Generating Functions for f-vectors of Simple Weight Polytopes

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1. Introduction to Polytopes
2. Coxeter Group and Weight Polytopes
3. f-polynomials of Simple Weight Polytopes

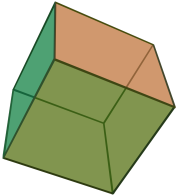
f-vector and f-polynomial

Definition (f-vector and f-polynomial)

Define the *f-vector* of a r-dim Polytope P as $f(P) := (f_0, \dots, f_r)$, where f_i is the number of i-dimensional faces of P.

Define its *f-polynomial* as $f_P(t) = \sum_{i=0}^{r} f_i t^i$.

Example:



A cube has 8 vertices, 12 edges and 6 faces.

$$f(P) = (8, 12, 6, 1)$$

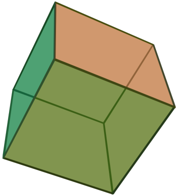
$$f_P(t) = 8 + 12t + 6t^2 + t^3$$

h-vector and h-polynomial

Definition (h-vector and h-polynomial)

Define the *h-polynomial* of a r-dim Polytope P as
$h_P(t) = f_P(t-1) = \sum_{i=0}^{r} f_i(t-1)^i$.
Assume $h_P(t) = \sum_{i=0}^{r} h_i t^i$, then define its *$h$-vector* as
$h(P) := (h_0, h_1, \dots, h_r)$.

Example:



A cube has $f_P(t) = 8+12t+6t^2+t^3$.
Replace t with $t-1$.

$$h_P(t) = f_P(t-1) = 1+3t+3t^2+t^3$$

$$h(P) = (1,3,3,1)$$

h-vector and h-polynomial

Definition (h-vector and h-polynomial)

Define the *h-polynomial* of a r-dim Polytope P as
$h_P(t) = f_P(t-1) = \sum_{i=0}^{r} f_i(t-1)^i$.
Assume $h_P(t) = \sum_{i=0}^{r} h_i t^i$, then define its *$h$-vector* as
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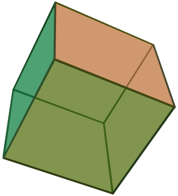
Example:



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$$h_P(t) = f_P(t-1) = 1+3t+3t^2+t^3$$

$$h(P) = (1,3,3,1)$$

Is this always symmetric?

Dehn-Somerville Equation

Definition (Simple Polytope)

A r-dimensional polytope is called a *simple polytope* if and only if each vertex has exactly r incident edges.

For example, a cube is a simple polytope.

Theorem (Dehn-Sommerville equation)

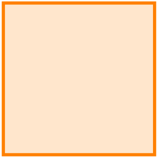
For any simple polytope P, its h-vector is symmetric.

Face Poset

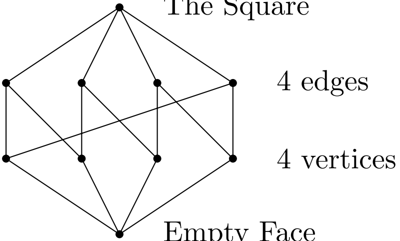
Definition (Face Poset)

The *face poset* of polytope P is the poset $\{\text{faces of } P\}$ ordered by inclusion of faces.

Example:



Polytope



Face Poset

*Note: A Face Poset is graded.

Summary

Methods to describe a polytope:

- f-polynomial/h-polynomial;
- face poset.

Section 2

Coxeter Group and Weight Polytopes

Finite Reflection groups

Definition (Finite Reflection Group)

A *finite reflection group* is a finite subgroup $W \subset \mathrm{GL}_n(\mathbb{R})$ generated by reflections, i.e. elements w such that $w^2 = 1$ and they fix a hyperplane H and negate the line perpendicular to H

Example: One example of a finite reflection group is the Dihedral Group $I_n = \{s, t \mid s^2 = t^2 = e, (st)^n = e\}$.

Coxeter groups

Definition (Coxeter Group)

A *Coxeter Group* is a group W of the form

$$W \cong \langle s_1, \dots, s_n \mid s_i^2 = e, (s_i s_j)^{m_{ij}} = e \rangle$$

for some $m_{ij} \in \{2, 3, 4, \dots\} \cup \{\infty\}$.
If W is finite, then W is called a *Finite Coxeter Group*.
$S = \{s_1, s_2, \dots, s_n\}$ is called the *Generating Set* of W.

Finite Coxeter Groups = Finite Reflection Groups

Here is a BIG theorem of Coxeter:

Theorem (Coxeter)

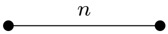
Finite Coxeter groups = Finite reflection groups.

Coxeter Diagram

Definition (Coxeter Diagram)

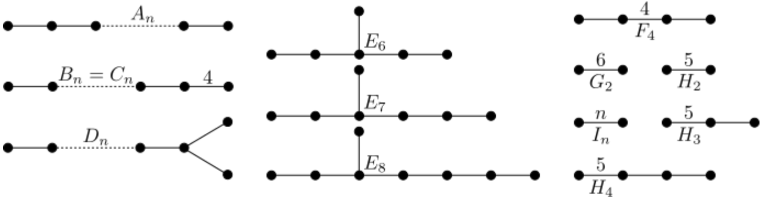
Given a Coxeter presentation (W, S), we can encapsulate it in the *Coxeter Diagram*, denoted $\Gamma(W)$, a graph with $V = S$ and if $m_{ij} = 3$, s_i and s_j are connected with no label and if $m_{ij} > 3$, s_i and s_j are connected with label m_{ij}.

Example: The dihedral group I_n has Coxeter diagram



Finite Coxeter Groups

Amazingly, finite Coxeter groups are classified! They come in four infinite families, A_n, B_n, D_n, and I_n, as well as a finite collection of exceptional cases. The Coxeter diagrams look as follows:



We will focus our energies on types A_n, B_n, D_n.

Weight Polytopes

Definition (Weight Polytope)

Given a finite Coxeter group W, $\lambda \in \mathbb{R}^n$, we define the *Weight Polytope* P_λ to be the convex hull of $\{w \cdot \lambda \mid w \in W\}$.

Weight Polytopes

Definition (Stabilizer)

Let $J(\lambda) = \{s \in S \mid s(\lambda) = \lambda\}$ be the *stabilizer* of λ.

Theorem (Maxwell)

The f-vector and face lattice of a weight polytope P_λ is only dependent on W, S and $J(\lambda)$.

Weight Polytope Example 1

Coxeter Group

$W = A_n =$ symmetric group S_{n+1}

Vector λ

$$\lambda = (\underbrace{0, \ldots, 0}_{n \text{ zeros}}, 1)$$

Weight Polytope Example 1

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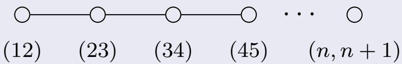

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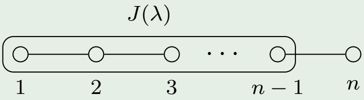
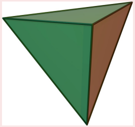


Polytope

Name: **Simplex**



Vertices: Set of vectors with n zeros and 1 one

Weight Polytope Example 2

Coxeter Group

$W = B_n =$ signed permutation group

Vector λ

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Weight Polytope Example 2

Coxeter Group

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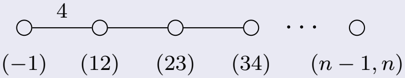

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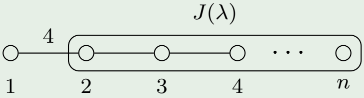
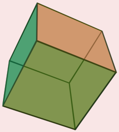


Polytope

Name: **HyperCube**



Vertices: Set of vectors with 1 and -1

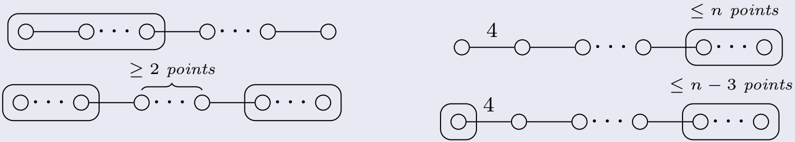
Section 3

f-polynomials of Simple Weight Polytopes

Renner's Classfication of Simple Polytopes

Theorem (Renner)

A type A_n or B_n weight polytope is simple iff its Coxeter diagram has one of the following structures.



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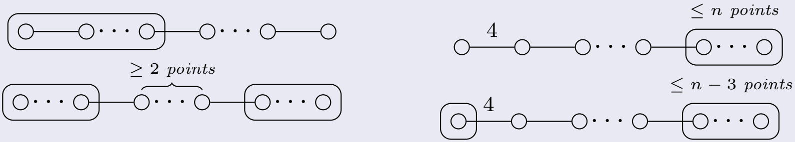
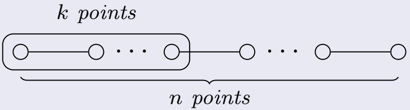


What are their f-polynomials?

Case 1

Theorem (Golubitsky)

Denote $F_{n,k}(t)$ as the f-polynomial for the f polytope of



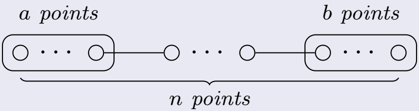
Then,

$$\sum_{n \geq k \geq 0} F_{n,k}(t) \cdot \frac{x^{n+1} y^k}{(n+1)!} = \frac{e^{xy}}{y-1} \cdot \left(y + \frac{e^{txy} - t - 1}{t + 1 - e^{tx}} \right) - 1.$$

Case 2

Theorem

Denote $F_{n,a,b}(t)$ as the f-polynomial for the f polytope of

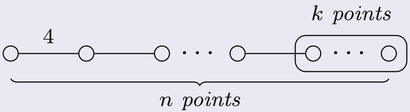


$$\text{Then, } \sum_{a,b\geq 0}\sum_{n>a+b} F_{n,a,b}(t)\cdot\frac{x^{n+1}y^a z^b}{(n+1)!} = \frac{1}{y^2-y}\Bigg(x+\frac{(xy-e^{xy}+1)(xz-e^{xz})}{y} + \frac{\left(tz+(t+1)e^{xz}-t-e^{(t+1)xz}\right)\left(\frac{ty+(t+1)e^{(xy)}-t-e^{((t+1)xy)}}{\left(t-e^{(tx)}+1\right)y}-e^{(xy)}\right)}{t(y-1)z} + \frac{e^{(xy+xz)}}{ty}+\frac{\left(ze^{(txy)}-ye^{(txz)}\right)e^{(xy+xz)}}{t(y-z)y}\Bigg).$$

Case 3

Theorem

Denote $F_{n,k}(t)$ as the f-polynomial for the f polytope of

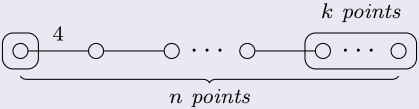


Then,
$$\sum_{n>k\geq 0} F_{n,k}(t)\cdot\frac{x^n y^k}{n!} =$$
$$\frac{1}{y-1}\left(e^{(t+2)xy}+\frac{e^{tx}\cdot\left(e^{2(t+1)xy}-(t+1)\,e^{2xy}+t-ty\right)}{(t+1-e^{2tx})y}\right).$$

Case 4

Theorem

Denote $F_{n,k}(t)$ as the f-polynomial for the f polytope of



Then,

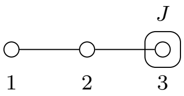
$$\sum_{n-2>k\geq 0} F_{n,k}(t)\frac{x^{n+1}y^k}{(n+1)!} = \frac{1}{y^2-y}\Bigg(xy + \left(y + \frac{(t+1)e^{(2\,xy)}}{t} - \frac{e^{(2\,(t+1)xy)}}{t} - 1\right)\left(\frac{(t+1)tx - te^{(tx)}}{t - e^{(2\,tx)}+1}+1\right) - x - \frac{\left((t+1)xy+\frac{1}{t}+1\right)e^{(2\,xy)} - \frac{e^{(2\,(t+1)xy)}}{t} - e^{((t+2)xy)}}{y}\Bigg).$$

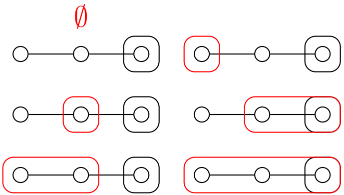
Ingredients of the Proof

Definition (J-minimal subset)

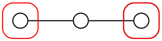
For a Coxeter diagram $\Gamma = (W, S)$ and subset $J \subseteq S$, a *J-minimal subset* is a subset $X \subseteq S$ such that no connected component of X on the Coxeter diagram lies entirely in J.

Example:





All six J-minimal subsets



Not J-minimal

Ingredients of the Proof

Theorem (Renner, Maxwell)

Consider the action of W on $\{faces\ of\ P_\lambda\}$, then there is a bijection

$$f : \{J(\lambda)\text{-}minimal\ sets\} \to \{orbits\ of\ the\ action\}.$$

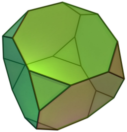
If X is $J(\lambda)$-minimal, then all faces in $f(X)$ are called X-type face. All X-type face has dimension $|X|$, and the number of X-type faces is

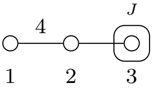
$$\frac{|W|}{|W_{X^*}|},$$

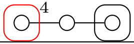
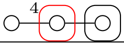
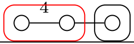
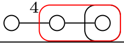
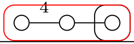
where $W_{X^*} \subseteq W$ is the subgroup generated by

$$\{s \in S | s \in X\ or\ s\ and\ X\ are\ not\ connected\}.$$

Example of Renner/Maxwell





X	Face	W_{X^*}	$\lvert W\rvert/\lvert W_{X^*}\rvert$
$\emptyset$	Vertices	$\{3\}$	$48/2 = 24$
	Long Edges	$\{1,3\}$	$48/4 = 12$
	Triangle Edges	$\{2\}$	$48/2 = 24$
	Octagons	$\{1,2\}$	$48/8 = 6$
	Triangles	$\{2,3\}$	$48/6 = 8$
	Truncated Cube	$\{1,2,3\}$	$48/48 = 1$

Summary: What have we done?

	f-polynomial	Face Poset
General Simple Weight Polytopes	✓	Maxwell (we rewrote ✓)
Weyl Group Weight Polytopes	✓ (some done by Golubitsky)	Renner
Simplex	Known	Known

The End!

Thank You!