HOMOMESY AND ROWMOTION ON THE TRAPEZOID POSET

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Abstract. Rowmotion is an invertible action on the order ideals of a poset that has been studied by many different authors (including Fon-der-Flaass, Cameron, Striker, and Williams). The action of rowmotion on the (a, b)-rectangle poset is well-understood: Propp and Roby showed that the action of rowmotion is the same as the action of cyclic rotation on necklaces with a black beads and b white beads. The rowmotion orbits for the corresponding (a, b)-trapezoid poset, however, remain mysterious. Using a bijection between order ideals of the rectangle and trapezoid poset given by Hamaker, Patrias, Pechenik, and Williams, we prove that the action of rowmotion on the (a, b)-trapezoid is the same as for the (a, b)-rectangle. The notion of *weak K-Knuth equivalence*, introduced by Buch and Samuel, is central to our proof technique. We also show that rowmotion on the (a, b)-trapezoid poset is homomesic with respect to the down-degree statistic for the case $a \leq 4$ and any b, giving an approach that could be generalized to all a and b.

1. Introduction and Background

A *partially ordered set* (henceforth abbreviated a *poset*) is a set $\mathcal{P}$ with a binary relation $\leqslant$ that is reflexive, anti-symmetric, and transitive. Two elements $x, y \in \mathcal{P}$ are *comparable* if we have $x \leqslant y$ or $y \leqslant x$, and *incomparable* otherwise. We say y *covers* x if $y \geqslant x$ and there does not exist $z \in \mathcal{P}$ such that $y > z > x$; equivalently, we say that x *is covered by* y. The *Hasse diagram* of $\mathcal{P}$ is an undirected graph with vertex set $\mathcal{P}$, and an edge between y and x if they have a cover relation. A *graded poset* is a poset P with a rank function $rank : P \to \mathbb{Z}$ such that if $x < y$, $rank(x) < rank(y)$ and if y covers x, $rank(y) = rank(x) + 1$.

Given a poset $\mathcal{P}$, an *order ideal* I of $\mathcal{P}$ is a subset of $\mathcal{P}$ that is downward closed, i.e: if $x \in I$ and $y \leqslant x$ in $\mathcal{P}$, then $y \in I$ as well. A *filter ideal* I' of $\mathcal{P}$ is the complement of an order ideal, or equivalently, a subset of $\mathcal{P}$ that is upward closed. Denote the set of order ideals of $\mathcal{P}$ to be $J(\mathcal{P})$, which is also a poset with relation given by inclusion between order ideals.

The *dual poset* of a poset $\mathcal{P}$ is the poset $\mathcal{P}^*$ that has the same underlying set as $\mathcal{P}$, but with the binary relation reversed. Note that an order ideal of $\mathcal{P}$ corresponds to a filter ideal of $\mathcal{P}^*$ and vice versa.

Given an order ideal $I \in \mathcal{P}$, we can take the maximal elements of I to get an *antichain* A, i.e: a set of elements of $\mathcal{P}$ that are incomparable to one another. In general, the set of order ideals of P is in bijection with the set of antichains, with the reverse map sending A to the order ideal *generated by* A, i.e: the ideal $\{x \in \mathcal{P} \mid x \leqslant y \text{ for some } y \in A\}$. The *down-degree* of I is defined to be the number of maximal elements of I.

A *linear extension* of a poset $\mathcal{P}$ is a bijection $\rho : \mathcal{P} \to \{1, 2, \ldots, |\mathcal{P}|\}$ that is order-preserving, i.e: $\rho(x) < \rho(y)$ for all $x < y$ in $\mathcal{P}$. A *P-partition of $\mathcal{P}$ of height m* is a order preserving map from $\mathcal{P}$ to $[m] = \{0, \ldots, m\}$. Denote $\mathrm{PP}^m(\mathcal{P})$ the set of all P-partitions of $\mathcal{P}$ of height m. Note that there is a bijection between $\mathrm{PP}^m(\mathcal{P})$ and chains of order ideals of $\mathcal{P}$ of length m, given by sending a P-partition $\pi : \mathrm{PP}^m(\mathcal{P})$ to the chain $\pi^{-1}(\{0\}) \subset \pi^{-1}(\{0, 1\}) \subset \cdots \subset \pi^{-1}(\{0, \ldots, m-1\})$. With this bijection, it's clear that a P-partition of height 1 corresponds to an order ideal.

In our paper, we will mostly study the rectangle poset

$$R_{a,b} = \{(i, j) \in \mathbb{Z}^2 \mid 1 \leqslant i \leqslant a, 1 \leqslant j \leqslant b\}$$

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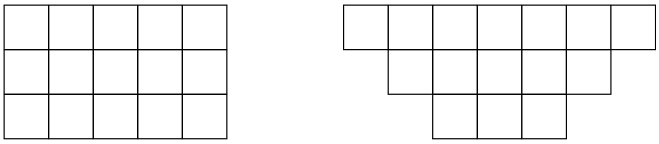


FIGURE 1. $R_{a,b}$ and $T_{a,b}$ for $a = 3$, $b = 5$

and the trapezoid poset

$$T_{a,a+b} = \{(i, j) \in \mathbb{Z}^2 \mid 1 \leqslant i \leqslant j \leqslant a + b - 1\}$$

with the relation $(i, j) \leqslant (i', j') \iff i \leqslant i'$ and $j \leqslant j'$. It was shown in [Sta86] that the number of P-partitions of height m is the same for the rectangle and trapezoid poset; in other words, we have: $\mathrm{PP}^m(R_{a,b}) = \mathrm{PP}^m(T_{a,a+b})$ for all a, b, m. Such a pair (P, Q) of posets satisfying $\mathrm{PP}^m(P) = \mathrm{PP}^m(Q)$ for all m is called a *doppelgänger pair*. It turns out that the rectangle and trapezoid poset (conjecturally) have a deeper connection than just being a doppelgänger pair; for a survey of these conjectures, see [Hop19].

2. Rowmotion and Minuscule Doppelgängers

2.1. Rowmotion on order ideals and piecewise linear rowmotion.

Definition 2.1. Let $\mathcal{P}$ be a poset, and $I \in J(\mathcal{P})$ an order ideal of $\mathcal{P}$. Then the rowmotion of I, denoted $\mathrm{row}(I)$ is the order ideal generated by the minimal elements that are not in I, i.e.

$$\mathrm{row}(I) = \langle a \in \mathcal{P} : a \in \min\{\mathcal{P} - I\}\rangle$$

Rowmotion can be viewed as composition of 'toggles' of poset elements. For $p \in \mathcal{P}$ and order ideal I, we denote $\tau_p(I)$ as toggling p on I which is defined as follows:

$$\tau_p(I) = \begin{cases} I \cup p & \text{if } p \notin I \text{ and } I \cup p \in J(\mathcal{P}), \\ I \setminus p & \text{if } p \in I \text{ and } I \setminus p \in J(\mathcal{P}), \\ I & \text{otherwise.} \end{cases}$$

Then rowmotion is just performing toggles row by row from top to bottom.

Proposition 2.2. [PR15] $\mathrm{row}(I) = \tau_{p_1} \circ \tau_{p_{n-1}} \circ \cdots \circ \tau_{p_n}(I)$ *where* $\{p_1, \cdots, p_n\}$ *is a linear extension of the poset* $\mathcal{P}$.

Rowmotion is generalized by Eisenstein and Propp to a piecewise linear action on P-partitions (or equivalently, order polytopes), which toggles are refined by a tropical exchange relation:

$$\tau(p) = \max\{a : a < p\} + \min\{b : b > p\} - p$$

We can identify a rowmotion as a height 1 P-partition where elements in the ideal are labeled as 0 and 1 otherwise. Then the classical Rowmotion is equivalent to piecewise linear rowmotion on this P-partition, thus we do not distinguish the notation between classical and piecewise linear rowmotion.

2.2. **Bijections between plane partitions of minuscule doppelgängers.** In this section we describe the bijection φ, given in [HPPW18], between P-partitions of the rectangle $R_{a,b}$ and the trapezoid $T_{a,a+b}$. Note that this bijection applies to all minuscule doppelgängerpairs shown in Figure 3, but we will mostly focus on the case of the rectangle and trapezoid. The construction is based on *k-jeu-de-taquin* slides, which are a K-theoretic analogue of the usual jeu-de-taquin. The discussion below is an adaptation of [HPPW18, Section 6.2].

Definition 2.3. An *increasing tableaux* of height ℓ on a poset $\mathcal{P}$ is a function $T : \mathcal{P} \to [\ell]$ such that whenever $x < y$ in $\mathcal{P}$, we have $T(x) < T(y)$. Denote the set of all increasing tableaux of height l to be $\mathrm{IT}^{[l]}(\mathcal{P})$. We say that x *covers* (resp. is covered by) $a \in \mathbb{N}$ in T if there exists $y \in \mathcal{P}$ covered by (resp. covering) x with $T(y) = a$. Let $C_T(x)$ denote the set of all a such that x either covers or is covered by a in T.

When $\mathcal{P}$ is a ranked poset with all maximal chains of the same length $\mathrm{ht}(\mathcal{P})$, [DPS17, Theorem 4.1] shows that there is a bijection $\mathrm{PP}^{[h]}(\mathcal{P}) \simeq \mathrm{IT}^{[h+\mathrm{ht}(\mathcal{P})]}(\mathcal{P})$. Since all of the miniscule Doppelgängers pairs are of this form, it suffices to find a bijection φ between increasing tableaux of such pairs. To define φ, we first need some preliminary definitions.

Definition 2.4. The *swap* of two numbers a, b in an increasing tableaux T is the function $\mathrm{swap}_{a,b}(T)$ such that for all $x \in P$:

$$\mathrm{swap}_{a,b}(T)(x) = \begin{cases} a & \text{if } T(x) = b \text{ and } a \in C_T(x) \\ b & \text{if } T(x) = a \text{ and } b \in C_T(x) \\ T(x) & \text{else} \end{cases}$$

After performing a swap, the resulting tableaux can be considered to still be increasing, but with the order of the numbers a and b switched. Next, we can describe K-jeu-de-taquin as a sequence of swaps, which turns a number a into the maximal number.

Definition 2.5. Suppose T is an increasing tableaux of height ℓ. Then define the *k-jeu-de-taquin slide* of $a \in \ell$ to be the tableaux

$$\mathrm{jdt}_a(T) := \left(\prod_{b=a+1}^{\ell} \mathrm{swap}_{a,b} \right) (T).$$

The resulting tableaux will still be increasing with the ordering that a is now the maximal value. Alternatively, we could make the tableaux increasing by replacing all instances of a with ℓ and decrease all $b \in [a+1, \ell]$ by one. However, unless stated otherwise, it is assumed that we don't relabel the entries.

We now define the bijection φ for the case of rectangle and trapezoid. For the other doppelgängers pairs, see [HPPW18].

Definition 2.6. Given an increasing tableaux $T \in IT^{[l]}(R_{a,b})$ of the rectangle, one obtains an increasing tableaux $\varphi(T) \in IT^{[l]}(T_{a,b})$ as follows:

(i) Create a larger poset $\mathcal{P} = R_{a,b} \cup \{(i,j) \mid 0 \leqslant i \leqslant a-1, -a+1 \leqslant j \leqslant 0, i+j \leqslant a-1\}$, and consider the tableaux $T \cup \overline{U}$ on $\mathcal{P}$ that is T on the part $R_{a,b}$, U is the minimal tableaux associated to the triangle shape, and $\overline{U}$ is U with all entries having a bar on top. Order the entries so that $\overline{1} < \overline{2} < \cdots < \overline{2a-3} < 1 < 2 < \cdots < l$.

(ii) Do K-jdt slide of $\overline{i}$ for all $i = 0, \ldots, 2a-3$.

(iii) Take the resulting increasing tableaux on the trapezoidal part $T_{a,b}$ of $\mathcal{P}$ to be the result $\varphi(T)$.

See Figure 2 for an example of φ. We are interested in how φ interacts with the action of rowmotion. In the case of order ideals, we will prove the following theorem.

Theorem 2.7. *For any of the minuscule doppelgängers pair*

$$(P, Q) \in \{(R_{a,b}, T_{a,a+b}), (OG(6,12), H_3), (\mathbb{Q}^{2n}, I_2(2n))\}$$

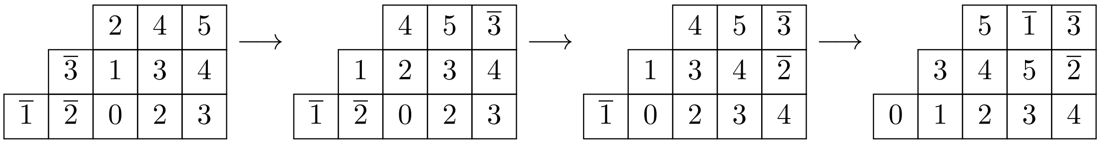


FIGURE 2. The map φ for an order ideal $I \in J(T_{3,3})$

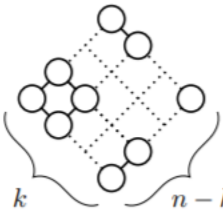
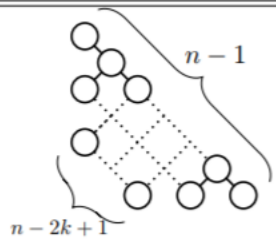
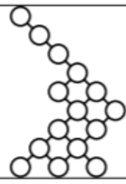
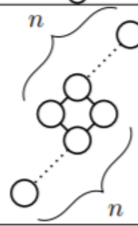
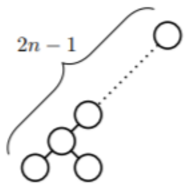
Label	Poset Name	Hasse Diagram	Hasse Diagram	Poset Name
(B)	$\Lambda_{\mathsf{Gr}(k,n)}$			$\Phi^+_{B_{k,n}}$
(H)	$\Lambda_{\mathsf{OG}(6,12)}$			$\Phi^+_{H_3}$
(I)	$\Lambda_{\mathbb{Q}^{2n}}$			$\Phi^+_{I_2(2n)}$

FIGURE 3. [HPPW18, Figure 1] Relevant miniscule Doppelgängers pairs

φ commutes with rowmotion on order ideals of the pair. In other words, we have a commutative diagram:

$$\begin{array}{ccc} J(P) & \xrightarrow{\varphi} & J(Q) \\ \big\downarrow \text{row} & & \big\downarrow \text{row} \\ J(P) & \xrightarrow{\varphi} & J(Q) \end{array}$$

Although Theorem 2.7 considers three minuscule doppelgängers pairs, the only hard part comes from the case of the rectangle and trapezoid. In particular, we have the following:

Proposition 2.8. *Theorem 2.7 is true for $(P, Q) \in \{(R_{a,b}, T_{a,a+b}), (OG(6, 12), H_3), (\mathbb{Q}^{2n}, I_2(2n))\}$ for all n, and all $a, b \leqslant 8$.*

Proof. The cases $(R_{a,b}, T_{a,a+b})$ for $a, b \leqslant 8$ and $(OG(6, 12), H_3)$ are checked by computer. For the last case, note that each poset only has 2 rowmotion orbits, one of size $2n-2$ and the other of size 2. It's easy to see that for $I \in J(OG(6, 12))$ in orbit of size 2, $\varphi(I)$ is also in orbit of size 2, hence rowmotion commutes with φ for such I. For any ideal I in the other rowmotion orbit of $J(OG(6, 12))$, I consists of all elements of $J(OG(6, 12))$ of rank $\leq m$ for some m. In this case, it can be seen that $\varphi(I)$ is the set of elements of $J(H_3)$ of rank $\leq m$ for some m. Observe that for a graded poset P with all maximal elements of P having rank $m + 1$, and the ideal $I \subseteq P$ consisting of all elements $\leq m$, the minimal elements of $P \setminus I$ are the elements of rank $m + 1$ thus $\text{row}(I)$ is either the ideal consisting of all elements of a poset P of rank $\leq m + 1$ or the empty ideal. Since $OG(6, 12)$ and H_3 both have the same maximal rank, we conclude that φ commutes with rowmotion on $(OG(6, 12), H_3)$ □

3. K-jdt Equivalence of Shifted and Standard Tableaux

The connection between the bijection of [HPPW18] and rowmotion is most apparent in their use of K-jdt slides. In this section, we will introduce some invariants of K-jdt and use these invariants to prove Theorem 2.7 through considering rowmotion and φ is terms of K-jdt slides. Throughout this section tableaux are assumed to be strictly increasing.

3.1. **K-jdt equivalence for tableaux of type A and B.** While the swaps involved in the bijection φ of [HPPW18] were performed between two integers, we could just as easily perform $\text{swap}_{\bullet,b}$ where we perform the swap with a set to a dummy variable "$\bullet$". A K-jdt slide is then the process of performing the swaps

$$\ldots \text{swap}_{\bullet,2} \circ \text{swap}_{\bullet,1} \circ \text{swap}_{\bullet,0}$$

for some tableaux where boxes are labeled either with nonnegative integers or with dots. We will be concerned with K-jdt on tableaux in two partially ordered sets. The first is tableaux in $\mathbb{N}^2$ with order $(a, b) \leq (c, d)$ when $a \leq c, b \leq d$. We will call a non-skew tableaux on this set a *standard tableaux*. A standard tableaux T's shape may be represented as a partition λ, with $(i, j) \in T$ if and only if $\lambda_j \geq i$. A tableaux of skew or non-skew shape on this set we will call *a tableaux of type A*. The second is tableaux in $\{(a, b) \in \mathbb{N}^2 | a \leq b\}$ with order $(a, b) \leq (c, d)$ when $a \leq c, b \leq d$. We will call a non-skew tableaux on this set a *shifted tableaux*. A shifted tableaux T's shape may be represented as partitions λ with $(i, j) \in T$ if and only if $\lambda_j \geq i - j$. A tableaux of skew or non-skew shape on this set we will call *a tableaux of type B*.

Two tableaux T, T' of type A or B are considered *K-jdt equivalent* if T can be reached from T' by a series of K-jdt slides on type A or B respectively. In particular, for a shifted tableaux T of shape $\lambda = (n, n-2, n-4 \ldots n-2r)$, $\varphi^{-1}(T)$ is K-jdt equivalent to T since φ can be written as a the composition of K-jdt slides. A useful consequence of the order that we perform the swaps, is that if we restrict two K-jdt equivalent tableaux T, T' to the same interval $[a, b]$ (i.e. remove all boxes from T and T' that are not in the interval $[a, b]$), then the same K-jdt slides which change T into T' restricted to the swaps within the interval $[a, b]$ will change $T|_{[a,b]}$ into $T'|_{[a,b]}$. Specifically,

Lemma 3.1 (Lemma 3.3 [BS16]). *If T and T' are K-jdt equivalent, then $T|_{[a,b]}$ and $T'|_{[a,b]}$ are K-jdt equivalent.*

In [BS16], Buch and Samuel show K-jdt equivalence for type A and type B tableaux can be described by K-Knuth and weak K-Knuth equivalent relations of reading words respectively.

Definition 3.2. The *row reading word* of a tableau T of type A or B is the reading word obtained by reading the rows of T from smallest element in the row to largest, starting with the largest row (under the increasing relations on boxes in the same column).

Example 3.3.

4	5	6
2	4	5
1	2	3

has row reading word 456245123

		6		
	3	4	5	
1	2	3	4	6

has row reading word 634512346

Theorem 3.4 (Theorem 6.2 [BS16]). *Tableaux T, T' of type A are K-jdt equivalent if and only if their row words are K-Knuth equivalent, where K-Knuth equivalence is the symmetric transitive closure of the following basic equivalences:*

- $uaav \equiv uav$ *for integers* a *and words* u, v
- $uabav \equiv ubabv$ *for integers* a, b *and words* u, v
- $uabcv \equiv uacbv$ *for integers* $b < a < c$ *and words* u, v
- $uabcv \equiv ubacv$ *for integers* $a < c < b$ *and words* u, v

Theorem 3.5 (Theorem 7.8 [BS16])**.** *Tableaux* T, T' *of type B are K-jdt equivalent if and only if their row words are weakly K-Knuth equivalent, where weak K-Knuth equivalence is the symmetric transitive closure of the basic equivalences of K-Knuth equivalence and the basic equivalence*

- $abv \equiv bav$ *for integers* a, b *and word* v*.*

The reader may notice weak K-Knuth and K-Knuth equivalence are similar and believe that K-jdt eqiuvalences of type A and type B are related. This is indeed true, though perhaps to less of an extent than one might believe. For a tableaux T of type B, we may construct a tableaux of type A by reflecting B across the diagonal. Concretely, we define T^2 to be the tableaux of type A where the entry in the (i, j) box of T^2 is the entry in the (i, j) box of T if $i \leq j$ and the entry in the box of (j, i) in T otherwise.

Proposition 3.6 (Proposition 7.1 [BS16])**.** *If* T *and* T' *are K-jdt equivalent tableaux of type B, then* T^2 *and* T'^2 *are K-jdt equivalent tableaux of type A.*

3.2. **Hecke permutations.** While weak K-Knuth and K-Knuth equivlance of row reading words completely describe K-jdt equivalence, these equivalences can be difficult to work with. Buch and Samuel [BS16] introduce a simpler yet cruder invariant of K-jdt on tableaux of type A and use this invariant to prove minimal tableaux are unique rectification targets (the reader is not expected to know what minimal tableaux or unique rectification targets are. The authors have not dove into unique rectification although an interesting project might be to see if the techniques we use in the next section would be useful for classifying some unique rectification targets). This invariant is the Hecke permutation. The *Hecke product* of a permutation u and a simple transposition $s_i = (i, i+1)$ is denoted $u \cdot s$ with

$$u \cdot s_i = \begin{cases} u & \text{if } u(i) > u(i+1) \\ us_i & \text{if } u(i) < u(i+1) \end{cases}$$

Definition 3.7 ([BS16])**.** The Hecke permutation of a tableaux T with reading word $u = a_1 a_2 a_3 \ldots a_k$, is the Hecke product

$$s_{a_k} \cdot (s_{a_{k-1}} \cdot (s_{a_{k-2}} \ldots (s_2 \cdot s_1) \ldots))$$

which is a permutation on $max(a_1, a_2 \ldots a_k) + 1$ elements. We will denote this permutation by $w(T)$ or $w(u)$.

The equivalence relation between tableaux's row reading words based on their Hecke permutations is a weakening of the K-Knuth equivalence relations such that if u and u' are K-Knuth equivalent, then $w(u) = w(u')$ (Buch and Samuel don't cite anybody for this... should find who originally did it/what to cite). In particular this implies

Corollary 3.8. *(Corollary 6.5* [BS16]*) The Hecke permutation of a tableaux of type A is invariant under K-jdt slides.*

3.3. **Almost minimal tableaux and the minimal ideal of a tableaux.** In section 2, we defined a bijection between plane partitions of height m and increasing tableaux of a certain height for ranked posets where all maximal chains have the same length. Given this bijection, it is natural to consider an order ideal as a specific case of an increasing tableaux.

Definition 3.9. An *almost minimal tableaux* is an increasing tableaux T such that for any square s, $T[s] - rank(s) \in \{0, 1\}$. Equivalently, $T - rank$ is an order ideal.

There is a clear bijection between almost minimal tableaux and order ideals from adding and subtracting rank. Although our proof of Theorem 2.7 will only involve almost minimal tableaux, our main theorem in this subsection will be more general and we will need more general notation:

Definition 3.10. Given an increasing tableaux T of shape λ/μ, its *minimal ideal* is the set of squares s such that $T[s] - rank(s) = 0$. This set is downward closed and thus an order ideal of λ/μ. For convenience will denote the *minimal ideal* of T by I_0 and the minimal ideal of T' by I_0'.

Our results in this section will come from analyzing the Hecke permutation of standard tableaux. We will specifically be interested in finding where elements i occur in the Hecke permutation formed from the row reading word of a tableaux.

Proposition 3.11. *Let T be a tableaux and $\overline{T_r}$ be the tableaux T without the first r rows, then for any i, $w(T)^{-1}(i) \geq w(\overline{T_r})^{-1}(i) - r$.*

Proof. Let $m = w(\overline{T_r})^{-1}(i)$. Each time we compute the Hecke product of w with a transposition s_i, only the i-th and $(i+1)$-th entries of w is changed. Since each row of T is increasing, m decreases by at most one, namely, when we compute Hecke product with s_{m-1} if $m-1$ is in the rth row. Thus

$$w(\overline{T_{r-1}})^{-1}(i) \geq w(\overline{T_r})^{-1}(i) - 1.$$

The proposition now follows by induction. □

In the following lemma $row(i)$ denotes the ith row of the tableaux T. This is the only place where we use row in such a way. Elsewhere, row refers to the action of rowmotion.

Lemma 3.12. *Let I_0 be minimal ideal of a standard (non-skew, non-shifted) tableaux T.*

i) If $|row(i) \cap I_0| < |row(i-1) \cap I_0|$, then $w(T)^{-1}(i) = |row(i) \cap I_0| + 1$

ii) If $|row(i) \cap I_0| = |row(i-1) \cap I_0|$, then $w(T)^{-1}(i) > |row(i) \cap I_0| + 1$

Proof. For each row r with $|row(r) \cap I_0| > 0$, the first element in the row is the first appearance of r in the row reading word of T. Using this, it is easy to check that the $|row(r) \cap I_0|$ part of the reading word in $w(\overline{T_{r-1}})$ simply moves the element r. This yields

$$w(\overline{T_{r-1}})^{-1}(r) = |row(r) \cap I_0| + r. \tag{1}$$

and for any integer a, if

$$r < w(\overline{T_r})^{-1}(a) \leq |row(r) \cap I_0| + r, \tag{2}$$

then

$$w(\overline{T_{r-1}})^{-1}(a) = |row(r) \cap I_0| - 1. \tag{3}$$

(i) We will prove by induction that for all $j \leq i$

$$w(\overline{T_j})^{-1}(i) = |row(i) \cap I_0| + j.$$

This will be enough to prove (i). The base case follows from equation 1. For the inductive step, suppose for $j \leq i$

$$w(\overline{T_j})^{-1}(i) = |row(i) \cap I_0| + j.$$

Notice since I_0 is an ideal,

$$j - 1 < |row(i) \cap I_0| + j \leq |row(i-1) \cap I_0| + j - 1 \leq |row(j-1) \cap I_0| + j - 1$$

Thus our argument around equations 2 and 3 finish our inductive step.
(ii)By equation 1,

$$w(\overline{T_{i-1}})^{-1}(i) = |row(i) \cap I_0| + i.$$

Also by equation 1, for all $j < i$,

$$w(\overline{T_{i-2}})^{-1}(j) < w(\overline{T_{i-1}})^{-1}(i).$$

Thus

$$w(\overline{T_{i-2}})^{-1}(i) \geq w(\overline{T_{i-1}})^{-1}(i),$$

and by proposition 3.11, we conclude

$$w(T)^{-1}(i) \geq |row(i) \cap I_0| + 2$$

□

Theorem 3.13. *Let T and T' be K-jdt equivalent standard (non-skew, non-shifted) tableaux with minimal ideals I_0 and I_0'. Then $I_0 = I_0'$.*

Proof. Suppose that $I_0 \neq I_0'$. Let r be the first row where I_0 and I_0' differ, then Lemma 3.12 implies that

$$w^{-1}(T)(r) \neq w^{-1}(T')(r).$$

Therefore the Hecke permutations of T and T' differ. Since by corollary 3.8, Hecke permutations are invariant under k-jdt slides for tableaux of type A, T and T' are not k-jdt equivalent. □

Since almost minimal tableaux T are completely described by their shape and their set of squares such that $T - rank = 0$, we conclude:

Corollary 3.14. *For a straight tableaux of shape λ, all almost-minimal tableaux of shape λ are in separate K-Knuth equivalence classes.*

To extend this result to result to shifted tableaux, we will use the connection K-jdt of tableaux T of type B and K-jdt of tableaux T^2 of type A. Notice in a standard tableaux, $rank(i,j) = rank(j,i)$. It follows for a shifted (non-skew) tableaux T, for any square $s = (i,j) \in T^2$, $T^2[s] - rank(s) = T^2[(j,i)] - rank(j,i)$. Thus T is almost minimal if and only if T^2 is almost minimal. Our above corollary combined with this observation and proposition 3.6 shows that:

Corollary 3.15. *For a shifted tableaux of shape λ, all almost-minimal tableaux of shape λ are in separate weak K-Knuth equivalence classes.*

Our last ingredient we will need to prove that the bijection of [HPPW18] commutes with row-motion on ordered ideals is the following corollary:

Corollary 3.16. *Let T, T' be two almost minimal tableaux of the same shape of type A or B with maximal rank r. Then $T|_{[1,r]}$ is k-jdt equivalent to $T'|_{[1,r]}$ if and only if $T = T'$.*

Proof. (⇐) If $T = T'$, then $T|_{[1,r]} = T'|_{[1,r]}$ by lemma 3.1.
(⇒) Suppose $T|_{[1,r]}$ is k-jdt equivalent to $T'|_{[1,r]}$. For type B, by proposition 3.6, $(T|_{[1,r]})^2 = T^2|_{[1,r]}$ is k-jdt equivalent to $(T'|_{[1,r]})^2 = T'^2|_{[1,r]}$. Define tableaux S, S' of type A, where $S = T, S' = T'$ for T, T' of type A and $S = T^2, S' = T'^2$ for T, T' of type B. Let I_0 be the minimal ideal of S and I'_0 the minimal ideal of S'. Notice that I_0, I'_0 are also the minimal ideals of $S|_{[1,r]}$ and $S'|_{[1,r]}$, respectively. Then Theorem 3.13 implies that $I_0 = I'_0$. Since almost minimal tableaux are completely determined by their shape and their ideal, $S = S'$ (they have the same shape and the same ideal), and therefore $T = T'$. □

3.4. **Rowmotion and φ in terms of K-jdt.** We may consider the bijection φ of [HPPW18] from section 2.2 as a bijection between increasing tableaux of the rectangle and trapezoid shape. Then φ descends to a bijection between almost minimal tableaux of the rectangle and trapezoid. Because φ is composed of K-jdt

Lemma 3.17. *For a rectangle shape λ, all almost minimal tableaux of shape λ are in separate weak K-Knuth equivalence classes. φ acting on order ideals can be described as matching each almost minimal tableaux of the rectangle with its unique weak K-Knuth equivalent almost minimal tableaux of the trapezoid.*

To progress to proving Theorem 2.7, we would now like to consider how rowmotion affects K-jdt equivalence. To do this we will describe rowmotion as a composition of K-jdt slides. This is done in [DPS17] under the name K-promotion.

Definition 3.18. For a straight or shifted tableaux T with all maximal elements of the same rank r (for rectangle and trapezoid, $r = a + b$), the *K-promotion* of T is defined as follows.

1) Turn the tableau into a skew tableau by replacing the minimal entry 1 by a dot • then subtract 1 from all other entries.
2) Turn the skew tableau into a straight shape by performing K-rectification, i.e. send
$$T \to \left(\Pi_{i=1}^{r} \text{swap}_{\bullet,i}\right).$$
3) Adding element $r + 1$ to the resulting tableau, so that the shape of tableau is preserved.

In the case where there does not exist a 1 in the tableaux, K-promotion simply decrements each entry by 1.

Lemma 3.19 (theorem 3.8 in [DSV19] does this up to showing K-bender knuth swaps correspond to inverse rowmotion, perhaps this is in the literature? Otherwise we will write out that step/that it follows closely from definitions)**.** *K-promotion on almost minimal tableaux is equal to the inverse of rowmotion on the corresponding order ideals.*

Example 3.20 (Inverse rowmotion as K-promotion)**.**

4	5	6
2	4	5
1	2	3

remove 1 →

3	4	5
1	3	4
•	1	2

K-rectify →

4	5	•
3	4	5
1	2	4

add maximum →

4	5	6
3	4	5
1	2	4

4	5	6
2	4	5
1	2	3

← add rank

1	1	1
0	1	1
0	0	0

← Rowmotion

1	1	1
1	1	1
0	0	1

← subtract rank

4	5	6
3	4	5
1	2	4

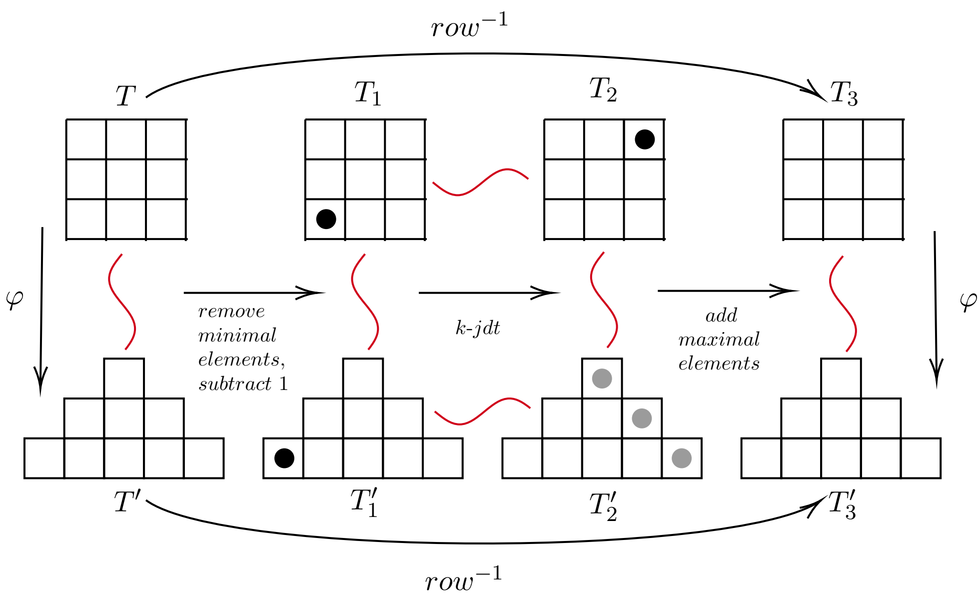


FIGURE 4. Commutative diagram for proof of Theorem 2.7. The red squiggles indicate weak K-Knuth equivalence. Note that where the large dot(s) ends up in the second to rightmost trapezoid will depend of the order ideal ideals.

Using this definition of rowmotion, we will prove Theorem 2.7 by showing that performing the above process preserves weak K-Knuth equivalence of order ideals of the rectangle and trapezoid i.e. if I is weakly K-Knuth equivalent to J, then $row^{-1}(I)$ is weakly K-Knuth equivalent to $row^{-1}(J)$. By lemma 3.17 this implies that rowmotion inverse commutes with φ and thus rowmotion commutes with φ.

Proof of Theorem 2.7 in the case of $(R_{a,b}, T_{a,b})$. Let T and T' be almost minimal tableaux of the rectangle and trapezoid respectively which are K-jdt equivalent (by Lemma 3.17 this is equivalent to $\varphi(T) = T'$). Let T_1, T_2, T_3 be the results of performing steps 1, 1 and 2 and 1,2 and 3 of k-promotion respectively on T and define T_1', T_2', T_3' similarly for T' (thus $T_3 = row^{-1}(T)$ and $T_3' = row^{-1}(T')$). By lemma 3.1, $T|_{2,maxrank+1}$ and $T'|_{2,maxrank+1}$ are K-jdt equivalent. Thus T_1 and T_1' are K-jdt equivalent. Performing K-jdt preserves K-jdt equivalence, thus T_2 and T_2' are K-jdt equivalent. By Lemma 3.17, T_3 is K-jdt equivalent to an almost minimal tableaux T^*. By lemma 3.1, $T^*|_{[1,maxrank]}$ is K-jdt equivalent to $T_2' = T_3'|_{[1,maxrank]}$. By corollary 3.16, $T^* = T_3'$. Thus

$$row^{-1}(T) = T_3 \overset{K\text{-jdt}}{\equiv} T^* = T_3' = row^{-1}(T').$$

Finally Lemma 3.17 implies that $\varphi(row^{-1}(T)) = row^{-1}(T') = row^{-1}(\varphi(T))$. □

The above proof completes the hard case of Theorem 2.7 with Proposition 2.8 covering the other cases.

4. TOGGLE SYMMETRY AND HOMOMESY OF DOWN-DEGREE

In this section, we will prove our second result about rowmotion on the trapezoid poset: that it is *homomesic* with respect to the *down-degree statistic* for the trapezoid $T_{3,n}$ and $T_{4,n}$. We first provide the relevant definitions.

Definition 4.1 ([PR15])**.** A statistic f on a set S is said to be *homomesic* with respect to an invertible operator $\Phi : S \to S$ if for all Φ-orbits $\mathcal{O}$,

$$\frac{1}{\#\mathcal{O}} \sum_{T \in \mathcal{O}} f(T) = \frac{1}{\#S} \sum_{T \in S} f(T).$$

Propp and Roby had rowmotion on the rectangle in mind when they formulated the term "homomesy." We will be concerned with when down-degree exhibits homomesy with respect to rowmotion for the rectangle and trapezoid. To approach this, we will focus on toggles as introduced by [CFDF95]. We define toggles as in [Hop17]. For an ideal I and an antichain A, we denote

$$\mathcal{T}_A^+(I) := \begin{cases} 1 & \text{if } A \notin I \text{ and } A \cup I \text{ is an ideal} \\ 0 & \text{otherwise} \end{cases}$$
$$\mathcal{T}_A^-(I) := \begin{cases} 1 & \text{if } A \in I \text{ and } I \setminus A \text{ is an ideal} \\ 0 & \text{otherwise} \end{cases}$$
$$\mathcal{T}_A(I) := \mathcal{T}_A^+(I) - \mathcal{T}_A^-(I)$$

We commonly use p to denote a single element antichain associated to the element p of the poset. We call a distribution μ on ideals in a poset *toggle-symmetric* if for any fixed p,

$$\mathbb{E}[\mu; \mathcal{T}_p(I)] = 0.$$

Some clear examples of toggle-symmetric distributions are the uniform distribution on all ideals or a uniform distribution on ideals contained in a single rowmotion orbit. The distribution defined by choosing a plane partition J_m uniformly at random, and then picking an ideal in J_m uniformly at random is a toggle symmetric distribution. The distribution defined by choosing a plane partition J_m uniformly at random from a rowmotion orbit, and then picking an ideal in J_m uniformly at random is also toggle-symmetric.

In slightly greater generality, we call a distribution μ on ideals in a poset *toggle on anitchains-symmetric* if for any fixed antichain A,

$$\mathbb{E}[\mu; \mathcal{T}_A(I)] = 0.$$

The uniform distribution on all ideals and a uniform distribution on ideals contained in a single rowmotion orbit are both toggle on anitchains-symmetric.

Remark 4.2. The distribution defined by choosing a plane partition J_m uniformly at random, and then picking an ideal in J_m uniformly at random is toggle on anitchains-symmetric. is not toggle on anitchains-symmetric (has sam proved this somewhere?).

In [CHHM17], Chan, Haddadan, Hopkins, and Moci show for any toggle-symmetric distribution μ on $\mathcal{R}_{a,b}$, $\mathbb{E}[\mu; \text{ddeg}] = \mathbb{E}[uni_{J(\mathcal{R}_{a,b})}; \text{ddeg}]$. In particular, they showed that the down-degree statistic is homomesic with respect to the action of rowmotion on the set of order ideals of the rectangle. To do this, they use the "rook" approach.

4.1. **The rook approach.** We will be concerned with rook arrangements on the trapezoid. For readers interested in rook arrangements on other shapes, see [Hop17]. Before we proceed we need a way of of labeling the elements of our posets. For the trapezoid, we use the Cartesian coordinates with the minimal element being $(0, 0)$ and the maximal elements having coordinates $(i, a + b - 1 - 2i)$ for $0 \leq i < a$, see 5 for an example. For notational convenience, (i, λ_i) will refer to the element $(i, a + b - 1 - 2i)$.

Definition 4.3. A *rook* on the (i, j) square of a trapezoid $T_{a,b}$ is a linear equation

$$R_{i,j} : \mathbb{R}^{J(T_{a,b})} \to \mathbb{R}$$

$$R_{i,j}(a \cdot I) = a \cdot \left(\sum_p \left(c_p^- \mathcal{T}_p^-(I) + c_p^+ \mathcal{T}_p^+(I) \right) + \sum_{i=0}^{a-2} c^-_{\{(i',\lambda_{i'}),(i'+1,\lambda_{i'+1})\}} T^-_{\{(i',\lambda_{i'}),(i'+1,\lambda_{i'+1})\}}(I) \right)$$

where

$$c^-_{(i',j')} = \begin{cases} 1 & \text{if } i' \geq i \text{ and } i'+j' \geq i+j \\ -1 & \text{if } i' < i \text{ and } i'+j' < i+j \text{ and } j' > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$c^+_{(i',j')} = \begin{cases} 1 & \text{if } i' \leq i \text{ and } i'+j' \leq i+j \\ -1 & \text{if } i' > i \text{ and } i'+j' > i+j \text{ and } j' > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$c^+_{\{(i',\lambda_{i'}),(i'+1,\lambda_{i'+1}\}} = \begin{cases} -1 & \text{if } i' \geq i \text{ and } i'+a+b-2-2i' \geq i+j \\ 0 & \text{otherwise} \end{cases}.$$

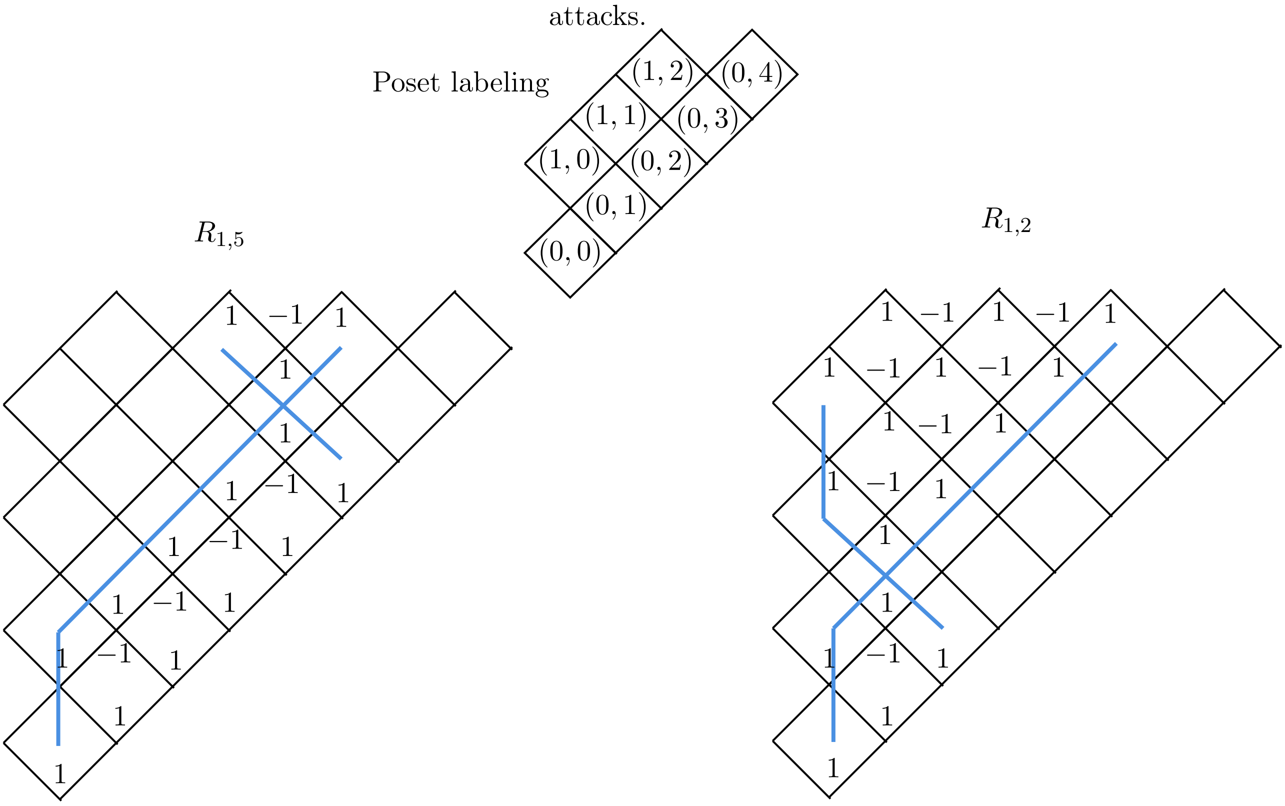
For each rook placed on the trapezoid, it *affects* an element p of our poset $c_p^- + c_p^+$ times. We say that the rook *attacks* a square p if $c_p^- + c_p^+ \neq 0$. As we see in Figure 5, a rook on the trapezoid will attack the squares that lie in the same row and column, with the exception that once we reach the leftmost or upmost part of the row/column, the rook starts attacking squares (or pairs of squares) that lie on the diagonal.

Our definition of a rook satisfy the following nice property:

Proposition 4.4 ([Hop17])**.** *For any order ideal $I \in J(T_{a,b})$, we have $R_{i,j}(I) = 1$.*

Figure 5. Rook Arrangements

In the below figure we write a number in a V-shaped nook with element p of our poset above it to denote to represent c_p^+ and a number in a Λ-shaped nook with an element of P below it to denote c_p^-. A number in a V-shaped nook at the top of our trapezoid refers to c_A^- were A is the antichain of the two elements bordering the nook. The blue line shows the squares the rook attacks.



For the case when $a = b$, Hopkins shows that all toggle-symmetric distributions have the same expected down-degree using the rook approach.

Theorem 4.5 ([Hop17], Theorem 4.2). *For any toggle-symmetric distribution μ on the trapezoid $T_{a,a}$,*

$$\mathbb{E}[\mu; \text{ddeg}] = \mathbb{E}[uni_{J(T_{a,b})}; \text{ddeg}] = \frac{ab}{a+b}$$

For a general trapezoid and a toggle-symmetric distribution μ, it is not necessarily true that we $\mathbb{E}[\mu; \text{ddeg}] = \mathbb{E}[uni_{J(T_{a,b})}; \text{ddeg}]$ (see [Hop17]). However, we can use the rook approach to to find the difference between $\mathbb{E}[\mu; \text{ddeg}]$ and $\mathbb{E}[uni_{J(T_{a,b})}; \text{ddeg}] = ab/(a+b)$:

Lemma 4.6. *For any toggle-symmetric distribution μ on the trapezoid $T_{a,b}$,*

$$\mathbb{E}[\mu; \text{ddeg}] = \frac{ab}{a+b} + \frac{a-b}{a+b}\left(\sum_{i=0}^{a}(a-1-i)\mathbb{E}[\mu; \mathcal{T}^{+}_{(i,0)}] - \sum_{i=0}^{a-2} i\mathbb{E}[\mu; \mathcal{T}^{-}_{(i,\lambda_i),(i+1,\lambda_{i+1})}]\right)$$

Proof. Consider the linear map $\varphi : \mathbb{R}^{J(T_{a,b})} \to \mathbb{R}$

$$\varphi := (2a - a^2)R_{(0,0)} + (b-a)\sum_{j=1}^{a} R_{(0,j)} + b\sum_{i=1}^{a-1} R_{(i,0)} + a\sum_{j=a+1}^{a+b-1} R_{(0,j)}.$$

We will compute $\mathbb{E}[\mu; \varphi]$ in two ways. On one hand, for any ideal I,

$$\varphi(I) = (2a - a^2)R_{(0,0)}(I) + (b-a)\sum_{j=1}^{a} R_{(0,j)}(I) + b\sum_{i=1}^{a-1} R_{(i,0)}(I) + a\sum_{j=a+1}^{a+b-1} R_{(0,j)}(I) = ab$$

thus $\mathbb{E}[\mu; \varphi] = ab$.

On the other hand, we see that the rook arrangement which φ defines attacks every element of $T_{a,b}$ $a+b$ times except for elements of form $(i, 0)$, which are attacked $(a+b) - (a-b)\cdot(a-1-i)$ times. Thus for some constants c_p,

$$\varphi = (a+b)\cdot \text{ddeg} - \sum_{i=0}^{a}(a-b)(a-1-i)\mathcal{T}^{+}_{(i,0)} + \sum_{i=0}^{a-2}(a-b)\mathcal{T}_{(i,\lambda_i),(i+1,\lambda_{i+1})} + \sum_{p} c_p\mathcal{T}_p$$

Over a toggle-symmetric distribution μ, $\mathbb{E}[\mu; \sum_p c_p\mathcal{T}_p] = 0$. Using linearity of expectation

$$\mathbb{E}\left[\mu; \sum_p c_p\mathcal{T}_p\right] = (a+b)\cdot\mathbb{E}[\mu; \text{ddeg}] - \sum_{i=0}^{a}(a-b)(a-1-i)\mathbb{E}\left[\mu; \mathcal{T}^{+}_{(i,0)}\right] + \sum_{i=0}^{a-2}(a-b)\mathbb{E}\left[\mu; \mathcal{T}^{-}_{(i,\lambda_i),(i+1,\lambda_{i+1})}\right]$$

Combining our two equations for $\mathbb{E}[\mu, \varphi]$ yields the desired result. □

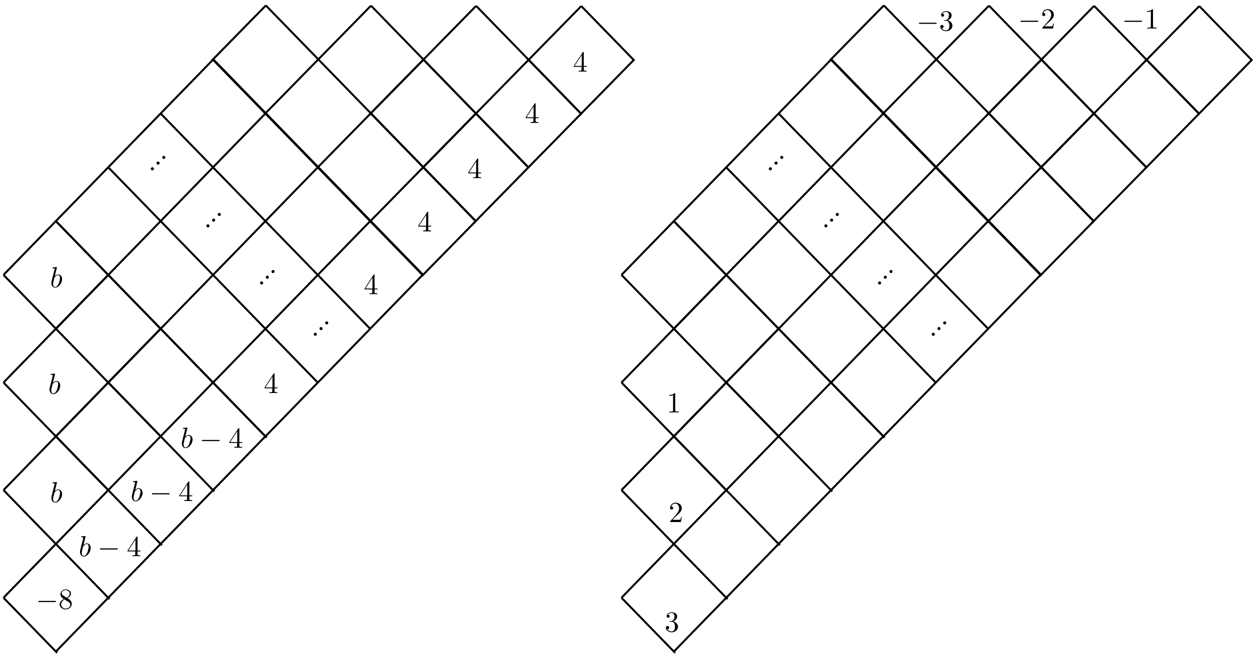
A way to evaluate the error term, that is:

$$\frac{a-b}{a+b}\left(\sum_{i=0}^{a}(a-1-i)\mathbb{E}[\mu; \mathcal{T}^{+}_{(i,0)}] - \sum_{i=0}^{a-2} i\mathbb{E}[\mu; \mathcal{T}^{-}_{(i,\lambda_i),(i+1,\lambda_{i+1})}]\right)$$

would yield many results about the trapezoid. Down-degree is homomesic with respect to an invertible operator Φ, if for any orbit $\mathcal{O}$ of Φ, and the error term for the distribution $uni_{\mathcal{O}}$ is 0. The error term is 0 for the distribution defined by choosing a standard shifted Young Tableaux on $T_{a,b}$ uniformly at random and then choosing an ideal consisting of all numbers $\leq m$ for some m if and only if a conjecture of Reiner, Tenner and Yong ([RTY18], conjecture 2.24) is true. The error term is 0 for the distribution defined by choosing a plane partition of height m uniformly at random and then choosing one of the m ideals from the plane partition uniformly at random if and only if a conjecture of Hopkins ([Hop19] Conjecture 3.10) is true.

In many general cases, we did not find a way to evaluate this error term. For rowmotion orbits, there is a way to trace out rowmotion orbits and consider a finite number of cases to show

FIGURE 6. Rook arrangement defined by φ for $T_{4,b}$ and error term scaled down by $(a+b)/(a-b)$



$\mathbb{E}[uni_{\mathcal{O}}; \text{ddeg}] = ab/(a+b)$ by locally (i.e. over some fixed number of rowmotion orbits) showing the error term is 0.

Theorem 4.7. *The action of rowmotion on the trapezoid exhibits For a rowmotion orbit $\mathcal{O}$ on $T_{3,n}$ or $T_{4,n}$,*

$$\mathbb{E}[uni_{\mathcal{O}}; \text{ddeg}] = \frac{ab}{a+b}.$$

For $T_{3,n}$, there were 3 cases we needed to consider and for $T_{4,n}$ there were 9 cases. We will not include this casework since there is a more powerful theorem we can show for $T_{3,n}$ and $T_{4,n}$:

Theorem 4.8. *For any toggle on anitchains-symmetric distribution μ on $T_{3,n}$ and $T_{4,n}$,*

$$\mathbb{E}[\mu; \text{ddeg}] = \frac{ab}{a+b}$$

In [Hop19], Hopkins proves a rowmotion orbit of order ideals is toggle on antichains-symmetric. Thus our above theorem implies theorem 4.7.

4.2. **Proof of Theorem 4.8.** Although Theorem 4.8 is only concerned with the $T_{3,n}$ and $T_{4,n}$ cases, we will state many of our lemmas more generally in hopes that this may help future readers can generalize our theorem.

Lemma 4.9. *For any toggle-symmetric distribution μ on the trapezoid $T_{a,b}$*

$$\sum_{i=0}^{a-1}(a-1-i)\mathbb{E}[\mu; \mathcal{T}^{-}_{(i,0)}] - \sum_{\substack{i+j\leq a-1 \\ j>0}} \mathbb{E}[\mu; \mathcal{T}^{-}_{(i,j)}] = 0$$

Proof. Define

$$\begin{aligned} a_n &:= \sum_{j=1}^{a-1-i} \mathbb{E}[\mu; \mathcal{T}^-_{(i,j)}]] \\ &= \Pr[\mu; I \text{ has a maximal element in } \{(i,j) | i+j=n, j>0\}] \\ b_n &:= \mathbb{E}[\mu; \mathcal{T}^-_{(i,j)}]] \\ &= \Pr[\mu; (n,0) \text{ is a maximal element in } I] \end{aligned}$$

Importantly, notice that

$$\begin{aligned} a_n + b_n &= \Pr[\mu; I \text{ has a maximal element in } \{(i,j)|i+j=n\}] \\ &= \Pr[\mu; I^C \text{ has a minimal element in } \{(i,j)|i+j=n+1|j>0\}] \\ &= \Pr[\mu; I \text{ has a maximal element in } \{(i,j)|i+j=n+1|j>0\}] \qquad \text{[by toggle-symmetry]} \\ &= a_{n+1} \end{aligned}$$

where I^C is the complement of the ideal I. The second to last equality follows from if an ideal is cut out with a Λ-shaped nook on a non-maximal element, then it must be followed by a V-shaped nook and vice versa. We now compute

$$\begin{aligned} \sum_{i=0}^{a-1}(a-1-i)\mathbb{E}[\mu; \mathcal{T}^-_{(i,0)}] - \sum_{\substack{i+j\le a-1 \\ j>0}} \mathbb{E}[\mu; \mathcal{T}^-_{(i,j)}] &= \sum_{i=0}^{a-1}(a-1-i)b_i - \sum_{i=1}^{a} a_i \\ &= \left(\sum_{i=0}^{a-1}(a-1-i)(a_{i+1}-a_i)\right) - \sum_{i=1}^{a} a_i \\ &= \sum_{i=1}^{a} a_i - \sum_{i=1}^{a} a_i = 0 \end{aligned}$$

□

In particular, the above lemma says that a toggle-symmetric distribution μ satisfies $\mathbb{E}[\mu; \text{ddeg}] = ab/(a+b)$ if and only if our error term form lemma 4.6 is equal to

$$\sum_{i=0}^{a-1}(a-1-i)\mathbb{E}[\mu; \mathcal{T}^-_{(i,0)}] - \sum_{\substack{i+j\le a-1 \\ j>0}} \mathbb{E}[\mu; \mathcal{T}^-_{(i,j)}] = 0.$$

Thus we have

Proposition 4.10. *A toggle-symmetric distribution μ satisfies $\mathbb{E}[\mu; \text{ddeg}] = ab/(a+b)$ if and only if*

$$\sum_{\substack{i+j\le a-1 \\ j>0}} \mathbb{E}[\mu; \mathcal{T}^-_{(i,j)}] = \sum_{i=0}^{a-2} i\mathbb{E}[\mu; \mathcal{T}^-_{(i,\lambda_i),(i+1,\lambda_{i+1})}].$$

To make progress towards proving this equality, we will first convert the LHS into a sum of toggles on antichains. Then we will try to use the toggle on antichains-symmetry property to move our toggles on antichains in the RHS from the maximal elements to toggles on antichains near the minimal elements of the trapezoid.

Lemma 4.11. *For a trapezoid $T_{a,b}$, there is an equality of functions on $J(T_{a,b})$:*

$$\sum_{\substack{(i,j)|i+j\le a-1 \\ j>0}} \mathcal{T}^+_{(i,j)} = \sum_{n\ge 2} \sum_{\substack{|A|=n \\ A\subset\{(i,j)|i+j\le a+1\}}} (-1)^n \mathcal{T}^-_A$$

Proof. Consider an ideal in the restricted shape $S = T_{a,b} \cap \{(i,j) | i+j \leq a, (i,j) \neq (a-1,1)\}$, with the same $<$ relations as in $T_{a,b}$. It is clear that the functions on $J(T_{a,b})$

$$\sum_{\substack{(i,j)|i+j\leq a-1\\ j>0}} \mathcal{T}^+_{(i,j)}, \sum_{n\geq 2} \sum_{\substack{|A|=n\\ A\in\{(i,j)|i+j\leq a+1\}}} (-1)^n \mathcal{T}^-_A$$

may be viewed as functions on restrictions of ideals in $J(T_{a,b})$ to $J(S)$. Any ideal in $J(S)$ corresponds to the lattice path consisting of up-left and down-left steps starting at the rightmost corner of our shape and ending with an up-left step at one of $a-1$ of the leftmost vertices of our shape, which cuts out the ideal i.e. the set of elements below our path is the ideal. $\sum_{\substack{(i,j)|i+j\leq a-2\\ j>0}} \mathcal{T}^+_{(i,j)}$ counts the number of Λ-shaped nooks with both edges on a square (i,j) with $i+j \leq a-2$ and $j > 0$ of such a lattice path. For each of these Λ shaped nooks, there must be a V-shaped nook after and before it. Conversely given any two V-shaped nooks, there is a there must be a Λ-shaped nook inbetween them, and the edges of this nook must be on a square (i,j) with (i,j) with $i+j \leq a-2$ and $j > 0$. Thus

$$\sum_{p\in S} \mathcal{T}^+_p = 1 + \sum_{\substack{(i,j)|i+j\leq a-1\\ j>0}} \mathcal{T}^+_{(i,j)}$$

and

$$\begin{aligned}\sum_{n\geq 2} \sum_{\substack{|A|=n\\ A\in\{(i,j)|i+j\leq a+1\}}} (-1)^n \mathcal{T}^-_A &= \sum_{n\geq 2} (-1)^n \binom{1+\sum_{\substack{(i,j)|i+j\leq a-1\\ j>0}} \mathcal{T}^+_{(i,j)}}{n} \\ &= \binom{1+\sum_{\substack{(i,j)|i+j\leq a-1\\ j>0}} \mathcal{T}^+_{(i,j)}}{1} + \sum_{n\geq 1} (-1)^n \binom{1+\sum_{\substack{(i,j)|i+j\leq a-1\\ j>0}} \mathcal{T}^+_{(i,j)}}{n} \\ &= 1 + \sum_{\substack{(i,j)|i+j\leq a-1\\ j>0}} \mathcal{T}^+_{(i,j)}\end{aligned}$$

□

Lemma 4.12. *For a trapezoid $T_{a,b}$, and a toggle on antichains-symmetric distribution μ,*

$$\sum_{i=0}^{a-2} \mathbb{E}[\mu; \mathcal{T}^-_{(i,\lambda_i),(i+1,\lambda_{i+1})}] = \sum_{i=1}^{a-1} \sum_{j<i} \mathbb{E}[\mu; T^-_{(j,i-j+1),(i,0)}] = \sum_{i=0}^{a-2} \mathbb{E}[\mu; \mathcal{T}^-_{(i,1)}]$$

Proof. Call an antichain A with two elements *adjacent* if $A = \{(i,j),(i',j')\}$ with $i+j = i'+j'+1$. Define the subsets of adjacent antichains

$$\begin{aligned} S^- &:= \{A = \{(i,j),(i',j')\} \in T_{a,b} | i+j = i'+j'+1, j \neq \lambda_i\}, \\ S^+ &:= \{A = \{(i,j),(i',j')\} \in T_{a,b} | i+j = i'+j'+1, 0 < j, j'\} \\ S &:= S^+ \cap S^-. \end{aligned}$$

To get the first equality, we consider the following equation on ideals

$$\begin{aligned} f &:= \sum_{A\in S^+} \mathcal{T}^+_A - \sum_{A\in S^-} \mathcal{T}^-_A \\ &= \sum_{A\in S} \mathcal{T}_A + \sum_{i=0}^{a-2} \mathcal{T}^+_{(i,\lambda_i),(i+1,\lambda_{i+1})} - \sum_{i=1}^{a-1} \sum_{j<i} \mu; T^-_{(j,i-j+1),(i,0)} \end{aligned}$$

We compute $\mathbb{E}[\mu; f]$ in two ways. First notice that if you can toggle in an antichain in S^+, then you can toggle out an antichain in S^-, thus for all ideals I, $f(I) = 0$ and so $\mathbb{E}[\mu; f] = 0$. Next

$$\begin{aligned}\mathbb{E}[\mu; f] &= \sum_{A\in S} \mathbb{E}[\mu; \mathcal{T}_A] + \sum_{i=0}^{a-2} \mathbb{E}[\mu; \mathcal{T}^+_{(i,\lambda_i),(i+1,\lambda_{i+1})}] - \sum_{i=1}^{a-1}\sum_{j<i} \mathbb{E}[\mu; T^-_{(j,i-j+1),(i,0)}] \\ &= \sum_{i=0}^{a-2} \mathbb{E}[\mu; \mathcal{T}^+_{(i,\lambda_i),(i+1,\lambda_{i+1})}] - \sum_{i=1}^{a-1}\sum_{j<i} \mathbb{E}[\mu; T^-_{(j,i-j+1),(i,0)}]\end{aligned}$$

We get the second equality from toggle on antichains-symmetry:

$$\begin{aligned}\sum_{i=0}^{a-2} \mathbb{E}[\mu; \mathcal{T}^-_{(i,1)}] &= \sum_{i=1}^{a-1}\sum_{j<i} \mathbb{E}[\mu; T^+_{(j,i-j+1),(i,0)}] \\ &= \sum_{i=1}^{a-1}\sum_{j<i} \mathbb{E}[\mu; T^-_{(j,i-j+1),(i,0)}]\end{aligned}$$

□

Proof of Theorem 4.8. **Case 1**: $T_{3,n}$. Between proposition 4.10 and lemma 4.12, all that remains, is to show

$$\mathbb{E}[\mu; \mathcal{T}^-_{(0,2)}] = \mathbb{E}[\mu; \mathcal{T}^-_{(1,\lambda_1),(2,\lambda_2)}].$$

Define S to be the set of anitchains

$$S := \{A \in T_{3,n} | A = \{(2, j), (1, j + 2)(0, j + 4)\} \text{ or } A = \{(2, j), (1, j + 2)(0, j + 5)\} \text{ and } j < \lambda_2\}$$

Then define a function of ideals,

$$f := \sum_{A\in S} \mathcal{T}_A - \mathcal{T}^-_{(0,2)} + \mathcal{T}_{(1,1)(0,3)} + \mathcal{T}_{(1,0)(0,2)} + \mathcal{T}_{(1,1)(0,4)} + \mathcal{T}_{(1,0)(0,3)} + \mathcal{T}^+_{(1,\lambda_1),(2,\lambda_2)}$$

We compute $\mathbb{E}[\mu; f]$ in two ways. First we prove that for any ideal I, $f(I) = 0$. This can be checked for any ideal with all maximal elements in $\{(i, j)|i + j > \lambda_2\}$. For smaller j, we can check by hand that for the restriction of the ideal to $T_{a,b} \cap \{(i, j)|2i + j < 5\}$ is zero, i.e:

$$\mathcal{T}^-_{(2,0),(1,2),(0,5)} + \mathcal{T}^-_{(2,0),(1,2),(2,4)} + \mathcal{T}^+_{(0,2)} + \mathcal{T}_{(1,1)(0,3)} + \mathcal{T}_{(1,0)(0,2)} + \mathcal{T}_{(1,1)(0,4)} + \mathcal{T}_{(1,0)(0,3)} = 0.$$

For other ideals, notice that

$$\mathcal{T}^+_{(2,j),(1,j+2)(0,j+4)}(I) = 1 \text{ for } j > 1 \text{ if and only if } \mathcal{T}^-_{(2,j-1),(1,j+1)(0,j+3)}(I) = 1.$$

Thus $f(I) = 0$. On the other hand, from toggle on antichains symmetry,

$$\mathbb{E}[\mu; f] = \mathbb{E}[\mu; \mathcal{T}^-_{(0,2)}] - \mathbb{E}[\mu; \mathcal{T}^-_{(1,\lambda_1),(2,\lambda_2)}]$$

Thus we conclude the theorem for the $T_{3,n}$ case.

Case 2: $T_{4,n}$. We will do this in three computations:

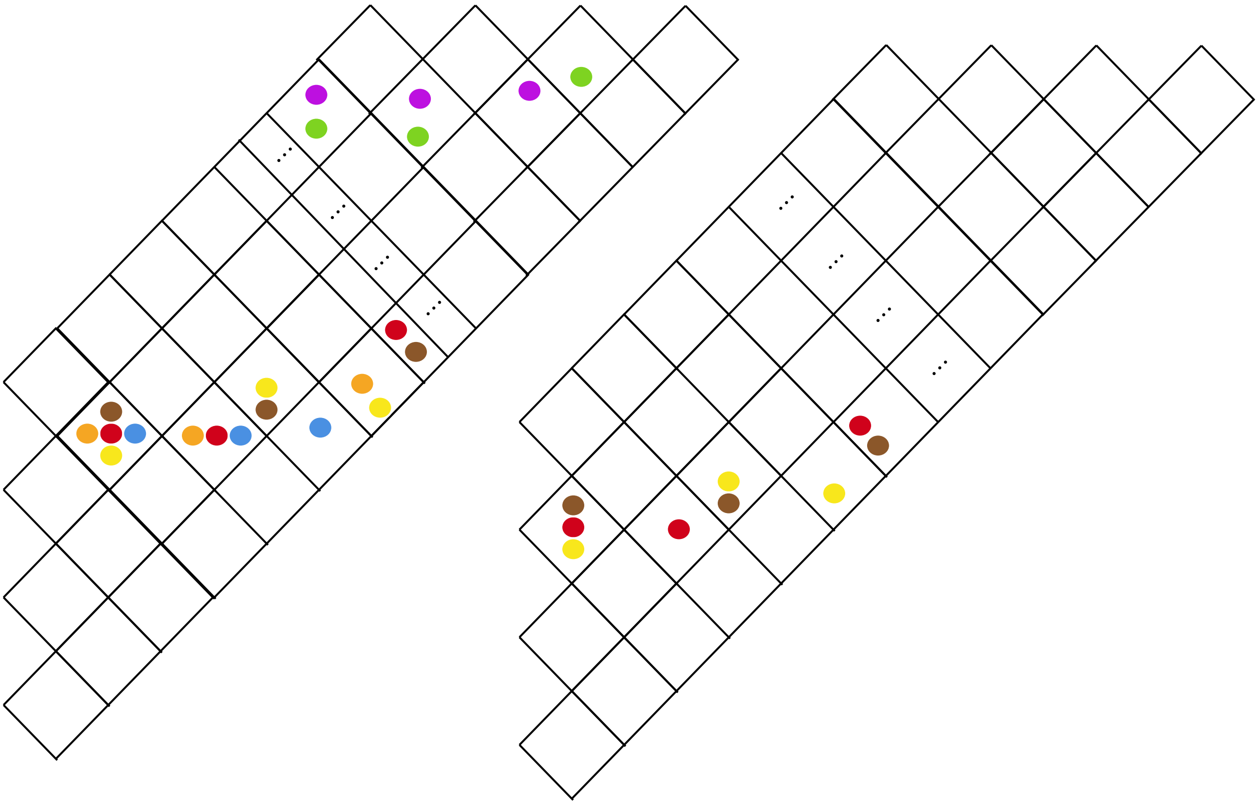
1) Let S be the set of 4 element antichains of $T_{4,n}$ of the form $(3, j), (2, j+2), (3, j+4), (4, j+6+m)$ for $0 \le m \le 2$ or of the form $(3, j), (2, j+2), (3, j+5), (4, j+6+m)$ for $0 \le m \le 1$ and $j < \lambda_4 - 1$. Let S' be the set of color coded antichains in figure 7.

Consider the function of ideals

$$f = \sum_{A\in S} \mathcal{T}_A + \sum_{A\in S'} \mathcal{T}_A + \mathcal{T}^+_{(2,\lambda_2),(3,\lambda_3)} - \mathcal{T}^-_{(1,1),(0,5)} - T^-_{(1,2),(0,5)} - T^-_{(1,2),(0,4)}$$

We compute $\mathbb{E}[\mu; f]$ in two ways. First I claim for any ideal I, $f(I) = 0$. This can be checked for any ideal with maximal elements (i, j) for $i + j > 8$ by noting if we can toggle in an antichain in S or S', then we can toggle our an antichain in S or S'. Moreover, in any ideal we can only

FIGURE 7. The antichains of S' coded by color



toggle in at most 1 antichain in S or S' and in any ideal we can toggle out at most 1 antichain from S or S'. We can check for any ideal with a maximal element with $i+j<8$, that these also satisfy $f(I)=0$. On the other hand, from toggle on antichains symmetry,

$$\mathbb{E}[\mu; f] = \mathbb{E}[\mu; \mathcal{T}^{+}_{(2,\lambda_2),(3,\lambda_3)}] - \mathbb{E}[\mu; \mathcal{T}^{-}_{(1,1),(0,5)} + T^{-}_{(1,2),(0,5)} + T^{-}_{(1,2),(0,4)}]$$

2)Let S be the set of three element antichains of the form $(i_1,j_1),(i_2,j_2),(i_3,j_3)$ where $i_1>i_2>i_3$, $i_1+j_1+2=i_2+j_2$, and $i_2+j_2+2\leq i_3+j_3\leq i_2+j_2+3$ with $i_1+j_1\leq 3+\lambda_3$ and $(i_2,j_2)\neq(2,\lambda_2)$. Let S' be the set of 4 element antichains of the form $(3,j),(2,j+2),(1,j+4),(0,j+6)$. Then define a function f on ideals

$$f = \sum_{A\in S}\mathcal{T}_A - \sum_{A\in S'}\mathcal{T}_A + \mathcal{T}^{+}_{(2,\lambda_2),(3,\lambda_3)} + \mathcal{T}^{+}_{(1,\lambda_1),(2,\lambda_2)} + \mathcal{T}^{-}_{(2,1),(1,3),(0,5)} - \sum_{i=1}^{2}\left(\sum_{0\leq x<i} T^{-}_{(i,1),(x,i+2-x)} + T^{-}_{(i,1),(x,i+3-x)}\right)$$

We compute $\mathbb{E}[\mu; f]$ in two ways. First, I claim for any ideal I, $f(I)=0$. For ideals which contain maximal elements (i,j) for $i+j>\lambda_3+2$, this can be checked since there are few since there are few possible nonzero terms in f. For ideals which contain multiple maximal elements (i,j) with $i+j\leq 8$, this can also be checked since there are few possible nonzero terms in f. For other ideals I, by chasing V and Λ nooks, we see you can toggle in an element of S' if and only if you can toggle out an element of S', and that the number of elements of S which can by toggled in equals the number of elements of S which can be toggled out. We conclude that $f(I)=0$. On the other hand, from toggle on antichains symmetry,

$$\mathbb{E}[\mu; f] = \mathbb{E}\left[\mu; \mathcal{T}^{+}_{(2,\lambda_2),(3,\lambda_3)} + \mathcal{T}^{+}_{(1,\lambda_1),(2,\lambda_2)} + \mathcal{T}^{-}_{(2,1),(1,3),(0,5)} - \sum_{i=1}^{2}\left(\sum_{0\leq x<i} T^{-}_{(i,1),(x,i+2-x)} + T^{-}_{(i,1),(x,i+3-x)}\right)\right]$$

3) By proposition 4.10, lemma 4.12 and adding our calculations in (1) and (2), showing

$$\mathcal{T}^-_{(1,1),(0,5)} + \mathcal{T}^-_{(1,2),(0,5)} + \mathcal{T}^-_{(1,2),(0,4)} + \sum_{i=1}^{2}\left(\sum_{0\le x<i} \mathcal{T}^-_{(i,1),(x,i+2-x)} + \mathcal{T}^-_{(i,1),(x,i+3-x)}\right)$$
$$-\mathcal{T}^-_{(2,1),(1,3),(0,5)} + \sum_{i=1}^{a-1}\sum_{j<i}\mathbb{E}[\mu; \mathcal{T}^-_{(j,i-j+1),(i,0)}] = \sum_{k\ge 2}\sum_{\substack{|A|=k\\ A\in\{(i,j)|i+j\le a+1\}}} (-1)^k \mathcal{T}^-_A$$

will imply our theorem for $T_{4,n}$. This can be checked for any ideal by restricting the ideal to $T_{4,n} \cap \{(i,j)|i+j \le 8\}$ and checking all ideals in this shape. ☐

Conjecture 4.13 ([Hop19, Conjecture 4.30]). *There is a size-preserving bijection φ between the rowmotion orbits of $\mathrm{PP}^\ell(R_{a,b})$ and $\mathrm{PP}^\ell(T_{a,a+b})$. Moreover, for all rowmotion orbits $\mathcal{O}$ for $\mathrm{PP}^\ell(R_{a,b})$, we have:*

$$\sum_{T\in\mathcal{O}} \operatorname{ddeg} T = \sum_{T\in\varphi(\mathcal{O})} \operatorname{ddeg} T.$$

Theorem 2.7 gives a candidate for the bijection φ for the case $\ell = 1$. Together with homomesy of rowmotion orbits of the trapezoid, this approach would be enough to prove the conjecture for the case $\ell = 1$.

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