THE BROKEN PTOLEMY ALGEBRA: A FINITE-TYPE LAURENT PHENOMENON ALGEBRA

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Abstract. Type A, or Ptolemy cluster algebras are a prototypical example of finite type cluster algebras, as introduced by Fomin and Zelevinsky. Their combinatorics is that of triangulations of a polygon. Lam and Pylyavskyy have introduced a generalization of cluster algebras where the exchange polynomials are not necessarily binomial, called Laurent phenomenon algebras. It is an interesting and hard question to classify finite type Laurent phenomenon algebras. Here we show that "breaking" one of the arrows in a type A mutation class quiver surprisingly yields a finite type LP algebra, whose combinatorics can still be understood in terms of diagonals and triangulations of a polygon.

Contents

Introduction

Fomin and Zelevinsky introduced cluster algebras, and classified the finite type ones. The classification is in correspondence with Cartan-Killing classification of semisimple Lie groups. The so called type A is a family of cluster algebras that can be realized by diagonal exchanges in polygon triangulations. The exchanges are given by the Ptolemy theorem; hence, the alternative name is Ptolemy cluster algebras.

In 2003, Fomin and Zelevinsky introduced the Ptolemy Algebra, a finite type cluster algebra in [3]. The cluster algebra can be described geometrically as a convex polygon P with $m+3$ edges where the cluster variables correspond to the diagonals of P and the clusters correspond to the triangulations of P. There are, accordingly, $\frac{(m+3)(m)}{2}$ cluster variables and C_{m+1} clusters where C_k is the kth Catalan number. The boundary edges correspond with coefficients from a tropological semigroup.

The mutation and exchange polynomials of the Ptolemy Algebra can also be given a geometric description. For a given triangulation T_0 of P, we mutate edge $x \in T_0$ in the following way. Edge x is the diagonal of unique quadrilateral in T_0. We mutate edge x by exchanging it with the other diagonal in the unique quadrilateral of T_0. This exchange is illustrated in Figure 1.

The exchange polynomial for edge x is given by multiplying the cluster variables of the opposite sides of the unique quadrilateral and then adding the two binomials together. For instance, in Figure 1, edge x has exchange polynomial $AC + BD$.

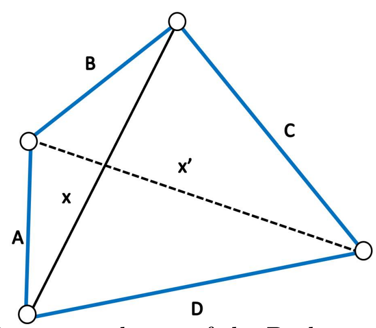

FIGURE 1. This is a general case of the Ptolemy exchange. Edges x and x' are the exchange pair and edges A, B, C and D are the edges of exchange quadrilateral $Q_{x,x'}$

Schiffler in 2008 in [1], gives a formula by way of T-paths for any diagonal in P in terms of an arbitary initial cluster, in the process proving positivity as conjectured by Fomin and Zelevinsky.

In 2012, Lam and Pylyavskyy in [2] reduce the requirements on Fomin and Zelevinsky's exchange polynomials. The polynomials are allowed to be irreducible polynomials as opposed to binomials. Lam and Pylyavskky give a method on how to create new irreducible polynomials analagously to the method given by Fomin and Zelevinsky. Lam and Pylyavskyy show that in their method, the Laurent Phenomenon holds, and accordingly call their algebra Laurent phenomenon or LP algebras. We describe this method of Lam Pylyavskyy in Section 1.5.

Cluster algebras of finite type are rare as are Laurent Phenomenon algebras of finite type. In this paper, we define the Broken Ptolemy Algebra by slightly altering an initial exchange relation, which can be thought of "breaking" an edge in the Ptolemy Algebra. The results, surprisingly so in an analagous Laurent Phenomenon Algebra. Similar to the Ptolemy Algebra, the Broken Ptolemy algebra is of finite type and has a spherical cluster complex. We prove finiteness by giving an analogue of Schiffler T-paths for a certain cluster. In contrast to the Ptolemy Algebra, the Broken Ptolemy algebra has fewer diagonals as cluster variables and has triangulations as clusters.

We believe that breaking an arbitrary edge in the Broken Ptolemy Algebra yields a Laurent Phenomenon Algebra of finite type, but we focus on breaking a specific edge in this paper. If this is the case, one is able to produce many LP algebras of finite type. Although we are able to prove finiteness and and sphericity for breaking a specific edge, we do not yet have a technique to generalize to other cases.

We also look into the simplicial complex of the Broken Ptolemy algebra. In another paper [4], Fomin and Zelevinksy describe in detail the fans of the simplicial complex of certain objects. The fan of the Ptolemy Algebra simplicial complex is shown to be homeomorphic to the unit sphere. We conjecture that the the fan of the Broken Edge LP Algebra simiplicial complex is very similar to that of the Ptolemy Algebra's and is also isomoprhic to that of a sphere.

1. Forming the Broken Ptolemy algebra

1.1. **Relation to the Ptolemy Algebra.** Cluster algebras can be represented by a combinatorial object known as quivers. We do not go into the innerworkings of a quiver but rather show how the quiver allows us to move from a cluster algebra to a LP algebra. On the lefthand side of Figure 2, we show the quiver for the A_4 Ptolemy algebra.

Each node of the quiver corresponds to a cluster variable and likewise an edge of P. We mutate a node k in quiver by the rules described in [5].

- Reverse orientation of all arrows adjacent to k
- Add an arrow from node i to node j for any path from node i to node j of length 2
- Remove all 2-cycles This mutation is identical to that of the Ptolemy exchange, which is the exchanging of diagonals of which we go into detail later in this paper.

The exchange polynomials of the quiver are also identical to that of the Ptolemy Algebra. Multiply the variables together to which the arrows face towards and add them to the variables to which the arrows point inwards. The exchange relations for variables a and c are

$$\{a,\ bY + cX\} \tag{1.1}$$

$$\{c,\ aT + bZ\} \tag{1.2}$$

To form our initial seed, we remove c from a's exchange relation in 1.1 to have new exchange relation given in 1.3,

$$\{a,\ bY + X\}. \tag{1.3}$$

We leave all other exchange relations, including equation 1.2 as is. Note the exchange polynomial in equation 1.1 is irreducible. As a result, we assymetrically break the arrow in the quiver that points from node c to node a in Figure equation 2. This is our "broken edge" quiver.

We give a geometric interpretation of the result of breaking this arrow in the quiver.

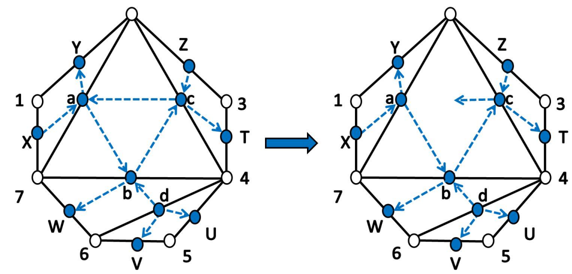

FIGURE 2. On the left is an illustration of a quiver for the A_4 Ptolemy Algebra. The right is visual representation of our "broken edge" quiver.

1.2. **The Initial Cluster.** Let P be a convex polygon with $m + 3$ edges. Let T_0 be any triangulation of P with given edges a, b, and c with vertices labeled $1, 2, \ldots, m + 3$ as shown in Figure 3.

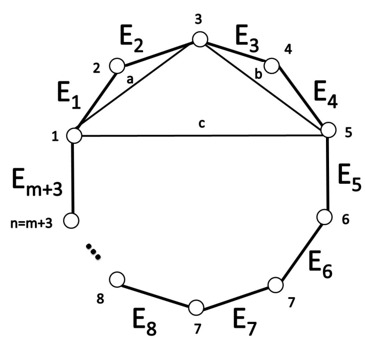

FIGURE 3. This is an example of an initial cluster. Only the required edges are shown. The rest can be filled with any valid triangulation of P

The intial exchange polynomials are given as the normal Ptolemy exchange as described by Schiffler [1]. However, there is one change. We change edge a's exchange polynomial to $\{a, bX + Y\}$, in essence "breaking" edge a's relation with edge c as described previously.

As a result, the following geometric interpretation, which refer to as the Broken Ptolemy algebra acts analagously to the Ptolemy Algebra. In the Broken Ptolemy algebra, cluster variables correspond to all the diagonals of P with the exception of edge M, the "missing" edge. Another way to think of this cluster is to glue "special" edge S together with edge M. Edge S can assume the role of edge M in certain cases. From this point on, we use the same variable to describe both the edge and the cluster variable as there is a bijection between the two. The clusters are those of triangulations of P and subtracting those triangulations containing both edges M and S. Since M is not a cluster variable, any triangulation including edge M is instead replaced by edge S. There are accordingly $C_{m+1} - C_{m-1}$ clusters in this Broken Ptolemy algebra, where C_k is the k^{th} Catalan number. Edge M and S have the specific positions shown in Figure 4.

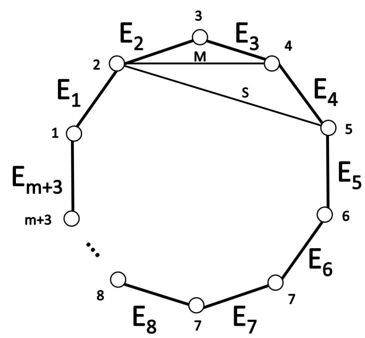

FIGURE 4. The cluster shows the fixed positions of missing edge M and the special diagonal S.

1.3. **Definitions.** We provide terminology to describe our mutation u_k, the mutation acting on edge k and exchange polynomials.

Definition 1.1. The *mutation partner* (MP) *of edge* k, denoted k' is the resulting edge of mutation on edge k. The mutation is described in detail in Section 1.6.

Definition 1.2. We call mutating edge k and its MP k' the *exchange pair* or EP.

Definition 1.3. The *exchange quadrilateral* is the unique quadrilateral that bounds the mutating edge k and its mutation partner k', with the 4 vertices being those of the vertices of k and k'. We denote the exchange quadrilateral as either $Q_{k,k'}$ or $Q(v_1, v_2, v_3, v_4)$ where v_i is a vertex of the exchange quadrilateral, depending on whether the situation is more easily described by the exchange pair or the vertices of the quadrilateral.

Definition 1.4. The $(T\text{-path})_k$ is defined as Schiffler's T-paths for edge k. The formula for this is found in section 1.4.

Definition 1.5. The *exchange polynomial* for an exchange pair is denoted $P_{k,k'}$

Definition 1.6. The *linearized* T-path over edge k, denoted $lin(T\text{-path})_k$ is $(T\text{-path})_k$ multiplied through by its least common denominator.

Definition 1.7. The *minimal triangulation* over edge k is a subset $T_k \subset T_0$, triangulation existing in the polygon P such that T_k is the set of the minimum number of edges that contains the vertices of edge k as well as the vertices of all the edges crossing edge k, with the exception S whose location is described in section insert.

Definition 1.8. A *crooked cluster* is a cluster such that the cluster is not a triangulation of P. In this specific Broken Ptolemy algebra, the crooked cluster contains the edge S and switching S with M yields a valid triangulation of P.

Definition 1.9. The *normal Ptolemy exchange* is that which is described by Schiffler and illustrated in Figure 1.

Definition 1.10. Let J_k denote the set of edges in the triangulation of P that cross edge k such that edge k is not in the cluster.

1.4. **Formula for Schiffler's T-paths.** In both our exchange polynomials rules and formula for putting the diagonals in terms of the initial cluster, we use Schiffler's T-paths. Below is a review of how to calculate the T-path for a diagonal in the polygon.

We use the formula for Schiffler's T-paths as given in [1]. For a polygon P with $n+3$ edges and a triangulation T_0 such that

$$T_0 = \{x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_{2n+3}\}$$

where $x_1, \ldots, x_n$ are the cluster variables corresponding to the diagonals of P and $x_{n+1}, \ldots, x_{2n+3}$ correspond to the boundary edges of P.

Let $M_{ab} \notin \{x_1, \ldots, x_n, x_{n+1}, \ldots, x_{2n+3}\}$, with vertices a and b in the polygon. Then the formula for M_{ab} in terms of T_0 is given in the following manner:

$$M_{ab} = \sum_{\gamma \in P(a,b)} X(\gamma). \tag{1.4}$$

In equation 1.4, γ refers to any sequence of edges in T_0, $(x_{j_1}, x_{j_2}, \ldots, x_{j_{l(\gamma)}})$ from vertex a to vertex b where $j_i \in \{1, \ldots, 2n+3\}$ that satisfies the following conditions:

- An edge in T_0 is used at most once
- Edges x_{j_i} and $x_{j_{i+1}}$ share exactly one vertex in P for $i = 1, 2, \ldots l(\gamma) - 1$
- $l(\gamma)$ is odd
- For any x_{j_i} such that i is even, edge x_{j_i} crosses edge M_{ab}
- If k and l are even with $k < l$ then intersection of x_{j_k} and M_{ab} is closer to vertex a than the intersection of x_{j_k} and M_{ab}

In words, γ is a path of odd length from vertex a to vertex b using edges in the triangulation such that edges on even steps cross M_{ab} and the crossings over edge M_{ab} progress strictly from vertex a to vertex b.

In equation 1.4, $X(\gamma)$ is the weight we give path γ as described in Equation 1.5.

$$X(\gamma) = \frac{\prod_{k \text{ odd}} x_{j_k}}{\prod_{k \text{ even}} x_{j_k}} \tag{1.5}$$

1.4.1. *Example of a T-path.* For an initial cluster in A_4 with diagonals a, b, c, and d as pictured in Figure 5, we find the T-path formula for edge M_{15}.

$$M_{15} = \frac{XU}{b} + \frac{XcV}{ad} + \frac{XcWU}{abd} + \frac{YbV}{ad} + \frac{YWU}{ad} \tag{1.6}$$

Each monomial in Equation 1.6 corresponds to a path from vertex 1 to vertex 5 that satisfies the aforementioned conditions.

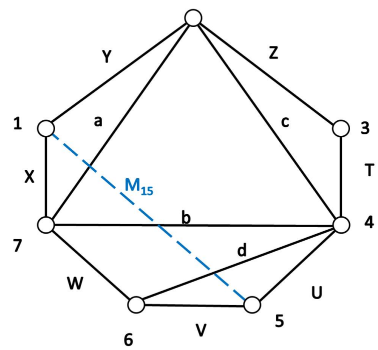

FIGURE 5. We find the T-path formula for edge M_{15} in Equation 1.6

1.5. **Lam and Pylyavskyy Exchange Polynomials.** In [2], Lam and Pylyavskyy give a mutation method for Laurent Phenomenon Algebras that we use to prove that our own exchange polynomial rules yield a LP algebra.

Given a seed $(\mathbf{x}, \mathbf{F})$, we can define *intermediate polynomials* by the following:

$$\hat{F}_j = x_1^{a_1}...\hat{x_j}...x_n^{a_n} F_j \tag{1.7}$$

with a_k chosen so that $\hat{F}_j|_{x_i \leftarrow \frac{\hat{F}_i}{x}}$ is a Laurent polynomial and is not divisible by F_i and x_j is not used in F_j. Let $(\mathbf{x}', \mathbf{F}') = \mu_i(\mathbf{x}, \mathbf{F})$ represent the tuple obtained by mutating $(\mathbf{x}, \mathbf{F})$ at node $i \in \{1, 2, \ldots, n|\}$. $\mu_i(\mathbf{x}, \mathbf{F})$ has cluster variables $x'_i = \frac{\hat{F}_i}{x_i}$ and $x'_j = x_j$ for $j \neq i$. To find its exchange polynomials, we define

$$G'_j = F_j|_{x_i \leftarrow \frac{\hat{F}_i|_{x_j=0}}{x'_i}}$$

Remove any common factors G'_j has with $F_i|_{x_j=0}$, and adjust by a monomial. The result is F'_j. $F'_i = F_i$. This completes the mutation, and forms a new seed.

1.5.1. *Example.* We give an example of the mutation process described above. Consider the seed with variables a, b, and c.

$$\{a,\ F_a = bB + (rs + pq)c\}$$
$$\{b,\ F_b = rs + pq\}$$
$$\{c,\ F_c = aA + bB\}$$

Then $\hat{F}_a = b^{-1} F_a$ because F_b divides $F_a|_{b=F_b}$ The new exchange for c after mutating at node a is

$$A\frac{bB}{ca'} + bB \rightarrow A + ca'$$

The new seed is thus

$$\{a',\ F_{a'} = bB + (rs + pq)c\}$$
$$\{b,\ F_b = rs + pq\}$$
$$\{c,\ F_c = A + ca'\}$$

1.6. **The Mutation.** Let T_0 be any initial cluster with form shown in Figure 3. Let edge k be in the initial cluster, and let the mutation acting on edge k, μ_k, act geometrically on k in the following manner with special attention given to edges M and S. This allows one to do the following exchange:

Case 1 The cluster is not crooked and the normal Ptolemy exchange results in triangulation of P without edge M. Then do the normal Ptolemy exchange on edge k.

Case 2 The cluster is not crooked and the Ptolemy exchange results in a triangulation with edge M.

(1) Do Ptolemy exchange on edge k to have intermediate step of triangulation with edge M.

(2) (a) If S is not in the triangulation, change M to S. This is shown in figure 7 when reading from right to left.

(b) If S is in the triangulation, first exchange the mutating edge from M to S. Then do Ptolemy exchange on the relocated M (it is now in the S position in Figure 6). This edge is the one with which edge k exchanges. Finally, let the relocated S fall back to the normal S location as the M edge is not allowed.

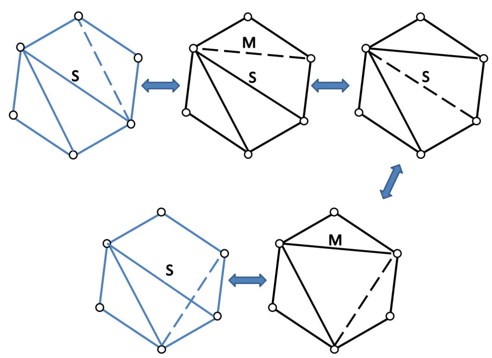

FIGURE 6. This is an example of the Case 2 cluster variable mutation. The dashed line is the edge moving in the mutation. Only the first and last steps are valid clusters.

Case 3 The cluster is a crooked. Edge S is thus in this cluster. Then

(1) Change edge S to edge M. If edge $k = S$ then, one is now mutating edge M. Do normal Ptolemy exchange on edge k. This is illustrated in Figure 7

(a) If edge S results in the normal Ptolemy exchange, switch the mutated edge from S and M and then do the Ptolemy exchange on the new M. This is shown in Figure 8

(b) Let edge M fall to edge S is necessary.

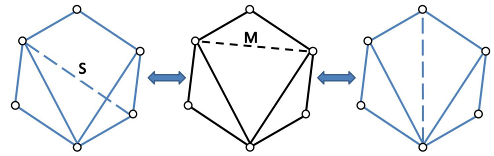

FIGURE 7. This is an example of a Case 3 mutation. The dashed edge is the edge is in the process of mutating. Only the first and last steps are valid clusters.

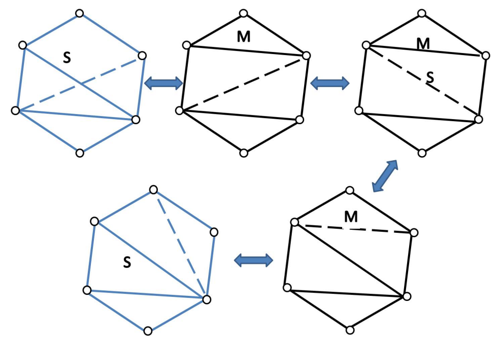

FIGURE 8. This is an example of a Case 3 mutation. The dashed edge is the edge is in the process of mutating. Only the first and last steps are valid clusters.

Remark 1.11. The mutation is an involution.

Remark 1.12. Given a cluster that does not result in a crooked cluster after a mutation on an edge, mutate at each cluster variable to create new clusters. Of these new clusters, at most one is a crooked triangulation of P.

Remark 1.13. The mutation is very similar in nature to the normal Ptolemy exchange. We do the Ptolemy exchange twice in order to skip past the triangulations containing both M and S.

1.7. **The Exchange Polynomials.** Label the vertices of P $1, 2, ..., n$ as shown in Figure 3. For any exchange pair x and x', let $Q(v_1, v_2, v_3, v_4)$ be the unique quadrilateral of which x and x' are diagonals. We describe a combinatorial construction of exchange polynomials with casework dependent on the vertices of x and x'.

The "c-rule" The unique quadrilateral is $Q(1, i, j, k)$ where $i \in \{2, 4\}$, 1 to i is an edge of the quadrilateral, and j and k are any other vertices such that $Q(1, i, j, k)$ is a possible exchange quadrilateral. Edge c is in the cluster as are all sides of $Q(1, i, j, k)$. Denote the edges of $Q(1, i, j, k)$ as A, B, C, D where edges A and C are not adjacent to any vertex.

An example of the "c"-rule is shown in Figure 9. Construct the exchange polynomial for x by multiplying c to the normal Ptolemy Exchange in the following way:

(1) Start with the regular Ptolemy Exchange, $AC + BD$

(2) Orient the triangles inside $Q(1, i, j, k)$ by connecting the midpoint of x with the midpoints of each side of the quadrilateral and orient counterclockwise

(3)
  (a) If x and c share a vertex, multiply c to the term in the Ptolemy exchange polynomial pointed to by the orientation of the triangles. For instance, if the arrows point out towards A and C then the EP is $cAC + BD$.
  (b) If x and c do not share a vertex, multiply c to the term of the Ptolemy exchange polynomial whose arrows point in towards x.
  (c) If both x and x' share a vertex with c, add c to the term that does not contain a c in the Ptolemy exchange polynomial.

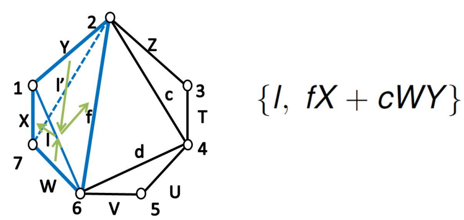

FIGURE 9. This is an example of the “c”-rule in the A_4 case.

Remark 1.14. Whenever S is being exchanged, we have that c is an edge of $Q_{x,x'}$. In this case, when we multiply c to a term, we have that both terms contain c, and so it is cancelled out of the polynomial as one removes common factors.

The “modified c-rule” The unique quadrilateral is $Q(1, i, j, k)$ where $i \in \{2, 4\}$, 1 to i is an edge of the quadrilateral, and j and k are any other vertices such that $Q(1, i, j, k)$ is a possible exchange quadrilateral. The diagonal c or some sides of Q_{xy} are not in the cluster. Label the edges of the quadrilateral as A, B, C, D with A and C having no vertices in common. An example is shown in Figure 11.

(1) Begin with the normal exchange polynomial as described in Case 1.7, including the c. For instance, the initial exchange is $cAC + BD$.
(2) Subsitute c with $(T\text{-path})_c$ over the minimal triangulation of c. This intermediate step results in $(T\text{-path})_c AB + CD)$.
(3) If edge $k \in \{A, B, C, D\}$ is missing from the cluster, we construct the exchange for this edge by substituting k with $(T\text{-path})_k$ over the minimal triangulation of k
(4) Remove common factors.
(5) Linearize the polynomial by multiplying through by the least common denominator.

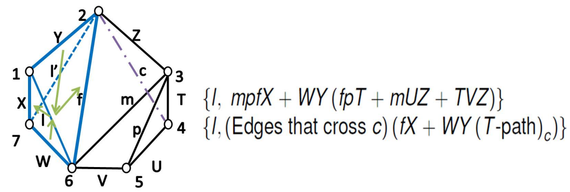

FIGURE 10. This is an example of the "modified c"-rule in the A_4 case.

The "crooked c" The unique exchange quadrilateral for x and x' is $Q(k, 2, 3, 4)$ where $k \neq 1$ and edge S is in the triangulation. Let $E_{i,j}$ be the edge between i and j and $P_{S,S'}$ be edge S's exchange polynomial.

(1) Consider the edges needed to make the Ptolemy exchange quadrilateral.
(2) The exchange polynomial is $E_{2,3}E_{4,k}S + E_{3,4}P_{S,S'}$. Note that Edges $E_{2,3}, E_{4,k}$, and $E_{3,4}$ are edges in the triangulation. An example of this exchange is shown in **??**

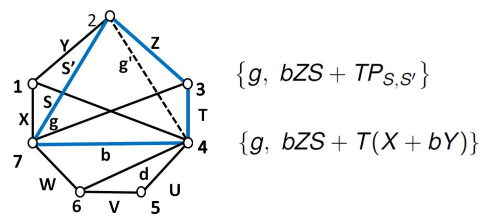

FIGURE 11. This is an example of the "crooked c" in the A_4 case.

The "1-3" rule The edge from vertex 1 to vertex 3 is an edge in $Q(1, 3, j, k)$ where $j < k \in \{5, 6, ..., n\}$. Edge M is in this quadrilateral. Label the other edges of Q as B, C, and D, and let edges M and C have no vertices in common.
Begin with the Ptolemy exchange, $MC + BD$. Because M is not in the cluster, we substitute M with $SE_3 + E_2E_4$, referring to Figure 4. The exchange polynomial is

$$(SE_3 + E_2E_4)C + BD. \tag{1.8}$$

An example of this is shown in 2.1

Remark 1.15. If edge M were in the cluster, the exchange 1.8 above, would be its regular Ptolemy exchange.

The "Ptolemy" Any valid exchange quadrilateral not mentioned above.
Label the edges of the exchange quadrilateral as A, B, C, and D where edges A and C have no common vertices. The exchange polynomial for edge x is $AC + BD$, the normal Ptolemy exchange.

2. Theorems and Results

Theorem 2.1. *The rules for the mutation we give in Section 1.6 and in Section 1.7 are consistent with the method by Lam and Pylyavskyy in* [2] *and described in Section 1.5 to produce new clusters and exchange polynomials.*

Remark 2.2. Theorem 2.1 tells us that our geometric construction yields a *LP* algebra. The cluster variables are the diagonals, minus edge M, and the clusters are the triangulations minus the triangulations with both edges M and S. Whenever M is in a trianglulation without edge S, we use S to "act in" for edge M. We call this algebra the Broken Edge algebra.

Theorem 2.3. *Given a polygon P, with non-crooked initial cluster including a, b, c, and the rest of the edges adjacent to vertex 4 as shown in Figure 12, edge k, any diagonal in P, excluding edge M can be expressed by the following formulas, which we call B-paths*

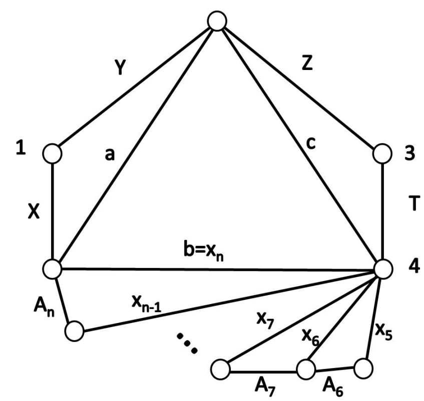

FIGURE 12. This is the general initial cluster in which we express the other edges.

Case 1 *Edge k such that k is in the cluster*

$$k = k$$

Case 2 *Edge S*

$$S = \frac{X}{a} + \frac{bY}{a} \tag{2.1}$$

Case 2 *Edge k with vertices $\{1,m\}$ where $5 \leq m \leq n-1$, i.e. edges that cross edge a and are not edge S.*

(2.2)

$$N_k = \frac{cbYx_m}{a}\left(\sum_{j=m+1}^{n-1}\frac{A_j}{x_jx_{j-1}}\right) + \frac{x_mX}{b} + \frac{x_mYA_n}{ax_{n-1}} + \frac{cx_mX}{a}\left(\sum_{j=m+1}^{n}\frac{A_j}{x_jx_{j-1}}\right)$$

Remark 2.4. Equation 2.2 is the T-path for edge k in Figure 12, with the addition of multiplying c to paths where a is in the denominator and c is not already in the numerator.

Case 3 *Any diagonal k in P not given above*

$$k = (T\text{-}path)_K \tag{2.3}$$

Theorem 2.5. *Our cluster algebra described in Theorem 2.1 above is finite. Moreover, for a polygon with $m+3$ vertices, there are $\frac{(m+2)(m-1)}{2} - 1$ cluster variables and $C_{m+1} - C_{m-1}$ clusters, where C_k is the kth Catalan number.*

2.1. **Examples.** We illustrate our mutation, exchange polynomials, and formulas in the A_4 case, a heptagon. The mutation of each diagonal is shown in Figure 13. The edge mutates from the edge within the quadrilateral to the dashed line. In Figure 13, edge f is a Case 1 mutation. The exchange polynomials are given by the following relations

$$\{l,\ mpfX + WY\,(fpT + mUZ + TVZ)\} \tag{2.4}$$

$$\{f,\ lp + Y(mU + TV)\} \tag{2.5}$$

$$\{m,\ fp + VZ\} \tag{2.6}$$

$$\{p,\ mU + TV\} \tag{2.7}$$

Edge l's exchange polynomial 2.4 is built with the modified c-rule. Edge f's exchange polynomial 2.5 is built with both the c-rules and an additional T-path. Relations 2.6 and 2.7 have normal Ptolemy exchange polynomials.

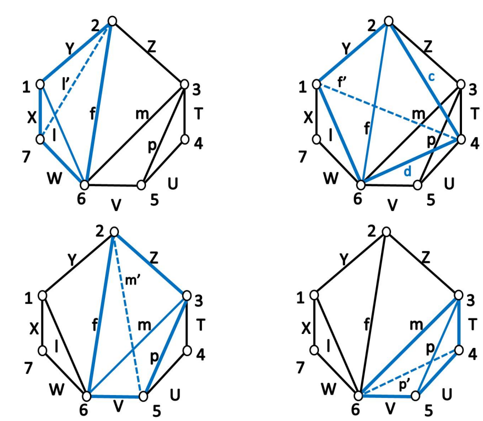

FIGURE 13. The above pictures show the same triangulation of a heptagon. Each of the four pictures shows a different exchange quadrilateral for the four diagonals of the heptagon.

We also give the formulas of the diagonals in A_4 shown in Figure 14 in terms of the initial cluster, also shown in Figure 14. These formulas are used from the formulas given in Theorem 2.3.

$$S = \frac{X}{a} + \frac{bY}{a}$$
$$f = \frac{ad}{b} + \frac{cW}{b}$$
$$g = \frac{aT}{c} + \frac{bZ}{c}$$
$$h = \frac{bV}{d} + \frac{UW}{d}$$
$$l = \frac{dX}{b} + \frac{cWX}{ab} + \frac{cWY}{a}$$
$$m = \frac{adT}{bc} + \frac{TW}{b} + \frac{dZ}{c}$$
$$n = \frac{aU}{b} + \frac{cV}{d} + \frac{cUW}{bd}$$
$$o = \frac{UX}{b} + \frac{cVX}{ad} + \frac{cUWX}{abd} + \frac{bcVY}{ad} + \frac{cUWY}{ad}$$
$$p = \frac{aTU}{bc} + \frac{TV}{d} + \frac{TUW}{bd} + \frac{UZ}{c}$$

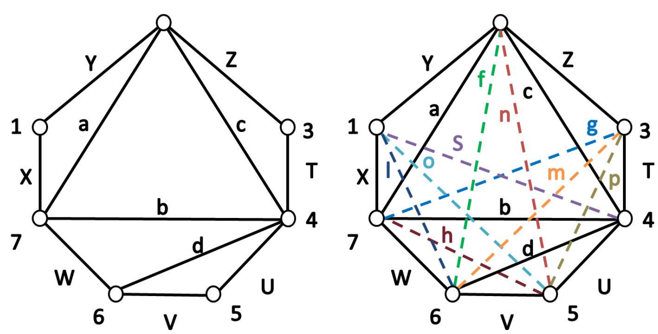

FIGURE 14. The illustration on the left is the specific initial cluster in A_4. The figure on the right is the labeling of all the diagonals in A_4.

2.2. **Conjecture that Cluster Complex is Spherical.** We conjecture the geometrical realization of the cluster complex of the Broken Ptolemy algebra is a $n-1$ dimensional sphere. We give partial results in a later section.

3. Proofs

3.1. Proof of Theorem 2.1 (Broken Edge → LP Algebra).

Proof of Theorem 2.1. Given an initial cluster and exchange polynomials as given above in Section 1.7, we prove that a mutation on a cluster variable as described in Pylyavskyy and Lam's paper [2] yields a new cluster with exchange polynomials described by our mutation and exchange rule described in Sections 1.6 and 1.7. We first prove two lemmas then directly prove our exchange through casework. □

Lemma 3.1 (Linearity of Exchange Polynomials)**.** *For any cluster with cluster variables* $(x_1, x_2, \ldots, x_n)$ *with mutation acting on* $x_k, k \in \{1, 2, \ldots, n\}$ *and boundary edges* $x_{n+1}, \ldots, x_{2n}$*, its exchange polynomial* $P(x_2, x_3, \ldots, x_n, x_{n+1}, \ldots, x_{2n})$ *as given by our rule in Section 1.7 is linear in every variable, including coefficients, the boundary edges..*

Proof of lemma. Looking at our exchange rule, the basis of each polynomial P for variable x, is the Ptolemy Exchange which is linear. It remains to show cases 1-3 are linear.

Case 1 If the c variable is being added to the normal Ptolemy Exchange of edge x, then the exchange polynomial is linear as c is not in the Ptolemy Exchange for x. If c is not in the exchange, then the linearized T-path around the minimal triangulation of c is multiplied to the appropriate term in the exchange polynomial.

There are different cases in which this can occur and are not written out in completion here. However, these cases can be checked in a similar fashion. It can be shown that the T-path factor will always be added to the side without the one edge that is in both the normal Ptolemy exchange and in the minimal triangulation, preserving linearity. All the variables in the denominators of the T-paths are not involved in the base Ptolemy Exchange, as will be shown in the next lemma, Lemma 3.2. Thus multiplying the EP through by the least common denominator of the T-path preserves linearity. Linearized T-paths are of course linear, including coefficients as each edge is only used once in any one path.

Case 2 The same minimal linearized T-path rules apply as they did in Case 1. In this case, c is being removed as a common factor of P. Linearity holds.

Case 3 This case follows from the reasoning used in Case 1.

□

Lemma 3.2 (Form of Intermediate Polynomials)**.** *For any cluster* T_0 *with cluster variables* $(x_1, x_2, \ldots, x_n) \in T_0$ *with mutation acting on* $x = x_k, k \in \{1, 2, \ldots, n\}$*, x's intermediate polynomial*

$$\hat{F}_x = \frac{P(x_1, x_2, \ldots, x_n)}{kJ_k}, \tag{3.1}$$

where J_x *is the subset of edges in the cluster that cross edges of* $Q_{x,x'}$ *that are not in the cluster* T_0*.*

Proof of lemma. One can verify that if edge $j \notin J_x$, $j \neq x$ then $Q_{j,j'}$ then $P_{j,j'}$ has at least one variable not in $P_{x,x'}$. As such, $P_{j,j'}$ does not divide $P_{x,x'}$. Thus, edge j does not appear in the denominator of $\hat{F}_x$.

If edge $j \in J_x, j \neq x$, then $P_{j,j'}$ divides $P_{x,x'}$ The minimal triangulations have a special configuration, shown in Figure 16 . All the edges that cross the edge in the desired T-path are adjacent to vertex 3. This is because of the configuration of the exchange polynomials in any given initial cluster and the mutation. There are 3 cases of where this can happen.

Case 1 This is the case where $Q_{x,x'} = Q(2,3,4,v)$, with $v \in \{5,\ldots,n\}$ and S in the cluster. Edge c is necessarily in the exchange pair. The exchange quadrilateral contains edge e_c with vertices 2 and n. S is the only edge that crosses e_c is e_c's subsitution. As a result, the exchange polynomial for edge c or its MP has a factor of S in both terms so S divides $P_{c,c'}$.

Case 2 The minimal triangulation is for the T-path over edge c. The only edges that cross c are those that are adjacent to vertex 3. The minimal triangulation then has the shape shown in Figure 17. In each T-path over c, each edge e in the cluster from vertex 3 is used exactly twice and always in the denominator. After linearizing the T-path, only the two paths that do not use e do not have e in the numerator. These two paths are visualized in Figure 19. Edge e has a normal Ptolemy exchange polynomial. After linearizing these paths, one can factor out the common variables on the top what is left is e's exchange polynomial. Therefore, e is in the denominator of F_k.

Case 3 The minimal triangulation is over a path used in Mutation Case 2(?). The edge e the T-path over is adjacent to vertex 4. The edges that cross e are adjacent to vertex 3 because the leftmost edges of the exchange quadrilateral are part of the triangulation and any edges that cross e would have to cross those two edges. Thus the configuration is the same as in Case 2 and can be seen as edge d in Figure 19.

□

Proof of Theorem 2.1. We directly insert the intermediate polynomials into a given cluster variable's exchange polynomial and verify that the new exchange polynomial is described by our rules in Section 1.7. For notation, we refer to Pylyvskyy and Lam's paper [2, section 2]. The primary cases can be identified through the mutated edge k and the edge e's exchange polynomial that $\hat{F}_k$ is being inserted into to create a new exchange polynomial.

First, we provide a few definitions for ease of notation.

Definition 3.3. We call the *c-rules exchange polynomials* those exchange polynomials that are built with either Case 1.7 or 1.7. These polynomials have an extra c or $(T\text{-path})_c$

Definition 3.4. We say an exchange polynomial has a c-$factor$ if either c or $(T\text{-path})_c$ is a common factor of the polynomial when initially building the exchange polynomial.

The cases are as follows:

(1) One or two exchange polynomials are built with the c-rules where the exchange polynomials have no c-factors that were removed and the rest are Ptolemy exchanges
(2) One polynomial is built with the c-rules and a c-factor was removed from the polynomial, and the rest are Ptolemy exchanges
(3) One polynomial is built with the c-rules without c-factors, one with the c-rules with c-factors, and the rest are Ptolemy exchanges

(4) One polynomial is built with the c-rules, one or two with the "1-3" rule, and the rest are Ptolemy exchanges
(5) One polynomial is built with the "1-3" rule, and the rest are Ptolemy exchanges.
(6) One polynomial is built with the "crooked c" rule. The rest are Ptolemy exchanges.

There are four main types of exchange polynomials: c-rules with c-factors, c-rules without c-factors, the "1-3" rule, and the normal Ptolemy Exchange. It suffices to show that the intermediate polynomials of one of the above inserted into the exchange polynomial of another variable above yields an exchange polynomial given by our rules. We describe the general outline of the proof of the different cases below. For the ease of notation and calculation, we sometimes work with the non-linearized T-paths and polynomials as there is a bijective correspondence between the non-linearized exchange polynomials and the linearized ones.

Case 1 The intial cluster has partial configuration as shown in Figure 15.

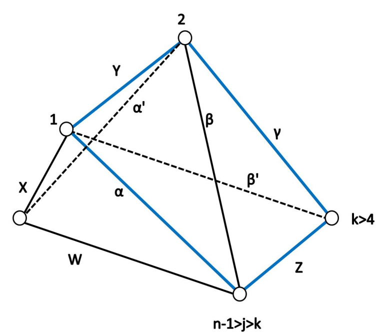

FIGURE 15. This is the base structure of P in this case. Note edge from vertex 1 to vertex 2 is in $Q_{\beta,\beta'}$ and is thus a case of the "c-rule".

The exchange polynomials of the variables with the c-rule are α and β. EPs are as follows:

$$\beta, \alpha\gamma J_c + YZlin(T\text{-paths})_c$$
$$\alpha, \beta J_c X + WYlin(T\text{-paths})_c$$

Note that the above equations are symmetric in nature, so it suffices to show that inserting $\hat{F}_\beta$ into α's EP and removing common factors yields the EP described by our mutation rules.

$$G_\alpha = \frac{J_cYZXlin(T\text{-paths})_c}{B'J_c} + WYlin(T\text{-paths})_c$$

Removing common factors and multiplying through by the least common denominator results in

$$\alpha, ZX + B'W,$$

which is what we would expect the new exchange polynomial to be after mutating edge β. Checking F_k when k is a normal exchange into a c-rule is straight forward, as is F_k when k is a c-rule into a normal exchange.

Case 2 This case will have the configuration partially given by Figure 16. The initial exchange polynomials are the following

$$\alpha,\beta XJ_c + YWlin(T\text{-paths})_c$$
$$\beta,\alpha J_d + Ylin(T\text{-paths})_d$$

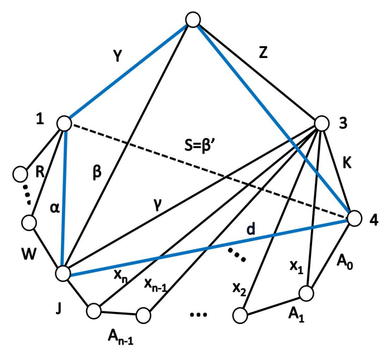

FIGURE 16. General Case of T-paths over edges c and d

To prove this case, we refer to the auxiliary diagram in Figure 17. Using this diagram, we can calculate the formula for the T-paths for c and the T-paths for edge d.

$$(T\text{-paths})_c = Z\left(\frac{A_0}{x_1} + \sum_{k=1}^{n-1}\frac{KA_k}{x_k x_{k+1}} + \frac{KJ}{\gamma x_n}\right) + \frac{\beta Y}{\gamma} \tag{3.2}$$

$$(T\text{-paths})_d = \gamma\left(\frac{A_0}{x_1} + \sum_{k=1}^{n-1}\frac{KA_k}{x_k x_{k+1}} + \frac{KJ}{\gamma x_n}\right) \tag{3.3}$$

The above equations 3.2 and 3.3 explicitly give T-paths and allow the one to do direct calculations. Note, that the formulas for these T-paths hold in all circumstances. From this and Lemmas 3.1 and 3.2, it is a matter directly computing new polynomials and verifying they are what we would expect.

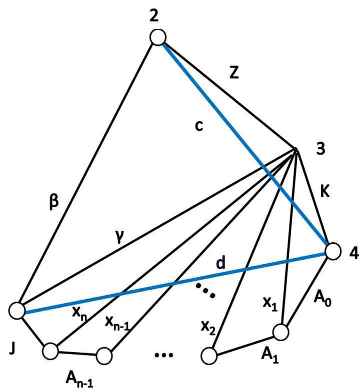

FIGURE 17. General Case of T-paths over edges c and d

Case 3 We again refer to Figure 16. When mutating at α, variable β's new exchange polynomial is the following, letting $\beta = 0$. Note, we have already modified for J_c and J_d.

$$\beta,\ YWZ\left(\frac{A_0}{x_1}+\sum_{k=1}^{n-1}\frac{KA_k}{x_kx_{k+1}}+\frac{KJ}{\gamma x_n}\right)+$$
$$\alpha' Y\gamma\left(\frac{A_0}{x_1}+\sum_{k=1}^{n-1}\frac{KA_k}{x_kx_{k+1}}+\frac{KJ}{\gamma x_n}\right)$$
$$\beta,\ YWZ\left(\frac{A_0}{x_1}+\sum_{k=1}^{n-1}\frac{KA_k}{x_kx_{k+1}}+\frac{KJ}{\gamma x_n}\right)+$$
$$\alpha' Y\gamma\left(\frac{A_0}{x_1}+\sum_{k=1}^{n-1}\frac{KA_k}{x_kx_{k+1}}+\frac{KJ}{\gamma x_n}\right)$$
$$\beta,\ WZ+\alpha'\gamma$$

Case 4 The corresponding Figure 18 diagram shows the possible configurations. Although not explicitly listed below, edge S is built with the c-rules. According to Figure 18, and our mutation rules, the initial exchange polynomials for α, β, and x_n are

$$\alpha,\ A_n(SZ+YT)+\beta x_n$$
$$\beta,\ Q(SZ+YT)+R\alpha$$
$$x_n,\ \alpha A_{n-1}+x_{n-1}A_n$$

Mutating at x_n yields

$$\alpha,\ x_n' A_n(SZ+YT)+\beta\left(\alpha A_{n-1}+x_{n-1}A_n\right),$$

We let $\alpha = 0$ in the exchange and remove common factors

$$\alpha,\ x_n'(SZ+YT)+\beta x_{n-1}$$

Mutating at the other edges is very similar.

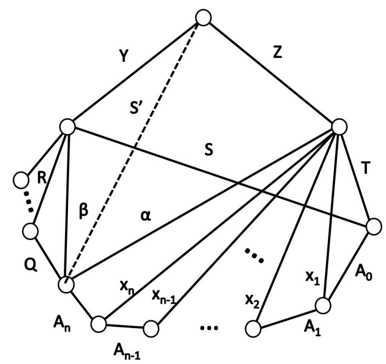

FIGURE 18. This is the base structure of P in Case 4.

Case 5 In this case, there is a variable j with EP $eJ_kAB+CD(lin(T\text{-path})_k)$ where k is some edge originating from vertex 4, and edge e, with one vertex at 3 and is in the triangulation and crosses k with a regular Ptolemy $P_{e,e'} = Xd + Zc$, where A, B, C, D (with $D = 1$ as a possibility by removing c factors) and X, Z, d, c are the edges of the respective exchange quadrilaterals of j and e, and where J_k is the set of edges that cross k, excluding e. Edges X and Z are the two edges closest to edge e with one vertex adjacent to vertex 3 and c, d are adjacent to X and Z. The set-up is partially shown in figure 19 (p2 insert). By Lemma 3.1, we know e's exchange polynomial is a factor of j. Thus, when we mutate at edge e, the following happens:

$$j,\ (Xd+Zc)J_kAB + CD\left(lin(T\text{-paths without edge j})_k(Xd+Zc) + (Xd+Zc)\frac{J_kabe'}{XZ}\right)$$

$$j,\ J_kAB + CD\left(lin(T\text{-paths without edge e})_k\right)$$

The above is the exchange we want as the two paths using edge e have been condensed into one path, with the rest of the exchange polynomial unchanged.

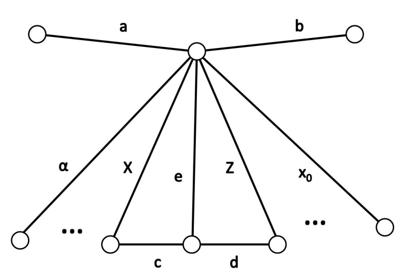

FIGURE 19. Case with Initial Cluster with $(T\text{-path})_k$, edge with regular Ptolemy Exchange crossing k with regular Ptolemy Exchange.

Case 6 Refer to Figure 20 for the first configuration. The relevant exchange polynomials are

$$S,\ Y\alpha+\gamma$$
$$\beta,\ T(Y\alpha+X)+Z\alpha S$$
$$\alpha,\ \beta A+TR$$

Note S is a factor of β and the calculations are directly done to find the desired exchange polynomials.

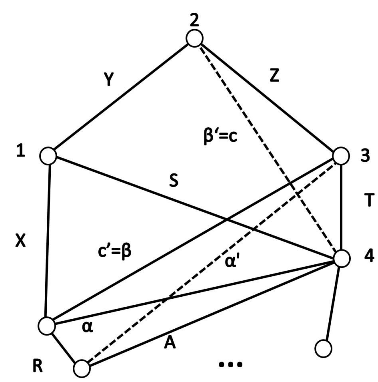

FIGURE 20. Case with Initial cluster One polynomial has exchange quadrilateral with vertices $2, 3, 4, k$ where $k \neq 1$. First configuration.

The second configuration is shown in figure 21. The relevant exchange polynomials are

$$S,\ Y\alpha+\gamma$$
$$\beta,\ T(Y\alpha+X)+Z\alpha S$$
$$\alpha,\ \beta A+TR$$

The calculations are direct as before.

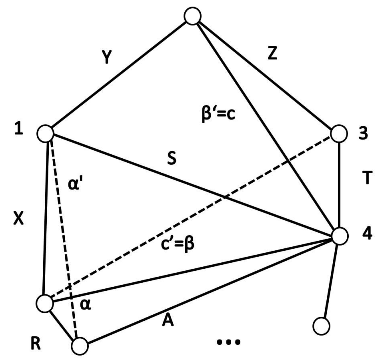

FIGURE 21. Case with Initial cluster One polynomial has exchange quadrilateral with vertices $2, 3, 4, k$ where $k \neq 1$. Second configuration.

Remark 3.5. Not all cases are explicitly calculated in this paper, especially when involving the normal Ptolemy exchange. This is because the calculation is straightforward and very similar with Lemmas 3.1 and 3.2. Any other case not fully given here uses one of the previous tools used in our proof.

□

3.2. **Proof of Theorem 2.3 (Formulas).**

Proof of Theorem 2.3. We verify these formulas by directly checking that they satisfy the exchange relations given by Theorem 2.1. If edge k is being exchanged with k' in quadrilateral $Q(v_1, v_2, v_3, v_4)$, depending upon the position of the vertices of Q, our exchange rule tells us what $k * k'$ is. Hence to verify the formulas given in Theorem 2.3, we must check all relevant cases of the location of $Q_{k,k'}$ by substituting in the formulas for all variables and making sure the equality holds.

Remark 3.6. If the unique exchange quadrilateral for k and k' is $Q(v_1, v_2, v_3, v_4)$ with $v_k \in \{5, 6, ..., n\}$. Schiffler proves in [1] that T-path formula holds for the normal Ptolemy exchange. All edges relevant to this situation fall under the case where $k = (T\text{-path})_k$, hence we must only check cases where at least one vertex of Q is in $\{1, 2, 3, 4\}$.

To simplify notation, we give the specific formula for important edges.
Edge k has vertices $\{2, m\}$ where $5 \leq m \leq n - 1$

$$P_m = \frac{ax_m}{b} + cx_m \left(\sum_{j=m+1}^{n-1} \frac{A_j}{x_j x_{j-1}} \right) + \frac{cx_m A_n}{bx_{n-1}} \tag{3.4}$$

Edge k has vertices $\{3, m\}$ where $5 \leq m \leq n - 1$

$$P_m = Tx_m \left(\sum_{j=m+1}^{n-1} \frac{A_j}{x_j x_{j-1}} \right) + \frac{Tx_m A_n}{bx_{n-1}} + \frac{Zx_m}{c} + \frac{Tax_m}{bc} \tag{3.5}$$

Edge k has vertices $\{m, p\}$ with $m, p \in \{5, 6, ..., n\}$ and $m < p$

$$R_{m,p} = x_p x_m \left(\sum_{j=m+1}^{p-1} \frac{A_j}{x_j x_{j-1}} \right) + \frac{A_p x_m}{x_{p-1}} \tag{3.6}$$

The rules described in Section 1.7 give us the cases we need to verify. These can be checked in a straightforward manner and the computations are left to the reader. The following paragraphs detail the method used to find the formulaic exchange polynomials in the cases where we do not use the normal Ptolemy exchange..

Edge k with vertices 3,p exchanging with edge k' with vertices 1,m where $5 \leq m < p \leq n$. The unique exchange quadrilateral for k and k' is $Q(1, 3, m, p)$. The exchange polynomial for k and k' is constructed using the "1-3" rule: since M is in the quadrilateral but cannot be in a cluster, we substitute $TY + SZ$ for M, resulting in the exchange relation $kk' = (TY + SZ)C + BD$ where C is the edge from p to m, B is the edge from 1 to p, and D is the edge from 3 to m. When we substitute in our formulas, we are left with

$$L_p N_m = (TY + SZ)R_{m,p} + N_p L_m$$

This equality can be easily verified through direct computation.

For the remaining cases, we just give the exchange quadrilateral and the corresponding exchange relation in terms of our formulas.

$Q(n, 1, 2, m)$:	$aN_m = XP_m + cYR_{m,n}$
$Q(1, 2, m, p)$:	$N_m P_p = P_m N_p + cYR_{m,p}$
$Q(1, 4, m, p)$:	$N_m x_p = N_p x_m + cSM_{m,p}$
$Q(1, 2, 3, m)$:	$SP_m = N_m + Yx_m$

□

3.3. **Proof of Theorem 2.5 (Finiteness).** This is a consequence of Theorem 2.3. Each edge can be represented uniquely in terms of the initial cluster. There are only finitely many possible combinations to create a B-path. The number of cluster variables and cluster mutation is counting the number of diagonals minus edge M and counting the number of triangulations that do not use both edge M and edge S.

3.4. **Partial Proof of Sphericity.** A *cluster complex* is the simplicial complex with the set of all cluster variables forming the ground set and the clusters forming the facets [2].

Partial Proof of Sphericity. We represent each diagonal by the formulas given in Section 2.2. If a diagonal is not in the initial cluster, we note that it is uniquely represented by its denominator, which represents the diagonals it crosses in the initial cluster. Hence, given an $m + 3$-gon, we can represent its diagonals as m-dimensional vectors, where each position in the vector represents a diagonal in the initial cluster, and a 1 is in that position if that diagonal is present in the denominator of the B-path of a variable. Using the labeling of Figure 9, the first entry of the vector corresponds to diagonal a, the second to b, the third to c, and the remaining entries correspond to diagonals x_{n-1} through x_6.

Remark 3.7. Diagonals in the initial seed have have 1 as their denominator; hence, we represent them in their inverse form by entering a -1 in their respective positions.

Example 3.8. Cluster $\{l, f, m, p\}$ in the A_4 case. Referring to the B-paths given in Section 2.1, we have that l has denominator ab, f has denominator b, m has denominator bc, and p has denominator bcd. Hence

$$\{l, f, m, p\} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

These vectors provide a geometric realization of a simplicial cluster complex. Each cluster is represented as a cone in $\mathbb{R}^m$. In [4], Fomin and Zelevinsky prove that the simplicial cones generated by all clusters in the Ptolemy algebra form a complete simplicial fan (Theorem 1.10). As a result, the geometric realization of the cluster complex of the Ptolemy algebra is an $(n-1)$-dimensional sphere (Corollary 1.11). This theorem is presented as a corollary of Theorem 3.11, which states that every element in $\mathbb{Z}^m$ has a unique representation as a positive integer linear combination of a simplicial cone in the fan. We prove the same is true for the Broken Ptolemy algebra.

Remark 3.9. To simplify notation, a bar on top of a variable means that it is appearing on the denominator of the B-path of a diagonal. Recall $m = n + 3$.

We compare the simplicial fan of the Ptolemy algebra to that of the Broken Ptolemy algebra. Missing edge M is represented in our formula with the denominator ac. Thus vector $\overline{ac}$ is not in the simplicial fan of the cluster complex of the Broken Ptolemy algebra, although it is in the Ptolemy algebra's cluster complex. Much like our geometric description of forming clusters in the Broken Ptolemy algebra by gluing edge M to S, we can think of the transition from the Ptolemy fan to the Broken fan as a "gluing" of $\overline{ac}$ to $\overline{a}$; the cones that include $\overline{ac}$ in the Ptolemy fan now all contain $\overline{a}$ instead and cones with both $\overline{a}$ and $\overline{ac}$ are removed.

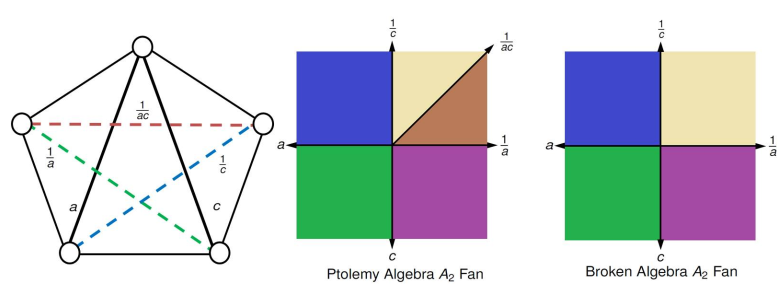

FIGURE 22. This shows how the fan changes from the Ptolemy algebra fan to the Broken Ptolemy algebra fan by the gluing of $\overline{ac}$ in the A_2 case.

Let $z \in \mathbb{Z}^m$. We know from Fomin and Zelevinsky [4] that z is uniquely represented as a linear combination of vectors that define a cone in the simplicial complex of the Ptolemy algebra with positive integer coefficients y_i, $i = 1, 2, \ldots, m$. We need to form a bijection between this representation of z and a unique representation of z as a linear combination of a cone in the simplicial complex of the Broken Ptolemy algebra, again with positive integer coefficients.

Remark 3.10. In the Ptolemy algebra fan, gluing $\overline{ac}$ to $\overline{a}$ only impacts those cones that have $\overline{ac}$ as a vector; Theorem 3.11 in [4] can still be applied to all cones unaffected by the movement of $\overline{ac}$. Cones without $\overline{ac}$ correspond to clusters in the Ptolemy algebra without edge M.

The affected cones correspond to triangulations of P that include edge M. There are four relevant cases of these triangulations. The first two cases deal with triangulations of P that have M but not S; our geometric description of clusters in the Broken Ptolemy algebra tells us that there is at least partial bijection between these clusters and clusters in the Broken Ptolemy algebra.

Case 1 M and the edge crossing only c are in the triangulation, which means that there is a $z \in \mathbb{Z}^m$ so that $z = y_1\overline{ac} + y_2\overline{c} + \sum_{i=3}^{m} y_i\overline{q_i}$ where $\overline{q_i}$ represents any other vector in the triangulation. To write z in terms of a cluster in the Broken Ptolemy algebra, we notice that a cluster with $\overline{a}$ and $\overline{c}$ is valid, and we can simply "move" the $\overline{c}$ from the $\overline{ac}$ term for the following representation in the Broken Ptolemy algebra: $z = y_1\overline{a} + (y_1 + y_2)\overline{c} + \sum_{i=3}^{m} y_i\overline{q_i}$.

Case 2 M is in the triangulation but $\overline{c}$ and $\overline{a}$ are not. This means that there must exist two edges that share some vertex $k \in \{5, 6, ..., m-1\}$ in order for the cluster to be a valid triangulation with M. These two edges must be $\overline{ab\_r_k}$ and $\overline{bc\_r_k}$ where $\_r_k$ represent the diagonals in the initial cluster that both edges cross. Figure 23 helps illustrate why both edges must cross the same diagonals. This means that there is a $z \in \mathbb{Z}^m$ so that $z = y_1\overline{ac} + y_2\overline{bc\_r_k} + y_3\overline{ab\_r_k} + \sum_{i=3}^{m} y_i\overline{q_i}$ in the Ptolemy algebra. Then, in the Broken Ptolemy algebra, we have that

$$z = \begin{cases} 2y_1\overline{a} + (y_1 + y_2)\overline{bc\_r_k} + (y_3 - y_1)\overline{ab\_r_k} + \sum_{i=3}^{m} y_i\overline{q_i} & \text{if } \ y_1 \leq y_3 \\ (y_1 + y_3)\overline{a} + (y_2 + y_3)\overline{bc\_r_k} + (y_1 - y_3)\overline{c} + \sum_{i=3}^{m} y_i\overline{q_i} & \text{if } \ y_1 \geq y_3 \end{cases}$$

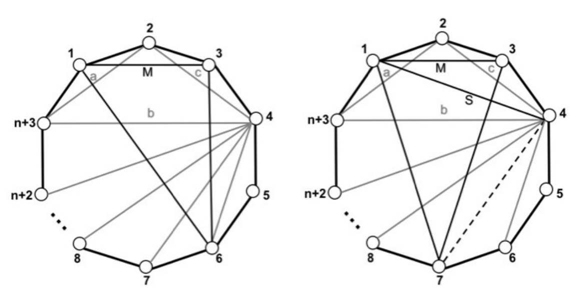

FIGURE 23. Examples of the different types of triangulations with edge M; the left figure shows how we get the edges discussed in case 2 and the right is an example of case 4.

The next two cases deal with triangulations of P that include both M and S. Our geometric description of clusters in the Broken Ptolemy algebra tell us that these triangulations do not have an analogue in the Broken Ptolemy algebra, hence it is more surprising that the following partial bijections exist.

Case 3 M, S, and b are all in the triangulation. Then there is a $z \in \mathbb{Z}^m$ so that $z = y_1\overline{ac} + y_2\overline{a} + y_3 b + \sum_{i=3}^{m} y_i\overline{q_i}$ in the Ptolemy algebra. Then, in the Broken Ptolemy algebra, $z = (y_1 + y_2)\overline{a} + y_3 b + y_1\overline{c} + \sum_{i=3}^{m} y_i\overline{q_i}$.

Case 4 M and S are in the triangulation, but b is not. This means that one triangle in the triangulation must include S, some x_k in the original cluster, and an edge from 1 to k, which will take the form $\overline{ab\_r_k}$, with $\_r_k$ representing the edges in the initial cluster between b and y_k (See Figure 23). We know that there is a cone in the Broken Ptolemy algebra that includes $\overline{ab\_r_k}$ and $\overline{bc\_r_k}$, so we have the following partial bijection: $z = y_1\overline{ac} + y_2\overline{a} + y_3\overline{ab\_r_k} + \sum_{i=3}^{m} y_i\overline{q_i}$ in the Ptolemy algebra. Then, in the Broken Ptolemy algebra,

$$z = \begin{cases} (y_2 + 2y_1)\overline{a} + y_1\overline{bc\_r_k} + (y_3 - y_1)\overline{ab\_r_k} + \sum_{i=3}^{m} y_i\overline{q_i} & \text{if } y_1 \leq y_3 \\ \\ (y_1 + y_2 + y_3)\overline{a} + y_3\overline{bc\_r_k} + (y_1 - y_3)\overline{c} + \sum_{i=3}^{m} y_i\overline{q_i} & \text{if } y_1 \geq y_3 \end{cases}$$

Hence all $z \in \mathbb{Z}^m$ represented in cones in the Ptolemy algebra including edge $\overline{ac}$ have a representation in the Broken Ptolemy algebra as linear combinations with positive integer coefficients of elements of a cone. □

We have shown that each integer point in $\mathbb{Z}^m$ has a representation in the Broken Ptolemy algebra. However, it remains to show that this representation is unique. We conjecture that the representation is unique, but at this time, we do not have a proof.

4. Acknowledgments

We would like to thank Pavlo Pylyavskyy for mentoring our research and giving us guidance. We would also like to thank Vic Reiner and Thomas McConville for their explanations of the simplicial complex and suggestions on how to proceed.

We are appreciative of the other mentors and TA of the University of Minnesota Math REU for all their support.

Thank you to the University of Minnesota Math Department for hosting the Summer 2013 REU.

This material is based upon work supported by the National Science Foundation Graduate Research Fellowship under Grant No. DMS-1067183 and DMS - 1148634.

References

[1] R. Schiffler, *A cluster expansion formula (A_n case).* E-JC **15**, (2008).
[2] P. Pylyavskyy and T. Lam, *Laurent phenomenon algebras*, preprint, (2012), arXiv:1206.2611.
[3] S. Fomin and A. Zelevinsky, *Cluster algebras II. Finite type classification*, Inventiones Mathematicae 154(1), (2003), 63-121.
[4] S. Fomin and A. Zelevinsky, *Y-systems and generalized associahedra*, Annals of Mathematics 158(3), (2003), 977-1018.
[5] Bernhard Keller, Cluster algebras, quiver representations and triangulated categories, May 2009 (http://front.math.ucdavis.edu/0807.1960).

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