Extended Nestohedra and their Face Numbers

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Introduction

Nestohedra are a class of simple convex polytopes with deep combinatorial structures related to cluster algebras. We introduce a new family of convex polytopes that extend the nestohedra, motivated by Laurent phenomenon algebras introduced in [1]. These new polytopes, called **extended nestohedra**, also generalize the graph cubeahedra introduced in [2]. We obtain several results for the extended nestohedra, including a polytopal realization, isomorphisms with nestohedra, and formulas for the extended nestohedra's face numbers.

Background

A **building set** $\mathcal{B}$ on a set S is a collection of nonempty subsets of S such that $\{i\} \in \mathcal{B}$ for all $i \in S$, and if $I, J \in \mathcal{B}$, $I \cap J \neq \varnothing$, then $I \cup J \in \mathcal{B}$.
The **connected components** of $\mathcal{B}$ are the maximal elements of $\mathcal{B}$, denoted $\mathcal{B}_{\max}$. We say $\mathcal{B}$ is **connected** if $\mathcal{B}_{\max} = \{S\}$. If G is an undirected graph on vertex set S, then the **graphical building set** $\mathcal{B}_G$ is defined to be $\{I \subseteq S \mid G|_I \text{ is connected}\}$.
An **extended nested collection** on a building set $\mathcal{B}$ on S is a collection $N = \{I_1, \ldots, I_m, x_{i_1}, \ldots, x_{i_r}\}$ of elements $I_j \in \mathcal{B}$ and x_i for $i \in S$ satisfying the following three properties:

1. For $i \neq j$, either $I_i \subseteq I_j$, $I_j \subseteq I_i$, or $I_i \cap I_j = \varnothing$,
2. For any collection $I_{i_1}, \ldots, I_{i_k} \in N$ of $k \geq 2$ pairwise disjoint elements of N, their union $\bigcup_{\ell=1}^{k} I_{i_\ell}$ is not an element of $\mathcal{B}$, and
3. For all $x_{i_\ell}, I_j \in N$, the set I_j does not contain i_ℓ.

A **nested collection** is an extended nested collection with no x_i elements. The **nested complex** $\mathcal{N}(\mathcal{B})$ for a building set $\mathcal{B}$ is the simplicial complex with vertices $\{I \mid I \in \mathcal{B} \setminus \mathcal{B}_{\max}\}$ and faces given by nested collections $\{I_1, \ldots, I_r\}$. The **extended nested complex** $\mathcal{N}^\square(\mathcal{B})$ is the simplicial complex with vertices $\{I \mid I \in \mathcal{B}\} \cup \{x_i \mid i \in S\}$ and faces given by extended nested collections $\{I_1, \ldots, I_m\} \cup \{x_{i_1}, \ldots, x_{i_r}\}$.
The nested complex $\mathcal{N}(\mathcal{B})$ is known to be isomorphic to the boundary of a simplicial polytope. The dual of this polytope is called the **nestohedron** $\mathcal{P}(\mathcal{B})$.
We show that the extended nested complex $\mathcal{N}^\square(\mathcal{B})$ is also isomorphic to the boundary of a simplicial polytope, and call the dual of this polytope the **extended nestohedron** $\mathcal{P}^\square(\mathcal{B})$.

Polytopality

We give an explicit realization of $\mathcal{P}^\square(\mathcal{B})$ in terms of Minkowski sums.
Theorem: For a building set $\mathcal{B}$ on $[n] := \{1, \ldots, n\}$, the extended nestohedron $\mathcal{P}^\square(\mathcal{B})$ is isomorphic to the polytope

$$\mathcal{P} := \sum_{i \in [n]} (0, e_i) + \sum_{I \in \mathcal{B}} (\{e_S \mid S \subsetneq I\}),$$

where $e_1, \ldots, e_n$ are the standard basis vectors of $\mathbb{R}^n$, and $e_S = \sum_{i \in S} e_i$ for all $S \subseteq [n]$.

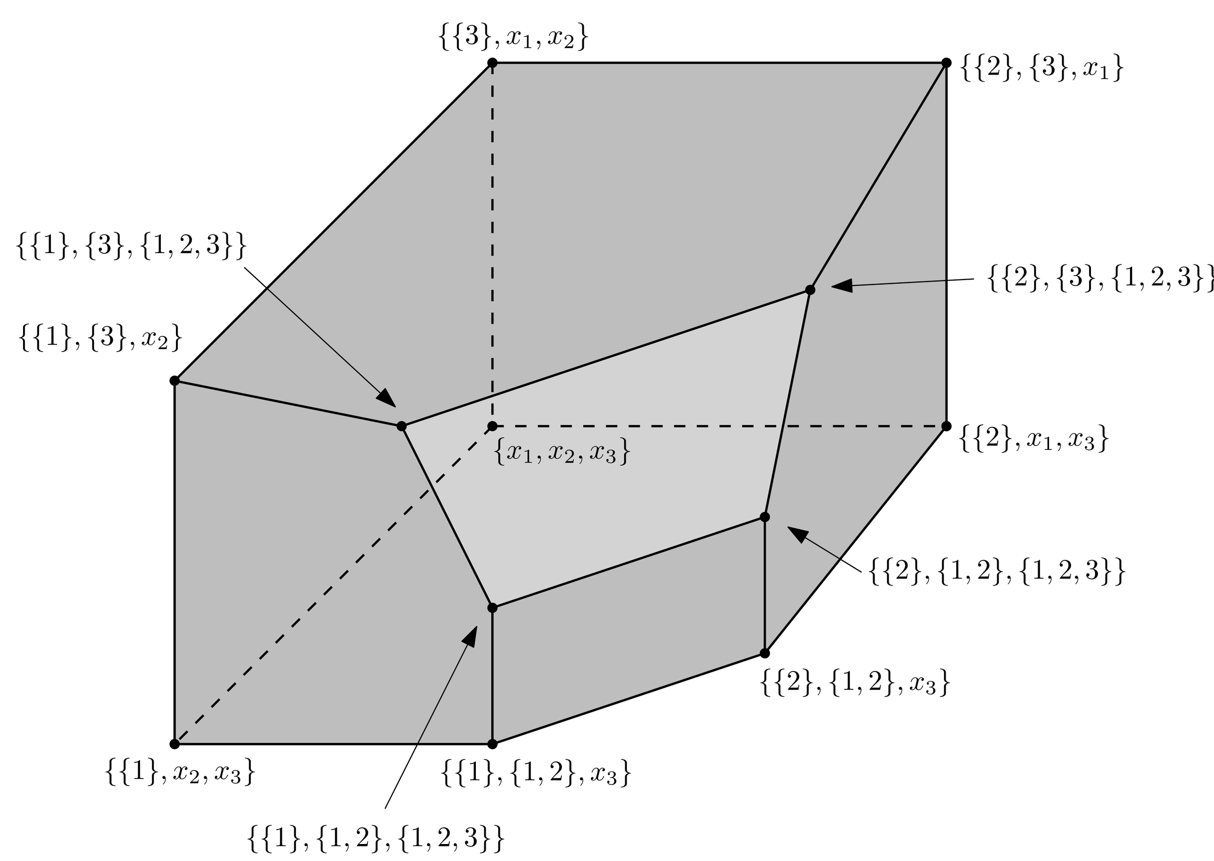

Figure 1: $\mathcal{P}^\square(\mathcal{B})$ for $\mathcal{B} = \{\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,2,3\}\}$.

One can also think of the realization as *shaving* faces corresponding to non-singleton elements of the building set from a n-cube whose faces correspond to singletons and x_i's.

Isomorphisms

We find isomorphisms of simplicial complexes of the forms

$$\mathcal{N}(\mathcal{B}) \simeq \mathcal{N}(\mathcal{B}'), \qquad \mathcal{N}^\square(\mathcal{B}) \simeq \mathcal{N}^\square(\mathcal{B}'),$$

and

$$\mathcal{N}^\square(\mathcal{B}) \simeq \mathcal{N}(\mathcal{B}'),$$

for connected building sets $\mathcal{B}$ and $\mathcal{B}'$. The first such isomorphisms are for connected building sets $\mathcal{B}$ on $[n]$ whose elements are all **intervals** of $[n]$.
Interval Isomorphism I: $\mathcal{N}^\square(\mathcal{B}) \simeq \mathcal{N}(\mathcal{B}')$, where $\mathcal{B}' = \mathcal{B} \cup \{\{n+1\}, \{n, n+1\}, \ldots, \{1, \ldots, n+1\}\}$.
The map sends $I \in \mathcal{B}$ to itself, and for every $i \in [n]$,

$$x_i \mapsto [i+1, n+1] := \{i+1, i+2, \ldots, n+1\}.$$

Interval Isomorphism II: If $[1, k] \in \mathcal{B}$ for all $k \in [n]$, then $\mathcal{N}(\mathcal{B}) \simeq \mathcal{N}(\mathcal{B}')$, where $\mathcal{B}'$ is the building set corresponding to the map

$$\begin{cases} [1,k] \mapsto [1, n-k] & \text{for } 1 \leq k < n, \\ [a,b] \mapsto [n+2-a, n+2-b] & \text{for } 1 < a \leq b \leq n. \end{cases}$$

Interval Isomorphism III: If $[1, k], [k, n] \in \mathcal{B}$ for all $k \in [n]$, then $\mathcal{N}^\square(\mathcal{B}) \simeq \mathcal{N}^\square(\mathcal{B}')$, where $\mathcal{B}'$ is the building set corresponding to the map

$$\begin{cases} x_k \mapsto [1, n+1-k] & \text{for } 1 \leq k \leq n, \\ [1,k] \mapsto x_{n+1-k} & \text{for } 1 \leq k \leq n, \\ [a,b] \mapsto [n+2-a, n+2-b] & \text{for } 1 < a \leq b \leq n. \end{cases}.$$

Next, we describe **spider** and **octopus building sets**, which are ways of gluing together different interval building sets. Given m interval building sets $\mathcal{B}_1, \ldots, \mathcal{B}_m$ on $[n_1], \ldots, [n_m]$ respectively, define the **spider building set** $\mathcal{B}_{\text{spi}}$ to be the building set on

$$\{v_{i,j} \mid i \in [m], j \in [n_i]\},$$

whose elements are either an element of some $\mathcal{B}_i$ under indentifying $v_{i,j} \leftrightarrow j$ (called a **leg set**), or an union of several such leg sets (called a **body set**).
Similarly, we define the **octopus building set** $\mathcal{B}_{\text{oct}}$ to be the building set on

$$\{*\} \cup \{v_{i,j} \mid i \in [m], j \in [n_i]\},$$

whose leg sets are the same as those of $\mathcal{B}_{\text{spi}}$, and body sets are unions of leg sets with the center $\{*\}$.

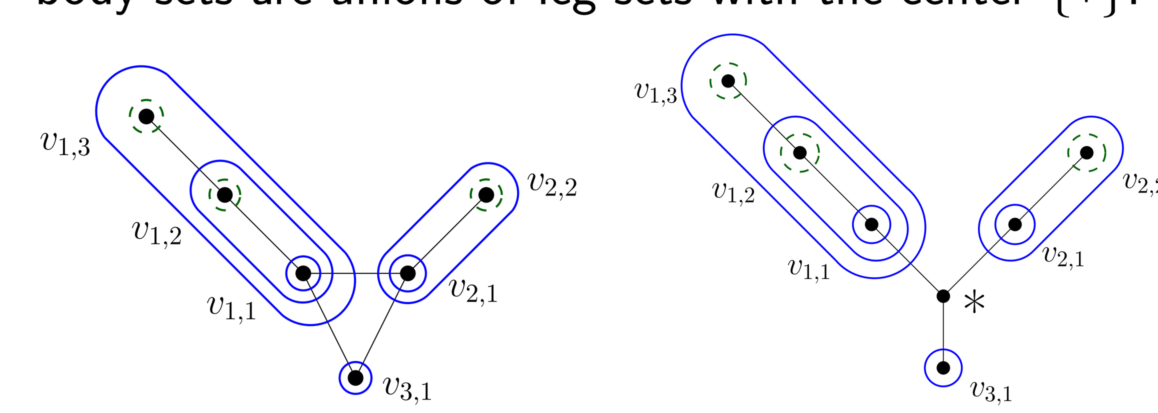

Figure 2: A spider with three legs and its corresponding octopus

The Interval Isomorphisms above now glue together to form the following isomorphisms:
Spider-Spider: $\mathcal{N}(\mathcal{B}_{\text{spi}}) \simeq \mathcal{N}(\mathcal{B}'_{\text{spi}})$
Spider-Octopus: $\mathcal{N}^\square(\mathcal{B}_{\text{spi}}) \simeq \mathcal{N}(\mathcal{B}'_{\text{oct}})$
Octopus-Octopus: $\mathcal{N}^\square(\mathcal{B}_{\text{oct}}) \simeq \mathcal{N}^\square(\mathcal{B}'_{\text{oct}})$
We show that the above isomorphisms between extended nested complexes are the only possible ones.
Theorem: If $\mathcal{N}^\square(\mathcal{B}) \simeq \mathcal{N}^\square(\mathcal{B}')$, then the isomorphism is an Octopus-Octopus isomorphism.
In the other two cases ($\mathcal{N}(\mathcal{B}) \simeq \mathcal{N}(\mathcal{B}')$ and $\mathcal{N}^\square(\mathcal{B}) \simeq \mathcal{N}(\mathcal{B}')$), there exist isomorphisms that are not Spider-Spider and Spider-Octopus isomorphisms.

Face Numbers

We find recursive formulas for the f-vector of an extended nestohedron in terms of the f-vectors of smaller nestohedra.
Theorem: If $\mathcal{B}$ is a building set on $[n]$, then

$$f_{\mathcal{P}^\square(\mathcal{B})}(t) = \sum_{S \subseteq [n]} (t+1)^{n-|S|} f_{\mathcal{P}(\mathcal{B}|_S)}$$

where $\mathcal{B}|_S = \{I \in \mathcal{B} \mid I \subset S\}$ is the **restriction** of $\mathcal{B}$ to S. Using this formula, we show that certain nestohedra and extended nestohedra have the same f-vectors.
Theorem: Let G be a forest graph and let $L(G)$ be the line graph of G. Then

$$f_{\mathcal{P}(\mathcal{B}_G)}(t) = f_{\mathcal{P}^\square(\mathcal{B}_{L(G)})}(t).$$

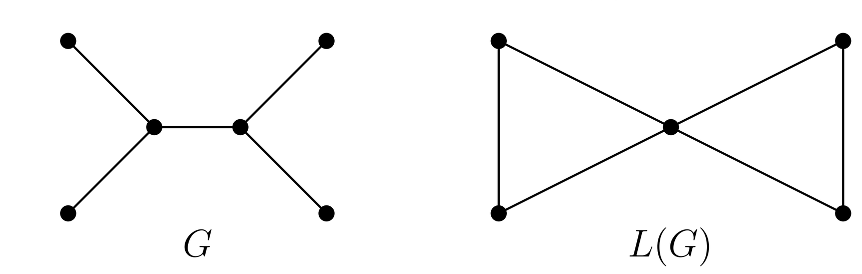

Figure 3: A forest G and its line graph $L(G)$.

Finally, we show Gal's conjecture [3] for extended nestohedra that are **flag** (its minimal non-faces are of dimension 1).
Theorem: If $\mathcal{P}^\square(\mathcal{B})$ is flag, then its γ-vector is nonnegative.

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