K-Knuth Equivalence for Increasing Tableaux

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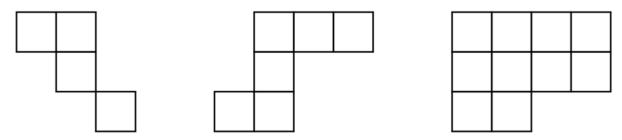
Introduction

A K-theoretic analogue of Schensted insertion for semistandard Young tableaux was first introduced in 2006. The resulting K-Knuth equivalence relations on words and increasing tableaux on [n] has prompted investigation into the equivalence classes of tableaux arising from these relations. Of particular interest are the tableaux that are unique in their class, which we refer to as unique rectification targets (URTs). We give several new families of URTs and a finite bound on the length of intermediate words connecting two K-Knuth equivalent words.

Background

Elements of the set $\mathbb{N} \times \mathbb{N}$ are called *boxes*, and a *shape* λ is any finite subset of $\mathbb{N} \times \mathbb{N}$. We say λ is a *straight shape* if whenever λ contains the box α it contains all boxes weakly northwest of α. A *skew shape* ν/μ is the set difference of two straight shapes $\nu \supseteq \mu$. Of the shapes below, the first is neither straight nor skew, the second is skew but not straight, and the third is straight.



A *filling* of a shape λ is any map $T : \lambda \to \mathbb{N}$, which assigns an integer to each box of λ, and the filling T is an *increasing tableau (of shape λ)* if the entries of T strictly increase down columns and from left to right along rows. Of the fillings below, only the third is an increasing tableau.

$$\begin{matrix}1&2&2&5\\3&4&5&5\\7&7&&\end{matrix} \qquad \begin{matrix}1&2&3&3\\3&4&5&6\\4&7&&\end{matrix} \qquad \begin{matrix}1&2&3&5\\3&4&5&6\\6&7&&\end{matrix}$$

In order to identify a tableaux solely by its filling, we will use the *row word* for T written $\mathfrak{row}(T)$, which is obtained by reading the entries of T from left to right along each row, starting from the bottom row and moving upward, so for example, the row word of the third tableau in the above example is 6734561235. For simplicity, we restrict ourselves to initial tableaux, where a tableau T *initial* if the set of labels of T is $[n]$ for some $n \in \mathbb{N}$.

Our problem centers around a K-theoretic analogue of the highly important algebraic combinatorial algorithm: RSK. The operation of the RSK algorithm involves the insertion of a positive integer into a straight semistandard tableaux, where a tableau is semistandard if it increases weakly along rows and strictly down columns.

K-Knuth Equivalence

Just as Hecke insertion is a K-theoretic analogue of the standard RSK insertion, *K-Knuth equivalence* is the corresponding analogue for Knuth equivalence, an equivalence relation on words extending to equivalence on semistandard tableau under RSK insertion. Two words are said to be *K-Knuth equivalent* if one can be obtained from the other via a series of applications of the K-Knuth relations:

$$\begin{aligned} xzy &\equiv zxy, && (x < y < z) \\ yxz &\equiv yzx, && (x < y < z) \\ x &\equiv xx, && \\ xyx &\equiv yxy. && \end{aligned}$$

By the third relation there are finitely many equivalence classes on any alphabet $[n]$ with infinitely many words in each class. The fourth rule implies that two words can be equivalent, but each letter could appear a different number of times in the two words. For example, $121 \equiv 212$, but 1 appears twice in the first word and once in the second. As an example, we see that $3124 \equiv 34124$ because

$$3124 \equiv 1324 \equiv 1342 \equiv 13422 \equiv 13242 \equiv 31242 \equiv 31424 \equiv 34124.$$

With these equivalence relations on words, one may ask how they relate to tableaux, which prompts discussion of Hecke insertion.

Hecke Insertion

Hecke insertion is an algorithm for inserting a positive integer into an increasing tableau, resulting in another increasing tableau, which may or may not be the same as the original. Defined by Buch and Samuel the *insertion class* of w is $\{w' : P(w') = P(w)\}$. If w and w' are in the same insertion class, then $w \equiv w'$. Hence Hecke insertion equivalence implies K-Knuth equivalence. To further understand this relationship between tableaux and row words let T and T' be two increasing tableaux. If $\mathfrak{row}(T) \equiv \mathfrak{row}(T')$, we say that T is *K-Knuth equivalent* to T' and write $T \equiv T'$, giving a K-Knuth equivalence on the set of increasing tableaux. Using the previous example and this fact, we say the tableaux found by Hecke inserting words 3124 and 34124 are K-Knuth equivalent, so $\begin{matrix}1&2&4\\3&&\end{matrix} \equiv \begin{matrix}1&2&4\\3&4&\end{matrix}$.

Unique Rectification Targets

In some instances an increasing tableau T has only a single straight tableau U in its equivalence class, in which case we define the tableau U to be a *unique rectification target* (URT) as described by Buch and Samuel. In other words, if T_1 is a URT, $T_1 \equiv T_2 \iff row(T_1) \equiv row(T_2)$. We can see that the tableaux mentioned in the previous example are not URTs because their straight tableaux are equivalent but not equal.

K-Knuth Equivalence Classes for Initial Tableaux

Now that we have the notion of an equivalence class of tableaux, we will provide several invariants under the K-Knuth equivalence relation. These will aid in proving results concerning the relations between tableaux in equivalence classes.

1. Restriction of a word (or a tableau) to an interval subalphabet

$$T_1 \equiv T_2 \implies T_1|_{[a,b]} \equiv T_2|_{[a,b]}$$

2. Outer hook of a tableau

$$T_1 \equiv T_2 \implies \text{outerhook}(T_1) = \text{outerhook}(T_2)$$

3. Transpose of a tableau

$$T_1 \equiv T_2 \iff T_1^t \equiv T_2^t$$

4. Hecke permutation

$$T_1 \equiv T_2 \implies w(T_1) \equiv w(T_2)$$

Data

With these invariants, we computed all equivalence classes of tableaux on $[n]$ for $0 \leq n \leq 7$. We were unable to obtain asymptotic bounds on the size of K-Knuth equivalence classes, but they seem to grow at least as quickly as $n!$.

Table 1: *Sets of Initial Tableaux*

Alphabet Size	Initial Increasing Tableaux	K-Knuth Classes of Initial Tableaux	URTs
0	1	1	1
1	1	1	1
2	3	3	3
3	13	13	13
4	87	79	71
5	849	620	459
6	11915	6036	3313
7	238405	70963	25904

Table 1 shows that the ratio of unique rectification classes of tableaux on $[n]$ to all K-Knuth classes of tableaux on $[n]$ decreases monotonically, and we expect the ratio to asymptotically tend to zero.

Results

Previously proven results are that minimal tableaux are URTs by Buch and Samuel and superstandard tableaux are URTs by Thomas and Yong. As an example, the first tableau is minimal, and the second is superstandard.

$$\begin{matrix}1&2&3&4\\2&3&4&\\3&4&5&\\4&&&\end{matrix} \qquad \begin{matrix}1&2&3&4\\5&6&7&\\8&9&10&\\11&&&\end{matrix}$$

We give two new families: right-alignable and hook-shaped. We also provide a finite bound on the length of intermediate words connecting two K-Knuth equivalent words.

Families of URTs

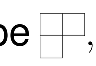
The first family of URTs we found are right-alignable, which are tableaux T such that the filling T' formed by right-aligning the rows of T is an increasing tableau. For example, the tableau

$T = \begin{matrix}1&2&3\\3&5&\\4&&\end{matrix}$ has right justification $T_R = \begin{matrix}1&2&3\\&3&5\\&&4\end{matrix}$.

so T is not right-alignable. We proved this though a property we call T-compatibility, a relationship between the location on a letter in the row word and its location in the tableau. Since superstandard and rectangular tableaux are right-alignable, we have discovered an alternate proof that those families are URTs.

The second family of URTs that we discovered are hook-shaped tableaux of certain fillings, where we define T of shape λ to be *hook shaped* if $\lambda = (m, 1^n)$. Let T be an initial hook-shaped tableau such that the first row of T is labeled $1, a_1, \ldots, a_n$, and the first column of T is labeled $1, b_1, \ldots, b_m$. Then T is a URT if and only if both of the sequences $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$ are consecutive. This is proven using the fact that K-Knuth equivalence is invariant under restrictions to subintervals. By this we see that, for example,

$T = \begin{matrix}1&2&3&4&5\\5&&&&\\6&&&&\end{matrix}$ is a URT but $\begin{matrix}1&2&3&4\\2&&&\\4&&&\end{matrix}$ is not.

We have also proven that if a shape λ is not rectangular or of the shape , then there exists a tableau of shape λ that is not a URT. We proved this sharpness using the prototypical non-URTs $\begin{matrix}1&2&4\\3&&\end{matrix}$ and $\begin{matrix}1&2&4\\3&4&\end{matrix}$.

Bound on Length of Words

At this point we were interested in finding a bound on the length of intermediate words between two words using K-Knuth relations. We know if $w \equiv w'$, then there exists a sequence $w = w_0, w_1, \ldots, w_r = w'$ of words such that w_i and w_{i+1} differ by one K-Knuth move. It is natural to ask whether it is always possible to find such a sequence so that the intermediate words w_i always have length at most that of the longer one. Surprisingly, the answer is no, as one can check by computer that $4235124 \equiv 4523124$ cannot be connected by words of length at most 7. However, it is possible to give an upper bound in terms of the size of the alphabet. Let w and w' be words and let ℓ be a positive integer. We say that $w \overset{\ell}{\equiv} w'$ if there exists a sequence $w = w_0 \equiv w_1 \equiv \cdots \equiv w_r = w'$ of words such that w_i and w_{i+1} differ by one K-Knuth move and each word w_i has length at most ℓ. With this, we have the result that if w is a word and $P(w) = T$, then $w \overset{|w|}{\equiv} \mathfrak{row}(T)$. Wtih these results we have that if $T_1 \equiv T_2$ are tableaux on $[n]$ and

$$\boldsymbol{N = \frac{1}{3}n(n+1)(n+2) + 3}$$

then $\mathfrak{row}(T_1) \overset{N}{\equiv} \mathfrak{row}(T_2)$. This bound on the number of letters bounds those which do something in the insertion. The usefulness of this bound lies in that now we have a bound on the length of intermediate words in using K-Knuth relations to test the equivalence of two words. While we have not classified all URTs, we have made valuable progress though finding new families and determining a bound on the length of words.

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