin{bmatrix} a_{i,1}-a_{i+1,2} \\ a_{i,2}-a_{i+1,2} \end{bmatrix}_q \prod_{i=1}^{r-1} \begin{bmatrix} a_{i,2}-1 \\ a_{i+1,2}-1 \end{bmatrix}_q^{-1}$$

$$= \frac{\frac{[N]!_q}{[N-1]!_q[1]!_q}}{\frac{[a_{1,2}]!_q}{[a_{1,2}-1]!_q[1]!_q}} \frac{\frac{[a_{r,2}]!_q}{[a_{r,2}-1]!_q[1]!_q}}{\frac{[a_{r,1}]!_q}{[a_{r,1}-1]!_q[1]!_q}} \begin{bmatrix} a_{r-1,2}-a_{r,2}-1 \\ a_{r,1}-a_{r,2}-1 \end{bmatrix}_q \frac{[a_{r,1}]!_q}{[a_{r,2}]!_q[a_{r,1}-a_{r,2}]!_q}$$

$$\prod_{i=1}^{r-1} \frac{\frac{[a_{i,2}]!_q}{[a_{i,2}-1]!_q[1]!_q}}{\frac{[a_{i,1}-a_{i+1,2}]!_q}{[a_{i,1}-a_{i+1,2}-1]!_q[1]!_q}} \frac{[a_{i,2}-1]!_q}{[a_{i+1,2}]!_q[a_{i,2}-a_{i+1,2}-1]!_q} \begin{bmatrix} a_{i-1,2}-a_{i,2}-1 \\ a_{i,1}-a_{i,2}-1 \end{bmatrix}_q \frac{\frac{[a_{i,1}-a_{i+1,2}]!_q}{[a_{i,2}-a_{i+1,2}]!_q[a_{i,1}-a_{i,2}]!_q}}{\frac{[a_{i,2}-1]!_q}{[a_{i+1,2}-1]!_q[a_{i,2}-a_{i+1,2}]!_q}}$$

$$= \frac{\frac{[N]!_q}{[N-1]!_q}}{\frac{[a_{1,2}]!_q}{[a_{1,2}-1]!_q}} \frac{1}{[a_{r,2}-1]!_q}[a_{r,1}-1]!_q \begin{bmatrix} a_{r-1,2}-a_{r,2}-1 \\ a_{r,1}-a_{r,2}-1 \end{bmatrix}_q \frac{1}{[a_{r,1}-a_{r,2}]!_q}$$

$$\prod_{i=1}^{r-1} \frac{[a_{i,2}]!_q}{[a_{i,2}-1]!_q} \frac{[a_{i,1}-a_{i+1,2}-1]!_q}{[a_{i+1,2}]!_q[a_{i,2}-a_{i+1,2}-1]!_q} \frac{[a_{i-1,2}-a_{i,2}-1]!_q}{[a_{i,1}-a_{i,2}-1]!_q[a_{i-1,2}-a_{i,1}]!_q} \frac{[a_{i+1,2}-1]!_q}{[a_{i,1}-a_{i,2}]!_q}$$

$$= \frac{\frac{[N]!_q}{[N-1]!_q}}{\frac{[a_{1,2}]!_q}{[a_{1,2}-1]!_q}} \frac{1}{[a_{r,2}-1]!_q}[a_{r,1}-1]!_q \begin{bmatrix} a_{r-1,2}-a_{r,2}-1 \\ a_{r,1}-a_{r,2}-1 \end{bmatrix}_q \frac{1}{[a_{r,1}-a_{r,2}]!_q}$$

$$= \prod_{i=1}^{r-1} \frac{[a_{i,2}]!_q}{[a_{i+1,2}]!_q} \frac{[a_{i+1,2}-1]!_q}{[a_{i,2}-1]!_q} \frac{[a_{i-1,2}-a_{i,2}-1]!_q}{[a_{i,2}-a_{i+1,2}-1]!_q} \frac{[a_{i,1}-a_{i+1,2}-1]!_q}{[a_{i,1}-a_{i,2}-1]!_q[a_{i-1,2}-a_{i,1}]!_q[a_{i,1}-a_{i,2}]!_q}$$

$$= \frac{\frac{[N]!_q}{[N-1]!_q}}{\frac{[a_{1,2}]!_q}{[a_{1,2}-1]!_q}} \frac{1}{[a_{r,2}-1]!_q}[a_{r,1}-1]!_q \frac{[a_{r-1,2}-a_{r,2}-1]!_q}{[a_{r,1}-a_{r,2}-1]!_q[a_{r-1,2}-a_{r,1}]!_q} \frac{1}{[a_{r,1}-a_{r,2}]!_q}$$

$$\frac{[a_{1,2}]!_q}{[a_{r,2}]!_q} \frac{[a_{r,2}-1]!_q}{[a_{1,2}-1]!_q} \frac{[N-a_{1,2}-1]!_q}{[a_{r-1,2}-a_{r,2}-1]!_q} \prod_{i=1}^{r-1} \frac{[a_{i,1}-a_{i+1,2}-1]!_q}{[a_{i,1}-a_{i,2}-1]!_q[a_{i-1,2}-a_{i,1}]!_q[a_{i,1}-a_{i,2}]!_q}$$

$$= \frac{(1-q^N)[N-a_{1,2}-1]!_q}{[a_{r,2}]!_q} \prod_{i=1}^{r} \frac{[a_{i,1}-a_{i+1,2}-1]!_q}{[a_{i,1}-a_{i,2}-1]!_q[a_{i-1,2}-a_{i,1}]!_q[a_{i,1}-a_{i,2}]!_q}$$

$$= \frac{(1-q^N)[N-a_{1,2}-1]!_q}{[N-a_{1,1}]!_q[a_{r,2}]!_q} \prod_{i=1}^{r} \frac{[a_{i,1}-a_{i+1,2}-1]!_q}{[a_{i,1}-a_{i,2}-1]!_q[a_{i,1}-a_{i,2}]!_q} \prod_{i=2}^{r} \frac{1}{[a_{i-1,2}-a_{i,1}]!_q}.$$

$$R = \frac{\begin{bmatrix} N \\ 1 \end{bmatrix}_q}{\begin{bmatrix} a_{1,1} \\ 1 \end{bmatrix}_q} \prod_{i=1}^{r} \frac{[a_{i,1}]_q}{[a_{i-1,1}-a_{i,2}]_q} \begin{bmatrix} a_{i,1}-1 \\ a_{i+1,1} \end{bmatrix}_q \begin{bmatrix} a_{i,1}-a_{i+1,1}-1 \\ a_{i,2}-a_{i+1,1} \end{bmatrix}_q \begin{bmatrix} a_{i-1,1}-a_{i,2} \\ a_{i,1}-a_{i,2} \end{bmatrix}_q \prod_{i=1}^{r-1} \begin{bmatrix} a_{i,1}-1 \\ a_{i+1,1}-1 \end{bmatrix}_q^{-1}$$

$$= \frac{\frac{[N]!_q}{[N-1]!_q[1]!_q}}{\frac{[a_{1,1}]!_q}{[a_{1,1}-1]!_q[1]!_q}} \frac{\frac{[a_{r,1}]!_q}{[a_{r,1}-1]!_q[1]!_q}}{\frac{[a_{r-1,1}-a_{r,2}]!_q}{[a_{r-1,1}-a_{r,2}-1]!_q[1]!_q}} \frac{[a_{r,1}-1]!_q}{[a_{r,2}]!_q[a_{r,1}-a_{r,2}-1]!_q} \frac{[a_{r-1,1}-a_{r,2}]!_q}{[a_{r,1}-a_{r,2}]!_q[a_{r-1,1}-a_{r,1}]!_q}$$

$$\prod_{i=1}^{r-1} \frac{\frac{[a_{i,1}]!_q}{[a_{i,1}-1]!_q[1]!_q}}{\frac{[a_{i-1,1}-a_{i,2}]!_q}{[a_{i-1,1}-a_{i,2}-1]!_q[1]!_q}} \frac{[a_{i,1}-1]!_q}{[a_{i+1,1}]!_q[a_{i,1}-a_{i+1,1}-1]!_q} \frac{[a_{i,1}-a_{i+1,1}-1]!_q}{[a_{i,2}-a_{i+1,1}]!_q[a_{i,1}-a_{i,2}-1]!_q} \frac{\begin{bmatrix} a_{i-1,1}-a_{i,2} \\ a_{i,1}-a_{i,2} \end{bmatrix}_q}{\begin{bmatrix} a_{i,1}-1 \\ a_{i+1,1}-1 \end{bmatrix}_q}$$

$$= \frac{[N]!_q}{[N-1]!_q} \frac{[a_{1,1}-1]!_q}{[a_{1,1}]!_q} \frac{[a_{r,1}]!_q}{[a_{r,2}]!_q} \frac{[a_{r-1,1}-a_{r,2}-1]!_q}{[a_{r,1}-a_{r,2}-1]!_q[a_{r,1}-a_{r,2}]!_q[a_{r-1,1}-a_{r,1}]!_q}$$

$$\prod_{i=1}^{r-1} \frac{[a_{i,1}]!_q}{[a_{i-1,1}-a_{i,2}]!_q} \frac{[a_{i-1,1}-a_{i,2}-1]!_q}{[a_{i+1,1}]!_q} \frac{1}{[a_{i,2}-a_{i+1,1}]!_q[a_{i,1}-a_{i,2}-1]!_q} \frac{\frac{[a_{i-1,1}-a_{i,2}]!_q}{[a_{i,1}-a_{i,2}]!_q[a_{i-1,1}-a_{i,1}]!_q}}{\frac{[a_{i,1}-1]!_q}{[a_{i+1,1}-1]!_q[a_{i,1}-a_{i+1,1}]!_q}}$$

$$= \frac{[N]!_q}{[N-1]!_q} \frac{[a_{1,1}-1]!_q}{[a_{1,1}]!_q} \frac{[a_{r,1}]!_q}{[a_{r,2}]!_q} \frac{[a_{r-1,1}-a_{r,2}-1]!_q}{[a_{r,1}-a_{r,2}-1]!_q[a_{r,1}-a_{r,2}]!_q[a_{r-1,1}-a_{r,1}]!_q}$$

$$\prod_{i=1}^{r-1} \frac{[a_{i,1}]!_q}{[a_{i+1,1}]!_q} \frac{[a_{i+1,1}-1]!_q}{[a_{i,1}-1]!_q} \frac{[a_{i,1}-a_{i+1,1}]!_q}{[a_{i-1,1}-a_{i,1}]!_q} \frac{[a_{i-1,1}-a_{i,2}-1]!_q}{[a_{i,2}-a_{i+1,1}]!_q[a_{i,1}-a_{i,2}-1]!_q[a_{i,1}-a_{i,2}]!_q}$$

$$= \frac{[N]!_q}{[N-1]!_q}\frac{[a_{1,1}-1]!_q}{[a_{1,1}]!_q}\frac{[a_{r,1}]!_q}{[a_{r,2}]!_q}\frac{[a_{r-1,1}-a_{r,2}-1]!_q}{[a_{r,1}-a_{r,2}-1]!_q[a_{r,1}-a_{r,2}]!_q[a_{r-1,1}-a_{r,1}]!_q}$$

$$\frac{[a_{1,1}]!_q}{[a_{r,1}]!_q}\frac{[a_{r,1}-1]!_q}{[a_{1,1}-1]!_q}\frac{[a_{r-1,1}-a_{r,1}]!_q}{[N-a_{1,1}]!_q}\prod_{i=1}^{r-1}\frac{[a_{i-1,1}-a_{i,2}-1]!_q}{[a_{i,2}-a_{i+1,1}]!_q[a_{i,1}-a_{i,2}-1]!_q[a_{i,1}-a_{i,2}]!_q}$$

$$= \frac{[N]!_q}{[N-1]!_q}\frac{1}{[a_{r,2}]!_q}\frac{[a_{r-1,1}-a_{r,2}-1]!_q}{[a_{r,1}-a_{r,2}-1]!_q[a_{r,1}-a_{r,2}]!_q}$$

$$\frac{[a_{r,1}-1]!_q}{[N-a_{1,1}]!_q}\prod_{i=1}^{r-1}\frac{[a_{i-1,1}-a_{i,2}-1]!_q}{[a_{i,1}-a_{i,2}-1]!_q[a_{i,1}-a_{i,2}]!_q}\prod_{i=2}^{r}\frac{1}{[a_{i-1,2}-a_{i,1}]!_q}$$

$$= \frac{(1-q^N)}{[a_{r,2}]!_q}\frac{[a_{r,1}-1]!_q}{[N-a_{1,1}]!_q}\prod_{i=1}^{r}\frac{[a_{i-1,1}-a_{i,2}-1]!_q}{[a_{i,1}-a_{i,2}-1]!_q[a_{i,1}-a_{i,2}]!_q}\prod_{i=2}^{r}\frac{1}{[a_{i-1,2}-a_{i,1}]!_q}$$

$$= \frac{(1-q^N)[N-a_{1,2}-1]!_q}{[N-a_{1,1}]!_q[a_{r,2}]!_q}\prod_{i=1}^{r}\frac{[a_{i,1}-a_{i+1,2}-1]!_q}{[a_{i,1}-a_{i,2}-1]!_q[a_{i,1}-a_{i,2}]!_q}\prod_{i=2}^{r}\frac{1}{[a_{i-1,2}-a_{i,1}]!_q}.$$

□

Acknowledgements

This research was conducted at the 2012 summer REU (Research Experience for Undergraduates) program at the University of Minnesota, Twin Cities, and was supported by NSF grants DMS-1001933 and DMS-1148634. We would like to thank Profs. Gregg Musiker, Pavlo Pylyavskyy, Vic Reiner,and Dennis Stanton, who directed the program, for their support, and express particular gratitude to Prof. Stanton both for introducing us to this problem and for his indispensable guidance throughout the research process. We also thank Alex Miller for his assistance in editing this report.

References

[1] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Enc. of Math. and its Appl., Vol. 96, Cambridge University Press, Cambridge, 2004.

[2] S. R. Ghorpade and S. Ram, Enumeration of splitting subspaces over finite fields, Arithmetic, Geometry, and Coding Theory, (Luminy, France, March 2011), Y. Aubry, C. Ritzenthaler, and A. Zykin Eds., *Contemporary Mathematics*, Vol. 574, American Mathematical Society, Providence, RI, 2012, 10pp., to appear.

[3] H. Niederreiter, The multiple-recursive matrix method for pseudorandom number generation, *Finite Fields Appl.* **1** (1995), 3-30.

Eric Chen, Princeton University, Princeton, NJ 08544
E-mail address: ecchen@princeton.edu

Dennis Tseng, Massachusetts Institute of Technology, Cambridge, MA 02139
E-mail address: DennisCTseng@gmail.com