PECKNESS OF EDGE POSETS

DAVID HEMMINGER, AARON LANDESMAN, ZIJIAN YAO

Abstract. For any graded poset P, we define a new graded poset, $\mathcal{E}(P)$, whose elements are the edges in the Hasse diagram of P. For any group, G, acting on the boolean algebra, B_n, we conjecture that $\mathcal{E}(B_n/G)$ is Peck. We prove that the conjecture holds for "common cover transitive" actions. We give some infinite families of common cover transitive actions and show that the common cover transitive actions are closed under direct and semidirect products. We also prove that $\mathcal{E}(B_n/G)$ is unimodal and symmetric in the case that G is a cyclic group or dihedral group.

Contents

1. Introduction

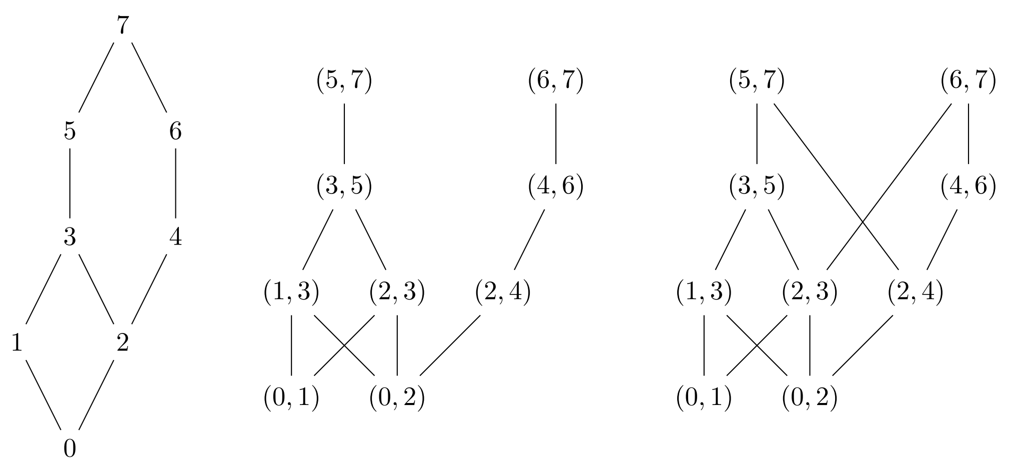
Let P be a finite graded poset of rank n. In this paper we study the structure of the edges in the Hasse diagram of P. To this end, we define an endofunctor $\mathcal{E}$ on the category finite graded posets with rank-preserving morphisms as follows

Definition 1.1. For $\mathcal{P}$ the category of graded posets, define the *functor of edges* $\mathcal{E}\colon \mathcal{P} \to \mathcal{P}$ as follows. The elements of the graded poset $\mathcal{E}(P)$ are pairs (x, y) where $x, y \in P$, $x \leq_P y$, and $\mathrm{rk}(y) = \mathrm{rk}(x)+1$. Define the covering relation $\lessdot_{\mathcal{E}}$ on $\mathcal{E}(P)$ by $(x, y) \lessdot_{\mathcal{E}} (x', y')$ if $x \lessdot_P x'$ and $y \lessdot_P y'$. Then define the relation $\leq_{\mathcal{E}}$ on $\mathcal{E}(P)$ to be the transitive closure of $\lessdot_{\mathcal{E}}$.

Let Q be a finite graded poset of rank n. Given a morphism $f\colon P \to Q$, define $\mathcal{E}(f)\colon \mathcal{E}(P) \to \mathcal{E}(Q)$ by $\mathcal{E}(f)(x, y) = (f(x), f(y))$.

We will show that $\mathcal{E}(P)$ is a well-defined graded poset in Section 3. Note that since an edge in the Hasse diagram of P can be identified with a pair $(x, y) \in P \times P$ such that $x \lessdot y$, the edges in the Hasse diagram are in bijection with elements $(x, y) \in \mathcal{E}(P)$ via this identification. With this in mind, we will frequently refer to $\mathcal{E}(P)$ as the *edge poset* of P.

Example 1.2. We give an example of an edge poset in Figure 1. Note that it is important we declare the relation $\leq_{\mathcal{E}}$ to be the transitive closure of $\lessdot_{\mathcal{E}}$. If instead we defined a relation $\leq_{\mathcal{E}'}$ on $\mathcal{E}(P)$ by $(x, y) \leq (a, b)$ if $x \leq a, y \leq b$, then $\mathcal{E}(P)$ would not necessarily be a graded poset. In Figure 1 we give an example of a poset P for which $\mathcal{E}(P)$ is not graded with the relation $\leq_{\mathcal{E}'}$. In the figure it is clear that $\mathcal{E}(P)$ is a graded poset, with $rk(x, y) = rk(x)$, but the Hasse diagram on the right represents a poset which does not have a grading.



P $\qquad$ $\mathcal{E}(P)$ $\qquad$ $\mathcal{E}(P)$ with relation $\leq_{\mathcal{E}'}$(not graded)

Figure 1. The definition of $\mathcal{E}$

We observe that when P has a nice structure, $\mathcal{E}(P)$ commonly has a nice structure as well. In particular we examine the *boolean algebra of rank n*, denoted B_n, which is defined to be the poset whose elements are subsets of $\{1, \ldots, n\}$ with the relation given by containment, i.e. for all $x, y \in B_n$, $x \leq y$ if x is a subset of y.

Throughout the paper we say a group G acts on P if it acts on the elements of P and that action is order-preserving and rank-preserving, i.e. for all $g \in G$ we have $x \leq y \Leftrightarrow gx \leq gy$ and $\mathrm{rk}(gx) = rk(x)$. By Theorem 2.6 and the fact that B_n is unitary peck, if G is any action on B_n, then B_n/G is Peck. We conjecture the following.

Conjecture 1.3. *If $G \subseteq Aut(B_n)$, then $\mathcal{E}(B_n/G)$ is Peck.*

We prove this conjecture holds whenever the group action of G on B_n has the following property.

Definition 1.4. A group action of G on P is *common cover transitive* (CCT) if whenever $x, y, z \in P$ such that $x \lessdot z$, $y \lessdot z$, and $y \in Gx$ there exists some $g \in Stab(z)$ such that $g \cdot x = y$.

Theorem 1.5. *If a group action of G on B_n is CCT, then $\mathcal{E}(B_n/G)$ is Peck.*

A large number of group actions on B_n have the CCT property. We first prove that some basic group actions on B_n are CCT. Throughout the paper we let a subgroup $G \subseteq S_n$ act on B_n by letting it act on the elements within subsets of $[n] := \{1, \dots, n\}$, i.e. $g \cdot x = \{g \cdot i \colon i \in x\}$ for all $g \in G$, $x \in B_n$. We also embed the dihedral group D_{2n} into S_n by letting it act on the vertices of an n-gon.

Proposition 1.6. *Let p be prime. The following actions are CCT.*

1. *The action of S_n on B_n.*
2. *The action of D_{2p} on B_p.*
3. *The action of D_{4p} on B_{2p}.*

We then show that cover transitivity is preserved under semidirect products, allowing us to describe several large families of CCT actions in Section 4.2.

Proposition 1.7. *Let $G \subseteq \mathrm{Aut}(P)$, $H \triangleleft G$, $K \subset G$ such that $G = H \rtimes K$. Suppose that the action of H is on P is CCT and the action of K on P/H is CCT. Then the action of G on P is CCT.*

The paper is organized as follows. In Section 2 we cover the necessary background for posets and Peck posets. In Section 3 we show that $\mathcal{E}$ is well-defined and prove Theorem 1.5 along with various other nice properties of $\mathcal{E}$. Section 4 contains the proofs of Propositions 1.6 and 1.7 as well as some examples of families of group actions shown to be CCT by these propositions. In Section 5, we demonstrate formulas for certain generalizations of the edge poset applied to the quotient of $B_{m \cdot l}/S_m \wr S_l$ and obtain as a corollary [4, Theorem 1.1]. Then, in Section 6, we use Polya theory to obtain and investigate formulas for the ranks of the generalized edge posets of B_n/G. In Section 7 we prove that $\mathcal{E}(B_n/G)$ is rank-unimodal for certain group actions that are not CCT. Then, in Section 8, we give a computational proof that $\mathcal{E}(B_n)$ is unitary Peck. Finally, in Section 9, we describe additional aspects of this problem we examined, an approach that didn't quite pan out, and further questions.

2. Background

In this section we review morphisms of graded posets, Peck posets, and a useful theorem about quotients of posets.

A *graded poset* P is a poset with a rank function $\mathrm{rk} : P \to \mathbb{Z}_{\geq 0}$ satisfying the following conditions.

1. If $x \in P_i$ and $x \lessdot y$, then $rk(x) + 1 = rk(y)$,
2. If $x < y$ then $\mathrm{rk}(x) < rk(y)$

Define the ith rank of P to be $P_i = \{x \in P \colon \mathrm{rk}(x) = i\}$. Additionally, if for all $x \in P, 0 \leq \mathrm{rk}(x) \leq n$, and there exists x, y with $\mathrm{rk}(x) = 0, rk(y) = n$, we say P is a graded poset of *rank n*.

Denote the category of finite graded posets by $\mathcal{P}$, and let P, Q be finite graded posets of rank n. Throughout the paper we write $x \leq_P y$ to denote that x is less than or equal to y under the relation defined on the poset P. When the poset is clear we will omit the P and simply write $x \leq y$.

A map $f \colon P \to Q$ is a *morphism* from P to Q if it is rank-preserving and order preserving, i.e. for all $x, y \in P$, $x \leq_P y \Rightarrow f(x) \leq_Q f(y)$ and $\mathrm{rk}(x) = \mathrm{rk}(f(x))$. We say that f is *injective/surjective/bijective* if it is an injection/surjection/bijection from P to Q as sets.

Remark 2.1. Note that we do not require the implication $f(x) \leq_Q f(y) \Rightarrow x \leq_P y$ in order for f to be a morphism. In particular this means a bijective morphism f need not be an isomorphism, since it will not necessarily have a two-sided inverse.

Write $p_i = |P_i|$. If we have

$$p_0 \leq p_1 \leq \ldots \leq p_k \geq p_{k+1} \geq \ldots \geq p_n$$

for some $0 \leq k \leq n$, then P is *rank-unimodal*, and if $p_i = p_{n-i}$ for all $1 \leq i \leq n$, then P is *rank-symmetric*. An *antichain* in P is a set of elements in P that are pairwise incomparable. If no antichain in P is larger than the largest rank of P, then P is *Sperner*. More generally, P is *k-Sperner* if no union of k antichains in P is larger than the union of the largest k ranks of P, and P is *strongly Sperner* if it is k-Sperner for all $1 \leq k \leq n$. We then make the following definition.

Definition 2.2. P is *Peck* if P is rank-symmetric, rank-unimodal, and strongly Sperner.

Let $V(P)$ and $V(P_i)$ be the complex vector spaces with bases $\{x : x \in P\}$ and $\{x : x \in P_i\}$ respectively. Note that we will frequently abuse notation and write P and P_i for $V(P)$ and $V(P_i)$ when our meaning is clear. In determining whether P is Peck, it is often useful to consider certain linear transformations on $V(P)$.

Definition 2.3. A map $U\colon V(P) \to V(P)$ is an *order-raising operator* if $U(V(P_n)) = 0$ and for all $0 \leq i \leq n-1$, $x \in P_i$ we have

$$U(x) = \sum_{y \gtrdot x} c_{x,y} y$$

for some constants $c_{x,y} \in \mathbb{C}$. We say that U is the *Lefschetz map* if all $c_{x,y}$ on the right hand side are equal to 1.

We then have the following well-known characterization of Peck posets.

Lemma 2.4 ([5], Lemma 1.1)**.** *P is Peck if and only if there exists an order-raising operator U such that for all $0 \leq i < \frac{n}{2}$, the map $U^{n-2i}\colon V(P_i) \to V(P_{n-i})$ is an isomorphism.*

Definition 2.5. If the Lefschetz map satisfies the condition for U in Lemma 2.4, then P is *unitary Peck*.

We say that a group G acts on P if the action defines and embedding $G \hookrightarrow \mathrm{Aut}(P)$, and define the *quotient poset* P/G to be the poset whose elements are the orbits of G, with the relation $\mathcal{O} \leq \mathcal{O}'$ if there exist $x \in \mathcal{O}$, $x' \in \mathcal{O}'$ such that $x \leq x'$. We will use the following result later in the paper.

Theorem 2.6 ([6], Theorem 1)**.** *If P is unitary Peck and $G \subseteq \mathrm{Aut}(P)$, then P/G is Peck.*

3. The Edge Poset Construction

In Section 3.1 we show that $\mathcal{E}$ as described in Definition 1.1 is well-defined and prove some useful properties of $\mathcal{E}$. Section 3.2 is devoted to the proof of Theorem 1.5. In Section 3.3 we discuss a possible generalization for $\mathcal{E}$.

3.1. Definition and Basic Properties.

First we show that $\mathcal{E}$ is well-defined in Lemmas 3.1 and 3.2. After showing that $\mathcal{E}$ is well-defined we then define a natural G action on $\mathcal{E}(P)$ and define a surjection $\mathcal{E}(P)/G \to \mathcal{E}(P/G)$ that will be important for the proof of Theorem 1.5. Then, we prove the simple result that $\mathcal{E}$ sends self-dual posets to self-dual posets. Finally, we note several equivalent definitions of CCT actions.

3.1.1. *Functoriality of $\mathcal{E}$ and Group Actions.* When the poset P is clear, we will simply use $\leq_{\mathcal{E}}$ and $\lessdot_{\mathcal{E}}$ to refer to $\leq_{\mathcal{E}(P)}$ and $\lessdot_{\mathcal{E}(P)}$. Similarly, in Subsection 3.2, we define an object $\mathcal{H}(B_n)$, and will use $\leq_{\mathcal{H}}$ and $\lessdot_{\mathcal{H}}$ in place of $\leq_{\mathcal{H}(P)}$ and $\lessdot_{\mathcal{H}(P)}$.

Lemma 3.1. *The relation $\leq_{\mathcal{E}}$ defines a partial order on $\mathcal{E}(P)$.*

Proof. We have that $(x, y) \leq_{\mathcal{E}} (x, y)$ and that $\leq_{\mathcal{E}}$ is transitive by definition. It remains to be shown that $\leq_{\mathcal{E}}$ is antisymmetric. Suppose $(x, y) \leq_{\mathcal{E}} (x', y')$ and $(x', y') \leq_{\mathcal{E}} (x, y)$. Then $x \leq_P x' \leq_P x$ and $y \leq_P y' \leq_P y$, so $x = x'$ and $y = y'$ by antisymmetry of $\leq_P$, hence $(x, y) = (x', y')$. □

Lemma 3.2. *For P a graded poset, the object $\mathcal{E}(P)$ is a graded poset.*

Proof. To show $\mathcal{E}(P)$ is graded, we must show that $(x, y) \lessdot_{\mathcal{E}} (x', y') \implies \mathrm{rk}(x, y) + 1 = \mathrm{rk}(x', y')$. This fact follows immediately from the definition of $\lessdot_{\mathcal{E}}$ and the definition $\mathrm{rk}_{\mathcal{E}}(x, y) = \mathrm{rk}_P(x)$. □

In order to define a group action, we will define more generally a way for morphisms of posets to induce morphisms of their edge posets.

Lemma 3.3. *Let $f\colon P \to Q$ be a morphism of finite graded posets, and define a map $\mathcal{E}(f)\colon \mathcal{E}(P) \to \mathcal{E}(Q)$ by $\mathcal{E}(f)(x, y) = (f(x), f(y))$ for all $(x, y) \in \mathcal{E}(P)$. Then*

(1) *$\mathcal{E}(f)$ is a morphism of finite graded posets*
(2) *$\mathcal{E}(\mathrm{id}_P) = \mathrm{id}_{\mathcal{E}(P)}$*
(3) *If $g\colon Q \to R$ is a morphism of finite graded posets, then $\mathcal{E}(g \circ f) = \mathcal{E}(g) \circ \mathcal{E}(f)$.*

Proof. Part (1) First, $\mathcal{E}(f)$ is rank-preserving, since for all $(x, y) \in \mathcal{E}(P)$ we have

$$\mathrm{rk}_{\mathcal{E}}(x, y) = \mathrm{rk}_P(x) = \mathrm{rk}_P(f(x)) = \mathrm{rk}_{\mathcal{E}}(\mathcal{E}(f)(x, y)).$$

Suppose $(x, y) \leq_{\mathcal{E}} (x', y')$. Then $x \leq_P x'$, $y \leq_P y'$, and since f is order-preserving, it follows that $f(x) \leq_P f(x')$, $f(y) \leq_P f(y')$. Hence $\mathcal{E}(f)(x, y) \leq_{\mathcal{E}} \mathcal{E}(f)(x', y')$. Thus $\mathcal{E}(f)$ is order-preserving and hence a morphism of finite ordered posets.

Part (2) This is trivial.

Part (3) For all $(x, y) \in \mathcal{E}(P)$ we have

$$\mathcal{E}(g \circ f)(x, y) = (g(f(x)), g(f(y))) = (\mathcal{E}(g) \circ \mathcal{E}(f))\,(x, y).$$

□

Remark 3.4. By Lemma 3.3, the edge poset construction $\mathcal{E}$ defines an endofunctor on the category finite graded posets with rank-preserving morphisms.

Given a group action of G on P, we can now easily define a natural group action of G on $\mathcal{E}(P)$ using Lemma 3.3. For all $g \in G$ we have that multiplication by g is an automorphism of P, so it follows that $\mathcal{E}(g)$ is an automorphism of $\mathcal{E}(P)$ and furthermore that this gives a well-defined group action by Lemma 3.3.

Definition 3.5. Given a G-action on P, define a G-action on $\mathcal{E}(P)$ by $g\cdot(x, y) = \mathcal{E}(g)(x, y) = (gx, gy)$.

We then have a well-defined quotient poset $\mathcal{E}(P)/G$. It is natural to ask whether the operation of quotienting out by G commutes with $\mathcal{E}$, that is, whether $\mathcal{E}(P/G) \cong \mathcal{E}(P)/G$. Unfortunately the two posets are rarely isomorphic, but there is always a surjection $\mathcal{E}(P)/G \to \mathcal{E}(P/G)$, and this surjection is also an injection precisely when the G-action on P is CCT, as will be shown in Lemma 3.14.

Proposition 3.6. *The map $q\colon \mathcal{E}(P)/G \to \mathcal{E}(P/G)$ defined by $q(G(x, y)) = (Gx, Gy)$ is a surjective morphism.*

Proof. Note that q is well defined because if $(x', y') = g(x, y) = (g \cdot x, g \cdot y)$ for some $g \in G$, then $x' \in Gx$ and $y' \in Gy$. Clearly q is rank-preserving and surjective, so it suffices to show that q is order-preserving. Suppose that $G(x, y) \leq G(w, z)$. Then there exist some $(x_0, y_0) \in G(x, y)$, $(w_0, z_0) \in G(w, z)$ such that $x_0 \leq w_0$ and $y_0 \leq z_0$. We then have that $(Gx, Gy) \leq (Gw, Gz)$ by definition, hence q is order-preserving. □

3.1.2. *The Opposite Functor and Self Dual Posets.* Next, we introduce the notion of a dual poset, given by applying the opposite functor, op, to a graded poset. We will show *op* commutes with $\mathcal{E}$. This will imply $\mathcal{E}(P)$ is self-dual if P is, which, in turn, will imply $\mathcal{E}(B_n/G)$ is self-dual for any group action of G on B_n.

Definition 3.7. Let $\mathcal{P}$ be the category of graded posets and let $op : \mathcal{P} \to \mathcal{P}$ be the opposite functor, on posets follows. For P a poset, the elements of P^{op} are the same as those of P with order relation $\leq_{P^{op}}$ defined by $x \leq_{P^{op}} y \Leftrightarrow x \geq_P y$. Induced maps on morphisms are given as follows: for P, Q graded posets with $f : P \to Q$, then $f^{op} : P^{op} \to Q^{op}$ is defined by $f^{op}(x) = f(x)$. The poset P^{op} is called the *dual* poset of P. elements A poset P is *self-dual* if there is an isomorphism of posets $P \cong P^{op}$.

Remark 3.8. Note that it is easy to check $op : \mathcal{P} \to \mathcal{P}$ is indeed a covariant functor. In more abstract terms, if we view P as a category, P^{op} is the opposite category. Additionally, *op* as defined in this way is actually a endofunctor on the category of all finite posets, which restricts to a functor on the subcategory of graded posets.

Lemma 3.9. *The functor $op : \mathcal{P} \to \mathcal{P}$ commutes with the functor $\mathcal{E} : \mathcal{P} \to \mathcal{P}$. That is, $\mathcal{E}(P^{op}) \cong \mathcal{E}(\mathcal{P})^{op}$.*

Proof. Observe that $\mathcal{E}(P^{op})$ is canonically isomorphic to $\mathcal{E}(P)^{op}$, as given by the map $F : \mathcal{E}(P^{op}) \to \mathcal{E}(P)^{op}, (x, y) \mapsto (x, y)$. The inverse to F is given by $G : \mathcal{E}(P)^{op} \to \mathcal{E}(P^{op}), (x, y) \mapsto (x, y)$. These maps are well defined because $\mathcal{E}(P)^{op}$ and $\mathcal{E}(P^{op})$ are the same as sets, and it follows from the definitions that these are both morphisms of graded posets, and so F defines an isomorphism. □

Proposition 3.10. *If P is a self-dual poset, then $\mathcal{E}(P)$ is also self-dual.*

Proof. Since P is self-dual, there is an isomorphism $f : P \to P^{op}$. By functoriality of $\mathcal{E}$, as shown in Lemma 3.3, we obtain that $\mathcal{E}(f) : \mathcal{E}(P) \to \mathcal{E}(P^{op})$ is an isomorphism. By Lemma 3.9, there is an isomorphism $\mathcal{E}(P^{op}) \cong \mathcal{E}(\mathcal{P})^{op}$. Then, letting $F : \mathcal{E}(P^{op}) \to \mathcal{E}(P)^{op}, (x, y) \mapsto (x, y)$ be the same isomorphism defined in the proof of 3.9, the composition $F \circ \mathcal{E}(f) : \mathcal{E}(P) \to \mathcal{E}(P)^{op}$ defines an isomorphism, so $\mathcal{E}(P)$ is self-dual. □

Example 3.11. While $\mathcal{E}(P)$ is commonly Peck when P is Peck, $\mathcal{E}(P)$ need not be Peck in general. Furthermore, adding the condition that P be self-dual does not change this fact. In Figure 3.1.2 we give an example of a poset P such that P is unitary Peck and self-dual, but $\mathcal{E}(P)$ is not rank-unimodal.

Remark 3.12. Whenever there is an action $\psi : G \times [n] \to [n]$, we obtain an induced action $\phi : G \times B_n \to B_n$ defined by

$$\phi(g, \{x_1, \ldots, x_k\}) = \{\psi(g, x_1), \ldots, \psi(g, x_k)\}.$$

It is easy to see that any action $\phi : G \times B_n \to B_n$ arises in this way. That is, for any action ϕ of G on B_n there exists an action ψ of G on $[n]$ such that $\phi(g, \{x_1, \ldots, x_k\}) = \{\psi(g, x_1), \ldots, \psi(g, x_k)\}$. This fact, that all actions on B_n are induced by actions on $[n]$ follows from the more general fact about atomistic lattices. Recall, L is an atomistic lattice if there exists a minimum element of L and any element of L can be expressed as a join of a subset of the elements, called atoms. Then, for L an atomistic lattice, any poset automorphism $f : L \to L$ restricts to an automorphism of the atoms of L, and f is uniquely determined by this restriction to atoms. In the case of B_n, the atoms are precisely the

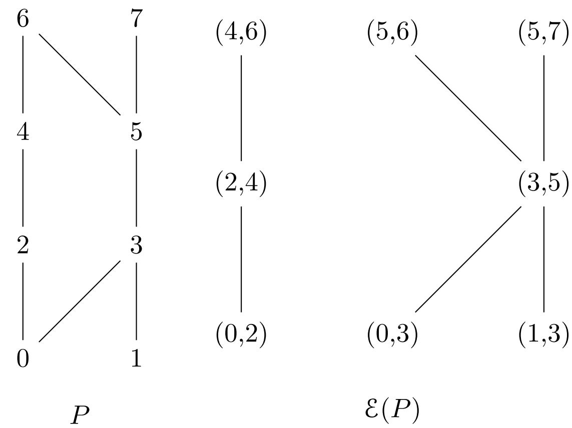


FIGURE 2. P is self-dual and unitary Peck, but $\mathcal{E}(P)$ is not Peck

singletons, and hence can be identified with $[n]$. Whenever an action ψ of G on $[n]$ is given, we refer to the action ϕ defined above as the *induced action* on B_n.

Corollary 3.13. *For any action $\phi : G \times B_n \to B_n$, both $\mathcal{E}(B_n/G)$ and $\mathcal{E}(B_n)/G$ are self-dual. In particular, they are both rank-symmetric.*

Proof. By Remark 3.12, any action $\phi : G \times B_n \to B_n$ is induced by an action $\psi : G \times [n] \to [n]$. Using this, observe that for any ϕ, the poset B_n/G is self-dual, as there is an isomorphism $f : B_n/G \to (B_n/G)^{op}, G \cdot x \mapsto G \cdot ([n] \setminus x)$. This map is well defined on G orbits because every action on B_n is induced by an action on $[n]$. Then, by Proposition 3.10, it follows that $\mathcal{E}(B_n/G)$ is self-dual.

It only remains to prove that $\mathcal{E}(B_n)/G$ is self-dual. However, from Proposition 3.10, $\mathcal{E}(B_n)$ is self-dual, with the isomorphism given by $\mathcal{E}(f) : \mathcal{E}(B_n) \to \mathcal{E}(B_n^{op}) \cong \mathcal{E}(B_n)^{op}$, that is, the map sending a pair of elements to the pair of their complements. Once again, since the action on B_n is induced by an action on $[n]$, this isomorphism descends to an isomorphism $\mathcal{E}(f)^G : \mathcal{E}(B_n)/G \to (E(B_n)/G)^{op}$, and so $\mathcal{E}(B_n)/G$ is self-dual. □

3.1.3. *Equivalent Definitions of CCT Actions.* We conclude this subsection by giving four equivalent definitions of CCT actions. The equivalence of (1) and (2) in the following Lemma 3.14 uses the notion of dual posets, while the equivalence of $(1), (3)$, and (4) use the fact that $q : \mathcal{E}(P)/G) \to (\mathcal{E}(P/G)), G(x, y) \mapsto (Gx, Gy)$ is always a surjection.

Lemma 3.14. *Let G be a group acting on a graded poset P. The following are equivalent:*

(1) *The action of G on P is CCT.*
(2) *Whenever $x \lessdot y, x \lessdot z$, and $y \in Gz$, there exists some $g \in Stab(x)$ with $gx = z$.*
(3) *The map $q \colon \mathcal{E}(P)/G \to \mathcal{E}(P/G)$ defined by $q(G(x, y)) = (Gx, Gy)$ is a bijective morphism (but not necessarily an isomorphism).*
(4) *For all i there is an equality $|(\mathcal{E}(P)/G)_i| = |(\mathcal{E}(P/G))_i|$.*

Proof. First, we show (1) $\Leftrightarrow$ (2). There is an isomorphism $f : B_n \to B_n^{op}, x \mapsto [n] \setminus x$. This defines a bijection between triples (x, y, z) with $x \lessdot z, y \lessdot z$ with $x \in Gy$, and triples (a, b, c) with $c \lessdot a, c \lessdot b$ with $a \in Gb$, given by $(x, y, z) \mapsto (f(x), f(y), f(z))$. Furthermore, some $g \in G$ satisfies $g \in Stab(z)$ and $gx = y$, if and only if it also satisfies $g \in Stab(f(z))$ and $g \cdot f(x) = f(y)$. If an action is CCT, all triples (x, y, z) satisfy these properties, and condition (2) says all triples (a, b, c) satisfy these properties. So, the above shows that (1) is equivalent to (2).

Second, we show (1) $\Leftrightarrow$ (3). Observe that q is a bijection exactly when there do not exist distinct orbits $G(x, y) \neq G(x', y')$ with $x' \in Gx$, $y' \in Gy$. Fix $(x, y), (x', y') \in \mathcal{E}(P)$ such that $x' \in Gx$ and $y' \in Gy$. Pick a $g \in G$ such that $g \cdot y' = y$. Then $(g \cdot x', y) \in G(x', y')$, so $G(x,y) = G(x',y')$ if and only if there exists some $g' \in G$ such that $g' \cdot x = g \cdot x'$ and $g' \cdot y = y$. Hence q is a bijection if and only if the G action is CCT.

Finally, we check (3) $\Leftrightarrow$ (4). Again using Proposition 3.6, the morphism q is always surjective. Since a morphism is always rank preserving, it must map $(\mathcal{E}(P)/G)_i$ surjectively onto $(\mathcal{E}(P/G))_i$. However, since the posets are finite, this surjection is a bijection if and only if the sets have the same cardinality. $\square$

Remark 3.15. While q is a bijection if and only if the action of G on P is CCT, it is *not* true that if the action of G on P is CCT, then q is an isomorphism. For example, take $G = D_{20} \subset S_{10}$ acting by reflections and rotations on $[10]$ and hence acting on B_{10}. From Proposition 1.6, this action is CCT. However, consider $x = \{2,4\}, y = \{1,2,4\}, a = \{2,4,7\}, b = \{2,4,6,7\}$. Then we may observe $(x,y),(a,b) \in \mathcal{E}(B_{10})$ and $Gx < Ga, Gy < Gb$, so $(Gx,Gy) <_{\mathcal{E}} (Ga,Gb)$. However, it is not true that $G(x,y) <_{\mathcal{E}} G(a,b)$.

3.2. **Proof of Theorem 1.5.** In this section we prove Theorem 1.5, which we recall here:

Theorem 1.5. *If a group action of G on B_n is CCT, then $\mathcal{E}(B_n/G)$ is Peck.*

The proof is largely motivated by the following Lemma.

Lemma 3.16. *Let P, Q two graded posets with a morphism $f : P \to Q$ that is a bijection (but not necessarily an isomorphism). If P is Peck, then Q is Peck.*

Proof. Let $\mathrm{rk}(P) = \mathrm{rk}(Q) = n$. Since P is Peck there exists an order-raising operator U such that $U^{n-2i} \colon P_i \to P_{n-i}$ is an isomorphism. Since f is a poset morphism, it follows that the map $f \circ U \circ f^{-1}$ is an order-raising operator on Q. We then have that $f \circ U^{n-2i} \circ f^{-1} = \left(f \circ U \circ f^{-1}\right)^{n-2i} : Q_i \to Q_{n-i}$ is an isomorphism since $U^{n-2i} \colon P_i \to P_{n-i}$ is an isomorphism and f is a bijection.

$\square$

By Lemma 3.16 and Proposition 3.6, in order to prove Theorem 1.5 it suffices to prove that $\mathcal{E}(B_n)/G$ is Peck. One way to do this is to prove that $\mathcal{E}(B_n)$ is unitary Peck and then apply Theorem 2.6. In fact, this approach generalizes to an arbitrary poset P.

Theorem 3.17. *If the action of G on P is CCT and $\mathcal{E}(P)$ is unitary Peck, then $\mathcal{E}(P/G)$ is Peck.*

Proof. Since the G-action is CCT, there is a bijection $q \colon \mathcal{E}(P)/G \to \mathcal{E}(P/G)$ by Lemma 3.6. Since $\mathcal{E}(P)$ is unitary Peck we have that $\mathcal{E}(P)/G$ is Peck by Theorem 2.6, hence $\mathcal{E}(P/G)$ is Peck by Lemma 3.16. $\square$

We prove that $\mathcal{E}(B_n)$ is unitary Peck in Section 8, but unfortunately the proof is messy and computational. Fortunately there is a simpler – albeit less direct – route. In order to avoid showing that $\mathcal{E}(B_n)$ is unitary Peck, we define a graded poset $\mathcal{H}(B_n)$ for all n such that $\mathcal{H}(B_n)$ is easily seen to be unitary Peck in Corollary 3.26. Furthermore, we define $\mathcal{H}(B_n)$ such that a group action of G on B_n induces a group action on $\mathcal{H}(B_n)$ (Lemma 3.23) and there is always a bijective morphism $f \colon \mathcal{H}(B_n)/G \to \mathcal{E}(B_n)/G$ (Lemma 3.24). By the above discussion, Theorem 1.5 readily follows.

Definition 3.18. For P a graded poset, define the graded poset $\mathcal{H}(P)$ as follows. Let the elements $(x,y) \in \mathcal{H}(P)$ to be pairs $(x,y) \in P \times P$ such that $x \lessdot y$. Define $(x,y) \lessdot_{\mathcal{H}} (x',y')$ if $x \lessdot x', y \lessdot y'$ and $x' \neq y$. Then, define $\leq_{\mathcal{H}}$ to be the transitive closure of $\lessdot_{\mathcal{H}}$. and define $\mathrm{rk}_{\mathcal{H}}(x,y) = \mathrm{rk}_P(x)$.

Example 3.19. We give an example of the poset $\mathcal{H}(B_3)$ in Figure 3. Observe that $\mathcal{H}(B_3)$ can be written as a disjoint union of three copies of B_2. This is a single case of the more general phenomenon proven in Proposition 3.25.

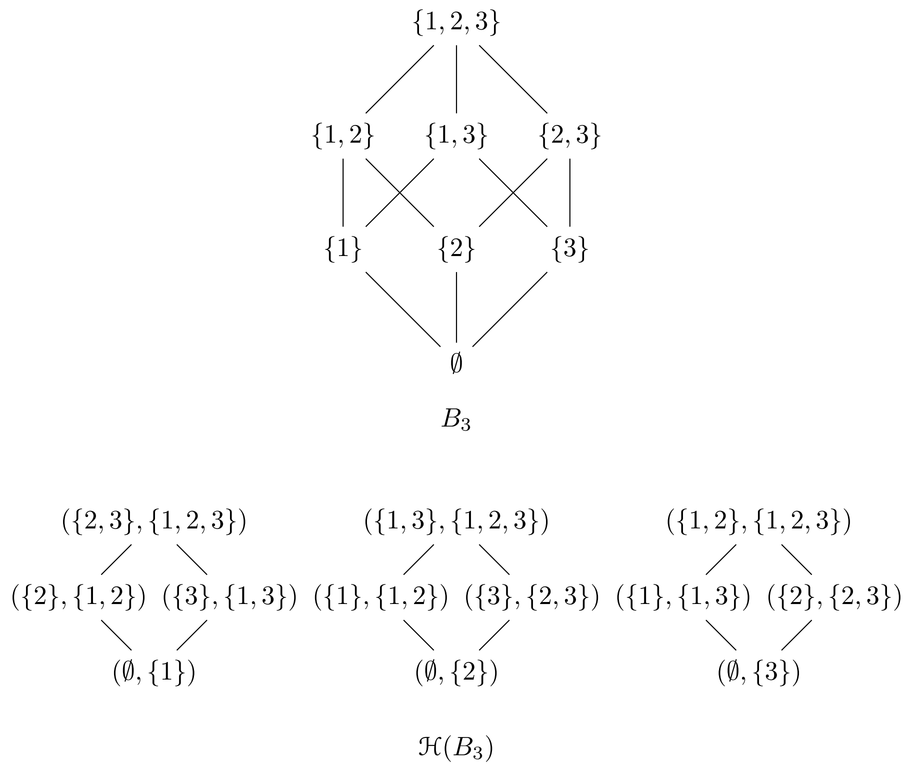


FIGURE 3. B_3 and $\mathcal{H}(B_3)$

Remark 3.20. Note that by definition $(x, y) \lessdot_{\mathcal{H}} (x', y')$ precisely when $(x, y) \lessdot_{\mathcal{E}} (x', y')$ and $x' \neq y$, hence $(x, y) \lessdot_{\mathcal{H}} (x', y') \Rightarrow (x, y) \lessdot_{\mathcal{E}} (x', y')$. In other words, $\mathcal{H}(P)$ has the same elements as $\mathcal{E}(P)$ but with a weaker partial order.

Lemma 3.21. *For P a graded poset, the object $\mathcal{H}(P)$, as defined in Definition 3.18 is a graded poset.*

Proof. This follows immediately from Remark 3.20 and the fact that $\mathcal{E}(P)$ is graded. □

Remark 3.22. While $\mathcal{E} : \mathcal{P} \to \mathcal{P}$ is a functor, $\mathcal{H}$ is not a functor. In particular, there is no possible definition of how $\mathcal{H}$ acts on morphisms. This is illustrated in 3.22. For example, suppose we took $f : P \to Q$ defined by $f(1) = 1, f(2) = f(3) = 2, f(4) = 3$. It is easy to see that there is no possible morphism $\mathcal{H}(f) : \mathcal{H}(P) \to \mathcal{H}(Q)$ because there are no morphisms at all between $\mathcal{H}(P) \to \mathcal{H}(Q)$.

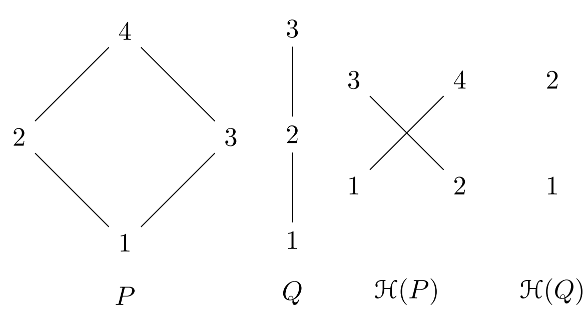


FIGURE 4.

Given an action of a group G on P, we define an action of G on $\mathcal{H}(P)$ as we did for $\mathcal{E}(P)$ by again defining $g \cdot (x, y) = (gx, gy)$ for all $(x, y) \in P$. We will then have a well-defined quotient poset $\mathcal{H}(P)/G$ with the same elements as $\mathcal{E}(P)/G$.

Lemma 3.23. *The function defined by $g \cdot (x, y) = (gx, gy)$ for all $g \in G$, $(x, y) \in \mathcal{H}(P)$ is a well-defined group action of G on $\mathcal{H}(P)$.*

Proof. Let $g \in G$. Since $\leq_{\mathcal{H}}$ is the transitive closure of $\lessdot_{\mathcal{H}}$ it suffices to show that for all $(x, y), (x', y') \in \mathcal{H}(P)$ we have $(x, y) \lessdot_{\mathcal{H}} (x', y') \Leftrightarrow g(x, y) \lessdot_{\mathcal{H}} g(x', y')$. Since g is an automorphism of P, we have $x \leq_P x' \Leftrightarrow gx \leq_P gx'$, $y \leq_P y' \Leftrightarrow gy \leq_P gy'$, and $y \neq x' \Leftrightarrow gy \neq gx'$, so the result follows from the definition of $\leq_{\mathcal{H}}$. $\square$

Lemma 3.24. *The map $f : \mathcal{H}(P)/G \to \mathcal{E}(P)/G$ defined by $G(x, y) \mapsto G(x, y)$ is a bijective morphism for any group action of G on P.*

Proof. The elements of $\mathcal{H}(P)/G, \mathcal{E}(P)/G$ are the same by definition, so it suffices to show that f is a morphism. Since f is clearly rank-preserving, it suffices to show f is order-preserving. This is immediate from Remark 3.20. $\square$

The remaining step in the proof of Theorem 1.5 is to show that $\mathcal{H}(B_n)$ is unitary Peck, which we do by generalizing Example 3.19 and showing that $\mathcal{H}(B_n)$ is isomorphic to the disjoint union of boolean algebras.

Proposition 3.25. *$\mathcal{H}(B_n)$ is isomorphic to n disjoint copies of B_{n-1}.*

Proof. Let the n disjoint copies of B_{n-1} be labeled $B_{n-1}^{(i)}$, $1 \leq i \leq n$, with the elements of $B_{n-1}^{(i)}$ labeled $x^{(i)}$, $x \subseteq \{1, \ldots, n-1\}$. We will show that the map

$$\begin{aligned} f \colon \mathcal{H}(B_n) &\longrightarrow \bigcup_{i=1}^{n} B_{n-1}^{(i)} \\ (x, x \cup i) &\longmapsto x^{(i)} \end{aligned}$$

is an isomorphism. Suppose we have $(x, y), (x', y') \in \mathcal{H}(B_n)$ with $(x, y) \lessdot (x', y')$. Let $j \in [n]$ such that $y' = y \cup \{j\}$, and let $i \in [n]$ such that $x' = x \cup \{i\}$. If $i \neq j$, then $x' \leq y$, contradicting the assumption that $(x, y) \lessdot_{\mathcal{H}} (x', y')$. Thus $x' = x \cup \{i\}$ and $y' = y \cup \{i\}$ for some $i \in [n]$.

Conversely we can easily check that if $i \notin y$, then $(x, y) \lessdot_{\mathcal{H}} (x \cup \{i\}, y \cup \{i\})$. It follows that for all subsets $w \subset [n]$ such that $|w| = 1$, there is an isomorphism from the subposet of elements $\{(x, y) \colon y \setminus x = w\}$ to B_{n-1} defined by $(x, y) \mapsto (x \setminus w, y \setminus w)$. Furthermore if $y \setminus x \neq y' \setminus x'$, then (x, y) and (x', y') are incomparable, so these subposets indexed by w are pairwise disjoint, and $\mathcal{H}(B_n)$ is isomorphic to n copies of B_n. $\square$

Corollary 3.26. *$\mathcal{H}(B_n)$ is unitary Peck for all $n \geq 0$.*

Proof. This follows immediately from Proposition 3.25 and the fact that B_{n-1} is unitary Peck. The latter is shown, for instance in [6, Theorem 2a] by noting that $B_k = (B_1)^k$ and that B_1 is clearly unitary Peck. $\square$

Corollary 3.27. *$\mathcal{H}(B_n)/G$ is Peck for any subgroup $G \subset S_n$.*

Proof. This follows from Corollary 3.26 and Theorem 2.6. $\square$

Corollary 3.28. *$\mathcal{E}(B_n)$ has a symmetric chain decomposition (SCD).*

Proof. $\mathcal{H}(B_n)$ has an SCD by Proposition 3.25 and the fact that B_{n-1} has an SCD, as shown in [3]. By Lemma 3.24 there is a bijective morphism $f \colon \mathcal{H}(B_n) \to \mathcal{E}(B_n)$, and since a bijective morphism takes an SCD to an SCD it follows that $\mathcal{E}(B_n)$ has an SCD. $\square$

Corollary 3.29. *$\mathcal{E}(B_n)/G$ is Peck for any subgroup $G \subset S_n$.*

Proof. By Corollary 3.27, $\mathcal{H}(B_n)/G$ is Peck. By Lemma 3.24, the map $f : \mathcal{H}(P) \to \mathcal{E}(P), G(x, y) \mapsto G(x, y)$ is a bijection. Then, by Lemma 3.16, it follows that $\mathcal{E}(B_n)/G$ is Peck. $\square$

Proof of Theorem 1.5. By Corollary 3.29, $\mathcal{E}(B_n)/G$ is Peck for any group action of G on B_n. Since the G-action is CCT, there is a bijective morphism from $\mathcal{E}(B_n)/G$ to $\mathcal{E}(B_n/G)$, hence $\mathcal{E}(B_n/G)$ is Peck by Lemma 3.16. $\square$

3.3. **A Generalization of $\mathcal{H}, \mathcal{E}$.** Although for the most part, we investigate $\mathcal{E}(P/G), \mathcal{E}(P)/G$, there is a natural generalization of $\mathcal{E}, \mathcal{H}$, to what we call $\mathcal{E}^{\overrightarrow{r}}, \mathcal{H}^{\overrightarrow{r}}$, where $\overrightarrow{r} = r_1, r_2, \ldots, r_k$ is an integer valued sequence. The results holding for these generalizations are analogous to those developed above. The purpose of developing these generalizations notion will be to give a more general application to Polya theory than could be given with $\mathcal{E}$.

Definition 3.30. Let $\overrightarrow{r} = r_1, \ldots, r_k$. For a graded poset P, define the graded poset $\mathcal{H}^{r_1,\ldots,r_k}(P)$, also notated $\mathcal{H}^{\overrightarrow{r}}(P)$, whose elements are formal symbols $(x_1, x_2, \cdots, x_{k+1})$ such that $rk(x_i) + r_i = rk(x_{i+1})$, for all $i \in [k]$. Say

$$(x_1, x_2, \cdots, x_{k+1}) \lessdot_{\mathcal{H}^{\overrightarrow{r}}} (y_1, y_2, \cdots, y_{k+1})$$

if $x_i \lessdot_P y_i, y_i \not\leq_P x_{i+1}$ for all $i \in [k+1]$. Then, define a relation $<_{\mathcal{H}^{\overrightarrow{r}}}$ on $\mathcal{H}^{r_1,\ldots,r_k}(P)$, to be the transitive closure of $\lessdot_{\mathcal{H}^{\overrightarrow{r}}}$. Finally, define

$$rk_{\mathcal{H}^{\overrightarrow{r}}(P)}(x_1, \ldots, x_{k+1}) = rk_P(x_1).$$

Definition 3.31. Let $\overrightarrow{r} = r_1, \ldots, r_k$, with $r_i \in \mathbb{N}$ for all $i \in [k]$. Let $\mathcal{P}$ be the category of graded posets. Define the functor $\mathcal{E}^{\overrightarrow{r}} : \mathcal{P} \to \mathcal{P}$, also notated $\mathcal{E}^{r_1,\ldots,r_k}$. For a graded poset P, define the graded poset $\mathcal{E}^{r_1,\ldots,r_k}(P)$, also notated whose elements are formal symbols $(x_1, x_2, \cdots, x_{k+1})$ such that $rk(x_i) + r_i = rk(x_{i+1})$, for all $i \in [k]$. Say

$$(x_1, x_2, \cdots, x_{k+1}) \lessdot_{\mathcal{H}^{\overrightarrow{r}}} (y_1, y_2, \cdots, y_{k+1})$$

if $x_i \lessdot_P y_i$ for all $i \in [k+1]$. Then, define a relation $\leq_{\mathcal{E}^{\overrightarrow{r}}}$ on $\mathcal{E}^{r_1,\ldots,r_k}(P)$, to be the transitive closure of $\lessdot_{\mathcal{E}^{\overrightarrow{r}}}$. Finally, define

$$rk_{\mathcal{E}^{\overrightarrow{r}}(P)}(x_1, \ldots, x_{k+1}) = rk_P(x_1).$$

Observe that under this notation, $\mathcal{E}^1 = \mathcal{E}, \mathcal{H}^1 = \mathcal{H}$.

Remark 3.32. For all $\overrightarrow{r}, \mathcal{E}^{\overrightarrow{r}}$ is a functor, for the same reasons that $\mathcal{E}$ is a functor. This generalization is essentially taking the nerves of the poset P. See [1] for some related constructions, although their constructions are not completely analogous.

In the next Lemmas, we cite analogous results which hold for $\mathcal{H}^{\overrightarrow{r}}$. The proofs are almost identical to those for $\mathcal{H}$.

Lemma 3.33. *Given a group action $\phi : G \times P \to P$ there are well defined group actions*

$$\phi_{\mathcal{E}} : G \times \mathcal{E}^{\overrightarrow{r}}(P) \to \mathcal{E}^{\overrightarrow{r}}(P),$$

$$\phi_{\mathcal{H}} : G \times \mathcal{H}^{\overrightarrow{r}}(P) \to \mathcal{H}^{\overrightarrow{r}}(P),$$

both given by

$$g \cdot (x_1, \cdots, x_{k+1}) = (g \cdot x_1, \cdots, g \cdot x_{k+1}).$$

Proof. The proof is analogous to that of Lemma 3.23. $\square$

Lemma 3.34. *The poset $\mathcal{H}^{r_1,\ldots,r_k}(B_n)$ is isomorphic to the multinomial coefficient $\binom{n}{r_1, r_2, \ldots, r_k}$ disjoint copies of $B_{n-\sum_{i=1}^k r_i}$. Consequently, $\mathcal{H}^{\overrightarrow{r}}(B_n)$ is unitary Peck and $\mathcal{H}^{\overrightarrow{r}}(B_n)/G$ is Peck.*

Proof. Once again, the proof of the above three statements is analogous to those of 3.25, Corollary 3.26, and Corollary 3.27. The reason there are multinomial coefficients here instead of binomial coefficients, is that an element $(x_1, x_2, \cdots, x_{k+1})$ lies in the copy of $B_{n-\sum_{i=1}^k r_i}$ determined by the ordered tuple of subsets $(x_2 \setminus x_1, x_3 \setminus x_2, \ldots, x_{k+1} \setminus x_k)$. The first consists of r_1 elements, the next of r_2 elements, up through the last which consists of r_k elements. Since the total number of ways to choose $r_1, \ldots, r_k$ in k distinct groups is $\binom{n}{r_1, r_2, \ldots, r_k}$, there are exactly this many disjoint copies of $B_{n-\sum_{i=1}^k r_i}$. □

4. Common Cover Transitive Actions

In this section, we develop the theory of CCT actions ϕ where G is a group, P is a poset, and $\phi : G \times P \to P$ is an action. Recall Definition 1.4, that ϕ is CCT if whenever $x, y, z \in P, x \lessdot z, y \lessdot z, x \in Gy$ then there exists $g \in \mathit{Stab}(z)$ with $gx = y$. We show that the CCT property is closed under semidirect products, in the appropriate sense. From Proposition 1.6, which will be proven in subsection 7.3, the action of S_n on B_n and the action of certain dihedral groups are CCT. We can then use these as building blocks to construct other CCT groups. In particular, we shall show in this section that automorphism groups of rooted trees are CCT.

Remark 4.1. Before continuing with the description of CCT actions, it is worth noting that the notion of cover transitivity generalizes to $\mathcal{E}$ and even $\mathcal{E}^{\overrightarrow{r}}$. Letting $\overrightarrow{r} = r_1, \ldots, r_k$, one would call a group action $\phi : G \times P \to P$, $\overrightarrow{r}$ *common cover transitive (CCT)* if for any chains $(x_0, \ldots, x_k), (y_0, \ldots, y_k)$, such that there exists $g_i \in G$ with $x_i = g_i y_i$, for all $i, 0 \leq i \leq k$, then there exists a single $g \in G$ with $x_i = gy_i$. It is trivial to see that 1 CCT agrees with our definition of CCT. Furthermore, it is also simple to see that an action is $\overrightarrow{r}$ CCT if and only if the natural surjection $q : \mathcal{E}^{\overrightarrow{r}}(P)/G \to \mathcal{E}^{\overrightarrow{r}}(P/G)$ is a bijection. Additionally, very many of the properties developed in this section will also hold for this generalized notion of $\overrightarrow{r}$ CCT.

Example 4.2. Two rather trivial examples of CCT actions are $\phi : S_n \times B_n \to B_n, \psi : G \times B_n \to B_n$ where G is arbitrary, ϕ is the action induced by S_n permuting the elements of $[n]$, and ψ is the trivial action. In the former case, $\mathcal{E}(B_n/S_l)$ is simply a chain with $n-1$ points, and so is $\mathcal{E}(B_n)/S_l$, since all (x, y) are identified under the S_l action. In the latter case, since G acts trivially by ϕ, both $\mathcal{E}(B_n/G) \cong \mathcal{E}(B_n)$ and $\mathcal{E}(B_n)/G \cong \mathcal{E}(B_n)$. So again, ψ is CCT.

4.1. Preservation Under Semidirect Products.

Let $G \subseteq \mathrm{Aut}(P)$, $H \triangleleft G$, $K \subset G$ such that $G = H \rtimes K$. Note that if $x, x' \in Hx$ we have $x' = h \cdot x$ for some $h \in H$, and since H is normal in G we have that $khk^{-1} = h'$ for some $h' \in H$, so

$$k \cdot x' = kh \cdot x = k(k^{-1}h'k) \cdot x = h' \cdot (k \cdot x)$$

Hence $k \cdot x$ and $k \cdot x'$ are in the same H-orbit, so we have a well-defined group action of K on P/H defined by $k \cdot Hx = H(k \cdot x)$.

Recall Proposition 1.7, as stated in the introduction, which says that the CCT property is preserved under semidirect products. We will use Proposition 1.7 to construct more examples of CCT group actions, in particular using it to give a simple proofs that CCT actions are preserved under direct products and wreath products.

Proposition 1.7. *Let $G \subseteq \mathrm{Aut}(P)$, $H \triangleleft G$, $K \subset G$ such that $G = H \rtimes K$. Suppose that the action of H is on P is CCT and the action of K on P/H is CCT. Then the action of G on P is CCT.*

Proof. Since $G = H \rtimes K$, every element $g \in G$ can be written uniquely as a product hk for some $h \in H$, $k \in K$. Let $x, y, z \in P$ such that $x \lessdot z$, $y \lessdot z$ and such that there exists some $h_0k_0 \in G$ such that $h_0k_0 \cdot x = y$. It suffices to show that there exists some $g \in \mathrm{Stab}(z)$ such that $g \cdot x = y$.

The orbits $Hx, Hy, Hz \in P/H$ satisfy $Hx \lessdot Hz$, $Hy \lessdot Hz$ such that $k_0 \cdot Hx = Hy$, so since the action of K on P/H is CCT there exists some $k_1 \in K$ such that $k_1 \in \mathrm{Stab}(Hz)$ and $k_1 \cdot Hx = Hy$. It follows that there exists some $h_1 \in H$ such that $h_1 k_1 h_0 \in \mathrm{Stab}(z)$ and $h_1 k_1 h_0 \cdot x \in Hy$.

Write $x' = h_1 k_1 h_0 \cdot x$. Since the group action of G must be order-preserving by definition, we have that $x' \lessdot z$. We already had that $y \lessdot z$ and $x' \in Hy$, hence there exists some $h_2 \in \mathrm{Stab}(z)$ such that $h_2 \cdot x' = y$ by the fact that the action of H on P is CCT. Then we have that $h_2 h_1 k_1 h_0 \cdot x = h_2 \cdot x' = y$ and $h_2 h_1 k_1 h_0 \cdot z = h_2 \cdot z = z$, as desired. □

Proposition 4.3. *For $\phi : G \times P \to P, \psi : H \times Q \to Q$ two CCT actions, then the direct product*

$$\phi \times \psi : (G \times H) \times (P \times Q) \to (P \times Q), (g, h) \cdot (x, y) \mapsto (gx, hy)$$

is also CCT.

Proof. First note that if either G or H acts trivially, then it can be easily checked that the action of $G \times H$ is CCT. Next, observe that $G \times H$ can be viewed as the semidirect product of $(G \times \{e\}) \rtimes (\{e\} \times H)$. Since the action of G on P is CCT, the action of $G \times \{e\}$ on $P \times Q$ is CCT. Also, since the action of H on Q is CCT, it follows that the action of $\{e\} \times H$ on $(P/G) \times Q$ is CCT. Therefore, the action of $(G \times \{e\}) \rtimes (\{e\} \times H)$ satisfies the conditions of Proposition 1.7 and so the action of $G \times H$ is CCT. □

Next, we use Proposition 1.7 to prove in Proposition 4.8 that the CCT property is preserved under wreath products with the symmetric group. First, we need some definitions of wreath product.

Definition 4.4. For G, H groups, with $H \subset S_l$, the *wreath product*, notated $G \wr H$, is the group whose elements are pairs $(g, h) \in G^l \times H$ with multiplication defined by

$$((g_1', \ldots, g_l'), h') \cdot ((g_1, \ldots, g_l), h) = ((g'_{h'(1)} g_1, \ldots, g'_{h'(l)} g_l), hh')$$

where H acts on $[l]$ via the embedding of H into S_l.

In other words, $G \wr H$ can be viewed as a certain semidirect product of $G^l \rtimes H$.

Definition 4.5. For any group G with a given action $\psi : G \times P \to P$, we obtain an induced action of $G \wr H$, $\phi : G \wr H \times P^l \to P^l$ defined by

$$((g_1, \ldots, g_l), h)(a_1, \ldots, a_l) = (g_{h^{-1}(1)} \cdot a_{h^{-1}(1)}, \ldots, g_{h^{-1}(l)} \cdot a_{h^{-1}(l)}).$$

Remark 4.6. Heuristically, one may think of the above action as obtained by first having G act separately on the l distinct copies of P, and then letting H act by permuting the copies.

Lemma 4.7. *For P a graded poset, the action*

$$\phi : S_l \times P^l \to P^l, \sigma \cdot (x_1, \ldots, x_l) = (x_{\sigma(1)}, \ldots, x_{\sigma(l)})$$

is CCT.

Proof. For $a \in P^l$ notate $a = (a_1, \ldots, a_l)$. Suppose $x, y, z \in P^l$ with $x \lessdot z, y \lessdot z$, and $x \in S_l y$. This means there is a unique i such that $x_i \lessdot z_i, x_k = z_k$ for $k \neq i$. Additionally, there is a unique j for which $y_j \lessdot z_j, y_k = z_k$ for $k \neq j$. Since $x \in S_l y$, we obtain the equality of multisets $\{x_1, \ldots, x_l\} = \{y_1, \ldots, y_l\}$. But for $k \neq i, j, x_k = z_k = y_k$, we also obtain equality of sets $\{x_i, x_j\} = \{y_i, y_j\}$. Since $rk(y_j) \lessdot rk(x_j)$, we obtain $y_j = x_i, y_i = x_j$. Then, taking the transposition $\sigma = (ij) \in S_l$, it follows that $\sigma \in Stab(z)$ and $\sigma \cdot x = y$. □

Proposition 4.8. *If $\psi : G \times P \to P$ is CCT, then $\phi : G \wr S_l \times P^l \to P^l$ where ϕ is the induced action defined in Definition 4.5 is also CCT.*

Proof. Note that the wreath product $G \wr S_l$ can be viewed as a semidirect product $G^l \rtimes S_l$. Since the action of G on P is CCT we obtain that the action of G^l on P^l is CCT by Proposition 4.3. Furthermore the action $S_l \times (P/G)^l \to (P/G)^l$ defined by $(\sigma, (x_1, \ldots, x_l)) \mapsto (x_{\sigma(1)} \ldots, x_{\sigma(l)})$ for $\sigma \in S_l$ and $x_i \in P/G$ is CCT by Lemma 4.7. Since $P^l/G^l \cong (P/G)^l$, it follows that the action ϕ satisfies the conditions of Proposition 1.7, so ϕ is CCT. □

4.2. **Examples of CCT Actions.** In this subsection, we describe several classes of CCT actions. First, we show that the automorphism group of any rooted tree is CCT. Second, we show linear automorphisms of simplices and octahedra are CCT. Third, we show that the left multiplication action is CCT if and only if the group is $\mathbb{Z}_2^k$, and that any action of $\mathbb{Z}_2^k$ on $[n]$ induces a CCT action on B_n.

4.2.1. *An application to rooted trees.* In this subsubsection, we prove that the automorphism group of rooted trees is always CCT. To do this we will apply Proposition 4.8 and Proposition 4.3, since the automorphism group of rooted trees is essentially built from direct products and wreath products with a symmetric group. To this aim, we first give definitions relating to rooted trees, then characterize their automorphisms, and finally show that such automorphism groups are always CCT. Our examination of of rooted trees automorphisms here, together with the examination of polytopes in subsubsection 4.2.2 was motivated by [2, Section 5].

Definition 4.9. A graded poset P is a *rooted tree* if there is a unique element $x \in P$ of maximal rank, called the *root*, and for all $x \in P$, other than the root, there exists a unique $y \in P$ with $y \gtrdot x$.

Example 4.10. In Figure 5 and Figure 6, we give two examples of rooted trees.

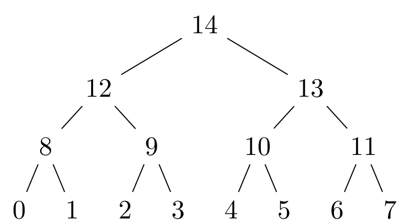


FIGURE 5. An example of a rooted tree with 8 leaves.

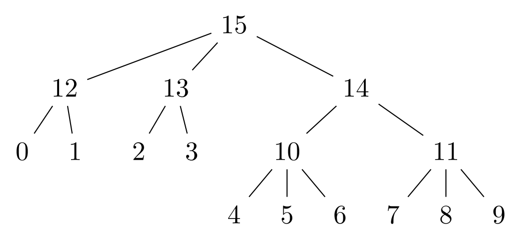


FIGURE 6. An example of a rooted tree with 10 leaves.

Definition 4.11. For P a rooted tree, an element $x \in P$ is a *leaf* if there is no $z \in P$ with $x > z$. Denote the set of all leaves of P by $L(P)$.

Lemma 4.12. *Let P be a rooted tree. Then, the action of $Aut(P)$ on P induces an action of $Aut(P)$ on $L(P)$. Furthermore, there is also an induced action of $Aut(P)$ on B_n where $n = |L(P)|$.*

Proof. First, we must show that $Aut(P)$ induces an action on $L(P)$. That is, we must show that for any $g \in Aut(P), x \in L(P)$, then $gx \in L(P)$. If $x \in L(P)$ but $gx \notin L(P)$, then there exists $y < gx$. However, then $g^{-1}y < x$, contradicting the assumption that x is a leaf. We then obtain the induced action $Aut(P) \times L(P) \to L(P), (g, x) \mapsto gx$.

Finally, to obtain the induced action of $Aut(P)$ on B_n identify $L(P) \cong [n]$ as sets, where $|L(P)| = n$, and from this we get of action of $Aut(P)$ on B_n. $\square$

Notation 4.13. For the rest of this section only, fix a rooted tree P and denote by G the group of automorphisms $Aut(P)$. Let G act on B_n, where $n = |L(P)|$, by the induced action $\phi : G \times L(P) \to L(P)$ described in the proof of Lemma 4.12.

Notation 4.14. For $x \in P$, denote $D(x) = \{y \in P : y \leq x\}$.

Proposition 4.15. *Let P be a rooted tree with root vertex labeled 0. Let $\{A_1, \ldots, A_m\}$ denote the set of isomorphism classes of $\{D(x) : x \lessdot r\}$. For $T \in A_j$, denote $G_j = Aut(T)$. Then,*

$$Aut(P) = (G_1 \wr S_{i_1}) \times (G_2 \wr S_{i_2}) \times \cdots \times (G_m \wr S_{i_m}), \tag{4.1}$$

In particular, $Aut(P)$ can be expressed as a sequence of direct products and wreath products of symmetric groups.

Proof. We proceed by induction on the rank of P. It is clear that if P is rank 0, then $Aut(P)$ is trivial. If the rank of P is greater than 0, label the vertices of P by $\{0, 1, \ldots, s\}$ such that the root is labeled 0 and the vertices just below the root are labeled $1, \ldots, k$. Let $A_1, \ldots, A_m$ denote the distinct isomorphism classes of trees in the set $\{D(1), \ldots, D(k)\}$. For $T \in A_j$, denote $G_j = Aut(T)$. Let $T_j = \{t : t \lessdot 0, D(t) \cong A_j\}$. Then, letting Q_j be the subtree of P whose elements lie in the set $0 \cup (\cup_{t \in T_j} D(t))$, it follows $Aut(Q_j) \cong G_j \wr S_{i_j}$, because after choosing a permutation of the elements of T_j, we are free to choose any element of G_j to permute each $D(t), t \in T_j$. If $t_1 \lessdot 0, t_2 \lessdot 0, g \cdot t_1 = t_2$, then it must be that $D(t_1) = D(t_2)$. Therefore, $Aut(P)$ must permute these isomorphism classes of trees, and the full automorphism groups is simply the direct product,

$$Aut(P) = (G_1 \wr S_{i_1}) \times (G_2 \wr S_{i_2}) \times \cdots \times (G_m \wr S_{i_m}), \tag{4.2}$$

Since each G_j is a sequence of direct products and wreath products with symmetric groups by the inductive assumption, it follows from (4.2) that so is $Aut(P)$. □

Example 4.16. Let P_1 be the rooted tree in Figure 5 and P_2 be the rooted in Figure 6, the proposition says that $Aut(P_1) = (S_2 \wr S_2) \wr S_2$; and $Aut(P_2) = (S_2 \wr S_2) \times (S_3 \wr S_2)$.

Corollary 4.17. *For P the automorphism group of a rooted tree, $Aut(P)$ is CCT.*

Proof. By Proposition 4.8, wreath products with symmetric groups preserve the CCT property, and by Proposition 4.3 the direct product of two CCT groups is again CCT. Therefore, by Proposition 4.15, all groups of the form $Aut(P)$ are built up from these operations, and so $Aut(P)$ is also CCT. □

4.2.2. *Automorphisms of Polytopes.* As another class of CCT actions, we describe several linear automorphism groups of polytopes whose actions are CCT. In particular, we prove that the linear automorphism groups of simplices and octahedra are CCT. Later in Proposition 1.6, we will also see that the action of the dihedral group on a regular n-gon is CCT for $n = p, 2p$. Since the dihedral group contains all linear automorphisms of the regular n-gon, this action gives another example of the linear automorphism group of a polytope being CCT.

Definition 4.18. Let M be a polytope, with a particular embedding in $\mathbb{R}^n$. The *group of linear automorphisms of M* is the subgroup of GL_n whose elements are $\{g \in GL_n : g \cdot M = M\}$.

First, we look at linear automorphisms of simplices. Let G be the group linear automorphisms of the $(n-1)$-simplex, whose vertices lie at the standard basis vectors in $\mathbb{R}^n$. The action of G on the $(n-1)$-simplex induces an action on $[n]$, given by identifying $[n]$ with the n vertices of the $(n-1)$-simplex. Hence, it induces an action on B_n.

Proposition 4.19. *The action of the group of linear automorphisms of the $(n-1)$-simplex, acting on B_n, as defined above, is CCT.*

Proof. The group of linear automorphisms in this case induces the usual action of S_n on B_n, because any permutation matrix defines a linear map on $\mathbb{R}^n$. However, we know the action of S_n on B_n is CCT from Example 4.2 □

Next, we look at linear automorphisms of octahedrons. Let G be the group of linear automorphisms of the n-octahedron, embedded inside $\mathbb{R}^n$, whose vertices are located at $\pm e_i$, where $e_1, \ldots e_n$ are the standard basis vectors of $\mathbb{R}^n$. Then, the action of G on the octahedron induces an action of G on the $2n$ vertices of the octahedron, and hence on B_{2n}.

Proposition 4.20. *The induced action of the group of linear automorphisms of the n octahedron on B_{2n} is CCT.*

Proof. It is simple to see that the group of linear automorphisms of the n-octahedron with vertices at $\pm e_i$, where $e_1, \ldots e_n$ is the hyperoctahedral group, since it is generated by the permutation matrices, together with the matrix A with $A_{1,1} = -1, A_{i,i} = 1, A_{j,k} = 0$ where $i \neq 1, j \neq k$. [1] It is well know that they hyperoctahedral group can be written as $S_2 \wr S_n$. Then, by Proposition 4.8, it follows that $S_2 \wr S_n$ is CCT. □

Remark 4.21. Let us give a brief recap of which linear automorphisms of polytopes are known to induce actions on B_n which are CCT. First, by the above lemmas, for octahedrons and simplices the induced action is CCT. By Proposition 1.6 and Proposition 7.21 the linear automorphism group of an n-gon induces a CCT action on B_n if and only if one of $n = 1, n = p, n = 2p$, for p a prime. Additionally, computers have verified that automorphisms of the 3-cube, with vertices at $(\pm 1, \pm 1, \pm 1)$ induces a CCT action. It is still unknown whether the linear automorphisms of n-cubes is CCT for $n > 3$, and also whether the remaining five exceptional regular polytopes (namely the dodecahedron and icosahedron in $\mathbb{R}^3$ as well as the 24-cell, 120-cell, and 600-cell polytopes in $\mathbb{R}^4$) induce CCT actions. These questions are repeated in Question 9.12 and Question 9.13.

4.2.3. *CCT Actions of $\mathbb{Z}_2^k$.* In this subsubsection, we show that any embedding of $\mathbb{Z}_2^k$ into S_n defines an action on B_n which is CCT. In particular, this implies that the left multiplication action of $\mathbb{Z}_2^k$ is CCT. However, it turns out that this is the only class of groups for which the left multiplication action is CCT.

Proposition 4.22. *Recall that G is an elementary abelian 2-group if $G \cong (\mathbb{Z}/2\mathbb{Z})^k$ for some k.*

(1) *For any $n \in \mathbb{N}$, and G an elementary abelian 2-group, every G action on B_n is CCT.*
(2) *For every finite group G which is not an elementary abelian 2-group, there exists at least one G action which is not CCT, namely the action of G on B_n induced by the left-regular action of G on itself.*

Proof. Part (1) Let $x, y, z \in B_n$ such that $x \lessdot z, y \lessdot z$, and $x = gy$ for some $g \in G$. Since $x \neq y$ we have $z = x \cup y$. Furthermore, since every element in $\mathbb{Z}_2^k$ has order 2 we have that $gy = g^2 x = x$ and thus $gz = gx \cup gy = y \cup x = z$. Hence $g \in \mathrm{Stab}(z)$ and thus ϕ is CCT.

Part (2) First, let us show $G \cong \mathbb{Z}_2^k \Leftrightarrow \forall g \in G, g^2 = e$. The forward implication is obvious. To see the converse, first note that if $\forall g \in G, g^2 = e$, then G is abelian because $aba^{-1}b^{-1} = abab = (ab)^2 = e$. Then, G is an abelian group, all of whose elements have order two, the structure theorem of finite abelian groups tells us $G \cong \mathbb{Z}_2^k$.

So, Suppose $G \ncong \mathbb{Z}_2^k$. Then, there exists $g \in G, g^2 \neq e$. Clearly $\{e\} \lessdot \{e \cup g\}, \{g\} \lessdot \{e \cup g\}, \{g\} \in G\{e\}$. So, to show the induced action $\phi : G \times B_n \to B_n$ is not CCT, it suffices to show there is no $h \in G, h \in Stab(\{e \cup g\}), h \cdot \{e\} = \{g\}$. If $h \in Stab(\{e \cup g\})$ then $h \cdot e = e$ or $h \cdot e = g$. In both cases, it is simple to see that $g^2 = e$. Therefore, there does not exist such an h and left multiplication is not CCT. □

[1] The hyperoctahedral group is commonly notated B_n, since it is the type B Coexeter group, but we do not use this here to avoid confusing with the boolean algebra.

5. Wreath Product of two Symmetric Groups

In this section, we prove a result similar to that of [4, Theorem 1.1]. We construct a certain sequence which is not only unimodal, but can even be exhibited as the ranks of a Peck poset. This construction gives an alternate proof of [4, Theorem 1.1] in the case that $r = 1$.

Notation 5.1. For this section, fix l, m with $n = l \cdot m$ and fix $G = S_m \wr S_l$. Let S_m act on B_m by the permutation representation, and then let G act on $B_m^l \cong B_{m \cdot l}$ by the action defined in Definition 4.5.

5.1. Recalling Pak and Panova's Result.

We first review the necessary definitions and then state [4, Theorem 1.1]:

A *partition* is a sequence of numbers $\lambda = (\lambda_1, \dots, \lambda_k)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$. If $\sum_{i=1}^k \lambda_i = n$ then λ is a *partition of* n, notated $\lambda \vdash n$. A *composition* is a sequence of numbers $\lambda = (\lambda_1, \dots, \lambda_k)$. That is, it is a partition where order matters. If $\sum_{i=1}^k \lambda_i = n$ then λ is a *composition of* n. Let $P_n(l, m)$ denote the set of partitions $\lambda = (\lambda_1, \dots, \lambda_k) \vdash n$, such that $\lambda_1 \leq m, k \leq l$. That is, $P_n(l, m)$ is the partitions which fit inside an $l \times m$ rectangle.

Notation 5.2. [4, Section 1] For λ a partition, let $\nu(\lambda)$ be the number of distinct nonzero part sizes of λ.

Notation 5.3. [4, Section 1] Let $p_k(l, m, r) = \sum_{\lambda \in P_k(l,m)} \binom{\nu(\lambda)}{r}$.

Theorem 5.4. [4, Theorem 1.1] *The sequence* $p_r(l, m, r), p_{r+1}(l, m, r), \dots, p_{l \cdot m}(l, m, r)$ *is unimodal and symmetric.*

5.2. A Proof of Theorem 5.4 for $r = 1$.

Now that we have managed to state Pak and Panova's Theorem, we can now conclude that Theorem 5.4 holds in the case of $r = 1$. In fact, we can do better, by realizing $p_i(l, m, 1)$ as ranks of a Peck poset.

Proposition 5.5. *There is an equality* $|(\mathcal{E}(B_n)/G)_i| = |\mathcal{E}(B_n/G)_i| = p_{i+1}(l, m, 1)$. *In particular, Theorem 5.4 holds in the case* $r = 1$.

Proof. First, observe that $S_m \wr S_l$ can be described as the automorphism group of a rooted tree of rank 2 with l elements at rank 1 and $m \cdot l$ elements at rank 2, such that each element at rank 1 is above m elements at rank 2. Then, by Corollary 4.17, it follows that the action of G on $B_{m \cdot l}$ is CCT and so $\mathcal{E}^r(B_n/G)$ is Peck.

Next, note that each equivalence class in B_n/G has a unique representative which is a Young Tableau. This is a well known fact, proven later in Lemma 5.11.

Now, let Gx, Gy be two G orbits with $\bar{x}$ the Young Tableau corresponding to x, and $\bar{y}$ the young tableau corresponding to y. Suppose $Gx \lessdot Gy$. Then, $\bar{x}$ must be contained in $\bar{y}$, but with a single box removed. Since $\bar{x}, \bar{y}$ are both Young Tableau, the removed box must be one of the corners of $\bar{y}$. Observe that the number of corners of a partition is precisely the number of distinct part sizes, and so $|\{Gx | Gx \lessdot Gy\}| = \nu(\bar{y})$. Therefore,

$$\begin{aligned}
|\mathcal{E}(B_n/G)_i| &= \sum_{(Gx,Gy) \in \mathcal{E}(B_n/G)_i} 1 \\
&= \sum_{Gy \in (B_n/G)_{i+1}} \left(\sum_{Gx \lessdot Gy} 1 \right) \\
&= \sum_{Gy \in (B_n/G)_{i+1}} \nu(\bar{y}) \\
&= \sum_{\lambda \in P_{i+1}(l,m)} \nu(\lambda).
\end{aligned}$$

Therefore, $|(\mathcal{E}(B_n)/G)_i| = |\mathcal{E}(B_n/G)_i| = p_{1+i}(l, m, 1)$. Since $\mathcal{E}(B_n)/G$ is Peck, $p_1(l, m, 1), p_2(l, m, 1), \dots, p_{l \cdot m}(l, m, 1)$ is unimodal, and hence Theorem 5.4 holds in the case $r = 1$. $\square$

5.3. **The Poset $\mathcal{E}^r(B_n)/G$.** The remainder of this section is devoted to computing the ranks of the posets $\mathcal{E}^r(B_n)/G$. Note that we will also obtain a different formula for the sizes of these ranks in Lemma 6.8, which will be used to prove an equality in Proposition 6.12.

Remark 5.6. It is worth noting that the proof given in Proposition 5.5 does not generalize to $r > 1$, as it is no longer true that $|\mathcal{E}^r(B_n/G)_i| = p_r(l, m, r)$. To see this, note that $\binom{\nu(\lambda)}{r}$ counts the number of corners of λ choose r, and so it only counts certain Young Diagrams contained in λ such that at most one block is removed from each distinct part size, whereas $|\mathcal{E}^r(B_n/G)_i|$ counts something much larger, as is described in Remark 5.17

To give our formula for the ranks of $\mathcal{E}^r(B_n)/G$, we will first describe representatives for the elements of B_n/G and then analogous representatives for vertices of $\mathcal{E}(B_n)/G$.

Definition 5.7. For $x \subset [l \cdot m]$ we say x is *left justified* if whenever $a \in x$, such that $a \neq 1 \pmod{l}$, then $a - 1 \in x$. In other words, when we pick out the boxes of the $l \times m$ rectangle which lie in x the resulting diagram is left justified.

Definition 5.8. For $x \subset [l \cdot m]$, define the *composition of* x, denoted $comp(x) = \lambda_1, \ldots, \lambda_l$ where $\lambda_i = |\{a \in x \colon \cdot (m-1) < a \leq i \cdot m\}|$. That is, if we pick out the boxes of the $l \times m$ rectangle which lie in x, the composition is just the sequence of the number of boxes in each row. Similarly, define the *partition of* x, notated $part(x)$ as follows. Let $\pi \in S_l$ be a permutation such that $\pi(i) \geq \pi(i+1)$ for all $i, 1 \leq i \leq l-1$. Then $part(x) = (\lambda_{\pi(1)}, \ldots, \lambda_{\pi(l)})$. That is, $part(x)$ is the $comp(x)$, written in decreasing order.

Lemma 5.9. *For any $g \in G, x \in B_n$, it follows that $part(x) = part(gx)$.*

Proof. It suffices to show this holds for any generator g. Then. G is generated by elements which permute rows, and elements that swap rows. It is clear that if g only permutes elements in a single row, then $comp(gx) = comp(x)$, so in particular, $part(gx) = part(x)$. If g swaps two rows, then $comp(gx)$ is simply a reordering of the parts of $comp(x)$, and so again $part(gx) = part(x)$. □

Definition 5.10. For $x \subset [l \cdot m]$, say x is a *Young Diagram* if x is left justified and $comp(x)$ is a partition.

Lemma 5.11. *For each $x \in B_{l \cdot m}$ there exists a unique representative $z \in Gx$ such that z is a Young Diagram.*

Proof. Uniqueness is clear, because for any $g \in G$, by Claim 5.9, $part(gx) = part(x)$. So, to we only have to show there is some $g \in G$ for which gx is a Young Diagram. Indeed, first, choose $h_1 \in G$ so that $h_1 x$ is left justified. This can be done because every permutation of a single row in the $l \times m$ rectangle lies in the wreath product, so, we can take h_1 to be the product of the elements that left justify each individual row. Then, let h_2 be the element that swaps the rows of $h_1 x$ so that they increase going down. Finally, taking $g = h_2 h_1$, it follows that gx is a Young Diagram. □

Notation 5.12. For $x \in B_n$ denote by $\overline{x}$ the unique element in Gx such that $\overline{x}$ is a Young Diagram.

Now that representatives for each G orbit in B_n have been described, we move on to describing representatives for each G orbit in $\mathcal{E}(B_n)$.

Notation 5.13. For $\lambda = (\lambda_1, \ldots, \lambda_k)$ a partition, introduce the alternate notation $\lambda = a_1^{b_1} \cdots a_s^{b_s}$ if the first b_1 parts of λ are equal to a_1, the next b_2 parts of λ are equal to b_2, and in general, for $1 \leq h \leq b_j$, the parts $\lambda_{h + \sum_{i=1}^{j-1} b_i}$ are all equal to a_j. Furthermore, all a_j must be distinct.

Lemma 5.14. *If $(x, y), (w, y) \in \mathcal{E}^r(B_n)_i$ then $(x, y) \in G(w, y)$ if and only if for all $g \in G$ such that gy is a Young Diagram, we have $(gx, gy) \in G(gw, gy)$. In particular, if $gy = \overline{y}$ then $(x, y) \in G(w, y)$ if and only if $(gx, \overline{y}) \in (gw, \overline{y})$.*

Proof. Clearly $g(x, y) \in G(g(w, y))$ if and only if $(x, y) \in G(w, y)$. $\square$

The point of the proceeding Lemma is that in order to determine when two elements are identified, we may assume that y is a Young Diagram.

Notation 5.15. For y a Young Diagram with $part(y) = comp(y) = a_1^{b_1} \cdots a_s^{b_s}$, and $x \subset y$, for all $i, 1 \leq i \leq s$, define y_i to be the $a_i \times b_i$ rectangle consisting of the rows $1 + \sum_{j=1}^{i-1} b_j, 2 + \sum_{j=1}^{i-1} b_j, \ldots, \sum_{j=1}^{i} b_j$. That is, y_i is just the rectangle of all parts of y which are of length a_i.

Proposition 5.16. *If y is a Young Diagram so that $part(y) = comp(y) = a_1^{b_1} \cdots a_s^{b_s}$, and $(x, y), (w, y) \in \mathcal{E}^r(B_n)_j$ then $(x, y) \in G(w, y)$ if and only if for all $i, 1 \leq i \leq s$, it holds that $part(x \cap y_i) = part(w \cap y_i)$.*

Proof. First, suppose for all $i, 1 \leq i \leq s$, that $part(x \cap y_i) = part(w \cap y_i)$. We know both $x \subset y, w \subset y$. Further, since y_i is a rectangle, there exists some $g \in \mathrm{Stab}(y_i)$ so that $gx = w$. Then, for each i, there exists some $g_{1,i} \in \mathrm{Stab}(y_i)$ so $g_{1,i}(x \cap y_i) = \overline{x \cap y_i}$, which only interchange elements in y_i. Similarly, there exists $g_{2,i} \in \mathrm{Stab}(y_i), g_{2,i}(w \cap y_i) = \overline{w \cap y_i}$. However, the assumption $part(x \cap y_i) = part(w \cap y_i)$ precisely means $\overline{x \cap y_i} = \overline{w \cap y_i}$. Therefore, $g_{2,i}^{-1} g_{1,i} \in \mathrm{Stab}(y)$ and $g_{2,i}^{-1} g_{2,i}(x \cap y_i) = (w \cap y_i)$. Applying this same procedure for all i and multiplying the corresponding group elements together gives an element $g \in \mathrm{Stab}(y)$ with $gx = w$. Therefore, $g(x, y) = (gx, gy) = (gx, y) = (w, y)$, so $(x, y) \in G(w, y)$.

Conversely, note that $(x, y) \in G(w, y)$ is equivalent to the existence of a $g \in \mathrm{Stab}(y)$ with $gx = w$. However, any $g \in \mathrm{Stab}(y)$ can only interchange rows of the same length. Therefore, we obtain the stronger result that we can write $g = h_1 \cdots h_s$ where $h_i \in \mathrm{Stab}(y_i)$, and $h_i(p) = p$ for all $p \notin y_i$. Then, it follows that $h_i(x \cap y_i) = w \cap y_i$ and so $part(x \cap y_i) = part(w \cap y_i)$, as claimed. $\square$

Remark 5.17. So, one way of viewing G orbits of an element $(x, y) \in \mathcal{E}^r(B_n)$ is as "outer" Young Diagrams, made up of a sequence of rectangles stacked on top of one another, each one wider than the next, which form the Young Diagram $\bar{y}$. Then, to each such rectangle we associate an "inner" Young Diagram. The inner Young Diagram corresponds to the elements in that rectangle in y but not in x. The sum of the sizes of all these inner Young Diagrams is r. Two elements are in the same G orbit if and only if their "outer" Young Diagram and "inner" Young Diagrams are all the same.

The next step is to give explicit formulas for the ranks of $\mathcal{E}^r(B_n)$.

Notation 5.18. For $p(x) = \sum_{i=0}^{N} c_i x^i$, a polynomial, define the notation $[p(x)]_r = c_r$.

Now, we briefly introduce notation for q binomial coefficients, so that we can state the next propositions. Let $q \in \mathbb{R}$ and let $[n]_q = \sum_{i=0}^{n-1} q^i$. Then, denote $[n]_q! = \prod_{i=1}^{n} [i]_q$. Finally, let $\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$.

Notation 5.19. For $\lambda = (a_1^{b_1} \cdots a_s^{b_s})$ a partition, denote $\eta(\lambda, r) = \left[\prod_{i=1}^{s} \binom{a_i + b_i}{a_i}_q\right]_r$.

Proposition 5.20. [7, Proposition 1.3.19] *There is an equality $[\binom{a+b}{b}_q]_j = |P_j(l, m)|$.*

Theorem 5.21. *The sizes of the ranks of $\mathcal{E}^r(B_n)/G$ can be written as*

$$|(\mathcal{E}^r(B_n)/G)_i| = \sum_{\lambda \in P_i(l,m)} \eta(\lambda, r).$$

Proof. By Proposition 5.16 each orbit (x, y) has a representative such that y is a Young Diagram, with $part(y) = \prod_{i=1}^{s} a_i^{b_i}$, and each $x \cap y_i$ is a Young Diagram. Therefore, for a fixed y, the number of orbits (x, y) with $|x| + r = |y|$ is exactly determined by the Young Diagrams of $\overline{x \cap y_i}$, for $1 \leq i \leq s$. Equivalently, it is determined by the Young Diagrams $[y_i \setminus x]$, for $1 \leq i \leq s$. So, we wish to calculate the number of tuples of Young Diagrams of the form $(\overline{y_1 \setminus x}, \overline{y_2 \setminus x}, \ldots, \overline{y_s \setminus x})$ so that $\sum_{i=1}^{s} |y_i \setminus x| = r$. Now, define j_i by $j_i = |y_i \setminus x|$, still of course with, $\sum_{i=1}^s j_i = r$. Then, by Proposition 5.20, the number of such partitions is $[\binom{a_i+b_i}{b_i}_q]_{j_i}$. Therefore, the number of elements $G(x,y)$ with $j_i = |y_i \setminus x|$, is simply the product $\prod_{i=1}^s [\binom{a_i+b_i}{b_i}_q]_{j_i}$. Therefore, since j_i were chosen arbitrarily, only subject to the constraint that $\sum_{i=1}^s j_i = r$, It follows that the total number of elements $G(x,y)$, for y fixed, is equal to

$$\sum_{\substack{(j_1,\ldots,j_s),\\ \sum_{i=1}^s j_i = r}} \prod_{i=1}^s \left[\binom{a_i+b_i}{b_i}_q\right]_{j_i} = \left[\prod_{i=1}^s \binom{a_i+b_i}{b_i}_q\right]_r = \eta(\lambda, r).$$

Then, summing this over all Young Diagrams y, gives that

$$|\mathcal{E}^r(B_n)/G|_i = \sum_{\lambda \in P_i(l,m)} \left[\prod_{i=1}^s \binom{a_i+b_i}{b_i}_q\right]_r = \sum_{\lambda \in P_i(l,m)} \eta(\lambda, r).$$

□

6. Relations to Polya Theory

In this section, we obtain summation formulas for the ranks of $\mathcal{E}^{\overrightarrow{r}}(B_n)/G$, in the case that the action of G on $\mathcal{E}^{\overrightarrow{r}}$ is induced by some action of G on $[n]$. These formulas are given by certain polynomials in Polya theory. In particular, we obtain polynomials from Polya theory which are the rank generating functions of $\mathcal{E}^{\overrightarrow{r}}(B_n)/G$, which implies that these polynomials have symmetric, unimodal coefficients. The result we prove is a generalization of [8, Corollary 7.16].

6.1. **A Brief Review of Polya Theory.** We follow the treatment from [8, Chapter 7]. First, we build up some definitions to state Polya's Theorem.

Definition 6.1. Let $G \subset S_n$ act on $[n]$ by the restriction of the action of S_n on $[n]$ by all permutations. For $\pi \in G$, the action of π on $[n]$ can be written in cycle notation so that there are c_i cycles of length i. Define the *cycle indicator* of π to be the monomial $Z_\pi(z_1, \ldots, z_n) = z_1^{c_1} z_2^{c_2} \cdots z_n^{c_n}$.

Definition 6.2. The *cycle indicator* for a group $G \subset S_n$, is the polynomial

$$Z_G(z_1, \ldots, z_k) = \frac{1}{|G|} \sum_{\pi \in G} Z_\pi(z_1, \ldots, z_k).$$

Definition 6.3. A *coloring* of a set S by the colors $R = \{r_1, \ldots, r_k\}$ is a map $S \to R$. Heuristically, a coloring of S can be thought of as an assignment of a "color" from the set R to each of the elements of S.

Notation 6.4. For A, B sets, let $A^B = Hom_{\text{sets}}(B, A)$. Let $G \subset S_n$ act on B_n. Then, G acts on $R^{B_n} = Hom_{\text{sets}}(B_n, R)$ by $g \cdot f(S) = f(g \cdot S)$ where $g \in G, S \in B_n, f \in R^{B_n}$. Then, let R^{B_n}/G denote the quotient of R^{B_n} by the action of G defined above.

Suppose $R = \{r_1, \ldots, r_k\}$, and define the map $c : R^S \to \mathbb{Z}^k$, by sending $f \in R^S$ to $c(f) = (i_1, \ldots, i_k)$ if $|\{s \in S \colon f(s) = r_j\}| = i_j$ for $1 \le j \le k$. Observe that if $f \in Gh$ then $c(f) = c(h)$, so the map *color* descends to a map on G orbits.

Define

$$\kappa(i_1, \ldots, i_k) = |\{Gf \in R^{B_n}/G \text{ such that } c(f) = (i_1, \ldots, i_k)\}|.$$

Finally, let

$$F_G(r_1, \ldots, r_k) = \sum_{i_1,\ldots,i_k} \kappa(i_1, \ldots, i_k) r_1^{i_1} \cdots r_k^{i_k}.$$

Remark 6.5. The above notation may seem extremely cumbersome. It is simply a formal way of saying that $\kappa(i_1, \ldots, i_k)$ is the number of inequivalent colorings of subsets of B_n, under the G action. Once again, see [8, Chapter 7] for a more lengthy exposition.

Theorem 6.6. *(Polya's Theorem) Let $G \subset S_n$ act on B_n. With F_G, Z_G as defined above, the following equality holds.*

$$F_G(r_1, \ldots, r_k) = Z_G\left(\sum_{j=1}^{k} r_j, \sum_{j=1}^{k} r_j^2, \ldots, \sum_{j=1}^{k} r_j^n\right).$$

6.2. **A Rank Generating Function.** Next, we produce rank generating functions for $\mathcal{E}^{i_2,\ldots,i_{k-1}}(B_n)/G$. The reason for starting at i_2 and ending at i_{k-1} is that we will include additional numbers i_1 and i_k to be colored by the first color and the kth color. Given an element $x_2, \ldots, x_k \in \mathcal{E}^{i_2,\ldots,i_{k-1}}$, the first color will be used to color the elements in x_2 and the kth color will be used to color the elements in $[n] \setminus x_k$.

Notation 6.7. Let $[f(x_1, \ldots, x_k)]_{x_{j_1}^{i_1} \cdots x_{j_l}^{i_l}}$ denote the coefficient of $x_{j_1}^{i_1} \cdots x_{j_l}^{i_l}$ in $f(x_1, \ldots, x_k)$, which may itself be a polynomial.

Lemma 6.8. *The number $\kappa(i_1, \ldots, i_k)$ is equal to $|(\mathcal{E}^{i_2,\ldots,i_{k-1}}(B_n)/G)_{i_1}|$. Consequently,*

$$\left[Z_G\left(\sum_{j=1}^{k} r_j, \sum_{j=1}^{k} r_j^2, \ldots, \sum_{j=1}^{k} r_j^n\right)\right]_{r_1^{i_1} \cdots r_k^{i_k}} = |(\mathcal{E}^{i_2,\ldots,i_{k-1}}(B_n)/G)_{i_1}|.$$

Proof. Define a map $m : (\mathcal{E}^{r_2,\ldots,r_{k-1}}(B_n))_{i_1} \to \{f \in R^{B_n} : c(f) = (i_1, \ldots, i_k)\}$ as follows. For $(x_2, \ldots, x_k) \in \mathcal{E}^{r_2,\ldots,r_{k-1}}(B_n)$, let $m(x_2, \ldots, x_k) = f$, where

$$f(t) = \begin{cases} r_1 & \text{if } t \in x_2 \\ r_j & \text{if } t \in x_{j+1} \setminus x_j \\ r_k & \text{if } t \notin x_k \end{cases}.$$

Observe that m is in fact a bijection, as we can easily define an inverse map by sending a coloring f to $(x_2, \ldots, x_k)$, where x_j is the set of all elements $t \in [n]$ for which there exists $l < j$ with $f(t) = r_l$. Next, the two group actions were defined so that so that,

$$g(x_1, \ldots, x_{k-1}) = (y_2, \ldots, y_{k-1}) \Leftrightarrow g \cdot m(x_1, \ldots, x_{k-1}) = m(y_2, \cdots, y_{k-1})$$

Therefore, m descends to a bijection $m^G : (\mathcal{E}^{r_2,\ldots,r_{k-1}}(B_n))_{i_1}/G \to R^{B_n}/G$, This implies $\kappa(i_1, \ldots, i_k)$ is equal to $|(\mathcal{E}^{i_2,\ldots,i_{k-1}}(B_n)/G)_{i_1}|$.

Then, by Polya's Theorem 6.6,

$$\left[Z_G\left(\sum_{j=1}^{k} r_j, \sum_{j=1}^{k} r_j^2, \ldots, \sum_{j=1}^{k} r_j^n\right)\right]_{r_1^{i_1} \cdots r_k^{i_k}} = \kappa(i_1, \ldots, i_k) = |(\mathcal{E}^{i_2,\ldots,i_{k-1}}(B_n)/G)_{i_1}|.$$

□

Now we arrive at our main result of the section. The following result provides a rank generating function for $\mathcal{E}^{\overrightarrow{r}}(B_n)/G$.

Proposition 6.9. *Let $z_j = \sum_{j=1}^{k} r_j^j$. Let*

$$Z_G^{i_2,\ldots,i_{k-1}}(r_1) = [Z_G(z_1, \ldots, z_{k-1}, 1)]_{r_2^{i_2} \cdots r_{k-1}^{i_{k-1}}} = \sum_t c_t r_1^t,$$

be a polynomial in r_1. Then, the coefficients c_t are the ranks of the Peck poset $\mathcal{E}^{i_2,\ldots,i_{k-1}}(B_n)/G$. In particular, the coefficients form a symmetric, unimodal sequence.

Proof. In Lemma 6.8, we saw

$$[Z_G(z_1, \ldots, z_k)]_{r_1^{i_1} \cdots r_k^{i_k}} = |(\mathcal{E}^{i_2,\ldots,i_{k-1}}(B_n)/G)_{i_1}|.$$

However, since $\sum_{j=1}^{k} i_j = n$, it follows that i_k is determined by the numbers $i_1, \ldots, i_{k-1}$, and so

$$[Z_G(z_1, \ldots, 1)]_{r_1^{i_1} \cdots r_k^{i_k}} = |(\mathcal{E}^{i_2,\ldots,i_{k-1}}(B_n)/G)_{i_1}|.$$

Next, write $Z_G^{i_2,\dots,i_{k-1}}(r_1) = \sum_t c_t(r_1)^t$. We have just shown that $c_t = |(\mathcal{E}^{i_2,\dots,i_{k-1}}(B_n)/G)_{i_1}|$. Since $\mathcal{E}^{i_2,\dots,i_{k-1}}(B_n)/G$ is a Peck poset by Lemma 3.34, its rank sizes form a symmetric and unimodal sequence. Therefore, the coefficients of $Z_G^{i_2,\dots,i_{k-1}}(r_1)$ form a symmetric, unimodal sequence. □

Remark 6.10. Note that in the case $\overrightarrow{r} = 0$, the above result precisely becomes [8, Corollary 7.16].

6.3. **Further Identities for the Wreath Product.** In this section, we draw on the methods developed earlier in this section to write down some interesting generating functions for the case that $G = S_m \wr S_l$. First, we will use Proposition 6.9 to obtain an explicit generating function for $p_i(l, m, 1)$, and then we will relate the sum $\sum_{\pi\in G}\binom{\mathrm{Fix}(\pi)}{t}$ to counting certain types of set partitions, which also count $|(\mathcal{E}^{\overrightarrow{r}}(B_n)/G)_0|$, where $\overrightarrow{r}$ is of the form $\overrightarrow{r} = 1, \dots, 1$.

Notation 6.11. For this section only, fix $m, l \in \mathbb{N}$ and fix $G = S_m \wr S_l$. Additionally, fix $n = m \cdot l$.

6.3.1. *A Generating Function for $p_i(l, m, 1)$.*

Proposition 6.12. *Let c_i be the number of i cycles in π and define*

$$W_\pi(z_1, \dots, z_n) = \begin{cases} \frac{Z_\pi(z_1,\dots,z_n)}{z_1} & \text{if } c_1 > 0 \\ 0 & \text{if } c_1 = 0 \end{cases}.$$

Then, there is an equality

$$\sum_{i=0}^{n} p_i(l, m, 1) r_1^i r_3^{n-i-1} = \sum_{\pi\in G} |\mathrm{Fix}(g)| W_\pi(r_1 + r_3, r_1^2 + r_3^2, \dots, r_1^n + r_3^n).$$

Proof. Recall from Proposition 5.5 that $p_i(l, m, 1)$ is the rank generating function of $\mathcal{E}(B_{l\cdot m})/G$.

As was seen in Proposition 6.9, it is also the case that

$$[Z_G(r_1 + r_2 + r_3, \dots, r_1^n + r_2^n + r_3^n)]_{r_2}$$

is the rank generating function of $|\mathcal{E}^1(B_n)/G|$. Therefore,

$$p_i(l, m, 1) = [Z_G(r_1 + r_2 + r_3, \dots, r_1^n + r_2^n + r_3^n)]_{r_2}.$$

So, to complete the proof, it suffices to show

$$[Z_G(r_1 + r_2 + r_3, \dots, r_1^n + r_2^n + r_3^n)]_{r_2} = \sum_{\pi\in G} |\mathrm{Fix}(g)| W_\pi(r_1 + r_3, r_1^2 + r_3^2, \dots, r_1^n + r_3^n).$$

To show this, it further suffices to show that for all $\pi \in G$,

$$Z_\pi(r_1 + r_2 + r_3, \dots, r_1^n + r_2^n + r_3^n)]_{r_2} = |\mathrm{Fix}(\pi)| W_\pi(r_1 + r_3, r_1^2 + r_3^2, \dots, r_1^n + r_3^n).$$

To see, we may note that r_2 must have a nonzero coefficient in the expansion of r_2 in Z_π. First, if r_2 has a nonzero coefficient, π must have some 1-cycle, as otherwise, no variable could appear in the expansion of Z_π raised only to the first power. Second, if π has some 1-cycle, then the coefficient of r_2 in

$$Z_\pi(r_1 + r_2 + r_3, r_1^2 + r_2^2 + r_3^2, \dots, r_1^n + r_2^n + r_3^n) = \prod_{i=1}^{n} (r_1^i + r_2^i + r_3^i)^{c_i}$$

is precisely

$$c_1 \cdot W_\pi(r_1 + r_3, r_1^2 + r_3^2, \dots, r_1^n + r_3^n) = |\mathrm{Fix}(\pi)| W_\pi(r_1 + r_3, r_1^2 + r_3^2, \dots, r_1^n + r_3^n)$$

because c_1 is the number of 1-cycles in π, which is by definition the number of fixed points of π. This is exactly what we wanted to show. □

6.3.2. *Bounded Partition Sizes.* This subsubsection is an application of Polya theory, and is moderately tangential to the rest of the paper. However, it relates to the functor $\mathcal{E}^{\overrightarrow{r}}$, in that it counts the $|(\mathcal{E}^{\overrightarrow{r}}(B_n)/G)_0| = |(\mathcal{E}^{\overrightarrow{s}}(B_n)/G)_1|$, as described in Remark 6.16.

Notation 6.13. Let $P_{[t]}[l,m]$ denote the set of partitions of the set $[t]$ into at most l blocks, such that each block has size at most m. Let $P[l,m] = \cup_{t\in\mathbb{N}} P_{[t]}[l,m]$. Denote $p_{[t]}[l,m] = |P_{[t]}[l,m]|$, and $p[l,m] = |P[l,m]|$.

Remark 6.14. The numbers $p_{[t]}[l,m]$ satisfy the recursive formula

$$p_{[t]}[l,m] = \sum_{k=1}^{l} \binom{t-1}{k-1} p_{[t-k]}[l-1,m].$$

This formula can be shown by splitting up the set $P_{[t]}[l,m]$ based on how many elements appear in the first row. The reason for including $t-1$ instead of t is that 1 must always appear in the first row.

Proposition 6.15. *There is an equality* $p_{[t]}[l,m] = \frac{t!}{l!(m!)^l} \sum_{\pi\in G} \binom{\mathrm{Fix}(\pi)}{t}$.

Proof. Using Polya's Theorem 6.6, we know

$$F_G(r_1,\dots,r_k) = \frac{1}{|G|} Z_G\left(\sum_{j=1}^{k} r_j, \sum_{j=1}^{k} r_j^2, \dots, \sum_{j=1}^{k} r_j^n\right).$$

In particular,

$$[F_G(r_1,\dots,r_k)]_{i_1 i_2\cdots i_t i_{t+1}^{m\cdot l-t}} = \left[\frac{1}{|G|} Z_G\left(\sum_{j=1}^{k} r_j, \sum_{j=1}^{k} r_j^2, \dots, \sum_{j=1}^{k} r_j^n\right)\right]_{i_1 i_2\cdots i_t i_{t+1}^{m\cdot l-t}}.$$

Of course, by definition of F_G, we know

$$[F_G(r_1,\dots,r_k)]_{i_1 i_2\cdots i_t i_{t+1}^{m\cdot l-t}} = \kappa(1,1,\dots,1,m\cdot l-t,0,\dots,0)$$

where there are t 1's in the above expression. By definition, κ is just the number of inequivalent ways to distinctly color t numbers in an $l\times m$ rectangle, up to the action of $G = S_m \wr S_l$. For any such coloring, let square a_i be colored with color r_i. Each G orbit of colorings has a unique representative such that the colors are sorted in increasing order along each row, and in increasing order down the first column. Then, the colorings defined above are in bijection with partitions of $[t]$ so that this partition has at most l blocks, and each block has at most m elements. The set of such partitions is exactly $P_{[t]}[l,m]$. Therefore, the number of such partitions is $p_{[t]}[l,m]$. Then, it follows that

$$p_{[t]}[l,m] = \kappa(1,1,\dots,1,m\cdot l-t,0,\dots,0) = \left[\frac{1}{|G|} Z_G\left(\sum_{j=1}^{k} r_j, \sum_{j=1}^{k} r_j^2, \dots, \sum_{j=1}^{k} r_j^n\right)\right]_{i_1 i_2\cdots i_t i_{t+1}^{m\cdot l-t}}.$$

Of course, $\frac{1}{|G|} = \frac{1}{l!(m!)^l}$ when $G = S_m \wr S_l$. So, to complete the proof, we just need to show that

$$\left[Z_G\left(\sum_{j=1}^{k} r_j, \sum_{j=1}^{k} r_j^2, \dots, \sum_{j=1}^{k} r_j^n\right)\right]_{i_1 i_2\cdots i_t i_{t+1}^{m\cdot l-t}} = t! \sum_{\pi\in G} \binom{\mathrm{Fix}(\pi)}{t}. \tag{6.1}$$

We reduce this problem to showing Z_G it is the sum over all the cycle indicators $\sum_{\pi\in G} Z_\pi(\sum_{i=1}^k r_i, \sum_{i=1}^k r_i^2, \dots, \sum_{i=1}^k r_i^n)$. So, to show Equation (6.1) holds, it suffices to show that

$$\left[Z_\pi\left(\sum_{j=1}^{k} r_j, \sum_{j=1}^{k} r_j^2, \dots, \sum_{j=1}^{k} r_j^n\right)\right]_{i_1 i_2\cdots i_t i_{t+1}^{m\cdot l-t}} = t! \binom{\mathrm{Fix}(\pi)}{t} \tag{6.2}$$

This is now apparent, because $Z_\pi = \prod_{i<m\cdot l}(\sum_j r_j^i)^{c_i}$, where c_i is the number of i cycles in π. The only way we can obtain a monomial of the form $r_1 r_2 \cdots r_t r_{t+1}^{m\cdot l-t}$ is if $c_1 \geq t$. Then, the coefficient of such a term will exactly be the number of ways to choose an ordered set of t elements from c_1 terms. That is, it is precisely $t!\binom{c_1}{t}$. However, c_1 is the number of 1-cycles, that is $c_1 = |\operatorname{Fix}(\pi)|$. Hence, (6.2) holds. $\square$

Remark 6.16. Thanks to Proposition 6.9, an equivalent way to state the above Proposition 6.15 is that $\sum_{\pi\in G}\binom{\operatorname{Fix}(\pi)}{t} = |(\mathcal{E}^{\overrightarrow{r}}(B_n)/G)_0|$, if $\overrightarrow{r}$ is the vector consisting of t ones. Also, letting $\overrightarrow{s}$ be the vector with $t-1$ ones, we may note $|(\mathcal{E}^{\overrightarrow{r}}(B_n)/G)_0| = |(\mathcal{E}^{\overrightarrow{s}}(B_n)/G)_1|$, since one may think of elements of $\mathcal{E}^{\overrightarrow{s}}(B_n)/G$ as tuples of $t-1$ elements, each contained in the next, whose lowest element is of rank 1, and one may think of elements of $(\mathcal{E}^{\overrightarrow{r}}(B_n)/G)_0$ as tuples of t elements, each contained in the next, whose lowest element is of rank 0. There is an obvious bijection between these two sets, given by adding or removing the element $\emptyset$ or rank 0 in B_n. Therefore, we also obtain $\sum_{\pi\in G}\binom{\operatorname{Fix}(\pi)}{t} = |(\mathcal{E}^{\overrightarrow{s}}(B_n)/G)_1|$.

We can also obtain a formula for the size of the whole set $p[l,m]$, by simply summing over all possible values of t.

Proposition 6.17. *Define the function $f : \mathbb{N}\cup 0 \to \mathbb{N}$, by*

$$f(x) = \begin{cases} \lfloor e\cdot n!\rfloor & \text{if } n>0 \\ 1 & \text{if } n=0 \end{cases}.$$

Then,

$$p[l,m] = \frac{1}{l!(m!)^l}\sum_{\pi\in G} f(|\operatorname{Fix}(\pi)|).$$

Proof. First, note that $\sum_{i=1}^k i!\binom{k}{i} = f(k)$. This can be seen fairly easily, because $\sum_{i=1}^k i!\binom{k}{i} = k!\sum_{i=0}^k \frac{1}{i!}$, while $k!\cdot e = k!\sum_{i=0}^\infty \frac{1}{i!}$. For $k>1$, it is easy to bound the difference $\sum_{i=k+1}^\infty \frac{1}{i!} < \frac{1}{k!}$, which implies $\sum_{i=1}^k i!\binom{k}{i} = \lfloor k!\cdot e\rfloor = f(k)$ for $k>1$.

By definition, $p[l,m] = \sum_{t=0}^{l\cdot m} p_{[t]}(l,m)$. Therefore, by Proposition 6.15,

$$\begin{aligned} p[l,m] &= \sum_{t=0}^{l\cdot m} p_{[t]}(l,m) \\ &= \sum_{t=0}^{l\cdot m}\frac{t!}{l!(m!)^l}\sum_{\pi\in G}\binom{\operatorname{Fix}(\pi)}{t} \\ &= \frac{1}{l!(m!)^l}\sum_{t=0}^{l\cdot m}\sum_{\pi\in G} t!\binom{\operatorname{Fix}(\pi)}{t} \\ &= \frac{1}{l!(m!)^l}\sum_{\pi\in G}\sum_{t=0}^{l\cdot m} t!\binom{\operatorname{Fix}(\pi)}{t} \\ &= \frac{1}{l!(m!)^l}\sum_{\pi\in G} f(|\operatorname{Fix}(\pi)|). \end{aligned}$$

$\square$

7. Unimodal Quotients

In this section, we show that several important posets $\mathcal{E}(B_n/G)$ are rank-unimodal. Note, they are always rank-symmetric by Corollary 3.13. First, we show unimodality for left multiplication actions. That is, if $|G| = n$, the action of G on itself by left multiplication induces an action of G on B_n such that $|(\mathcal{E}(B_n)/G)_i| = \binom{n-1}{i}$. Using this result, we show that the action of the cyclic group, C_n, on the vertices of an n-gon by rotation induces an action on B_n such that $\mathcal{E}(B_n/C_n)$ is rank-unimodal and rank-symmetric. Then, letting the dihedral group act on the vertices of an n-gon by rotations and reflections, we obtain an induced action on B_n so that $\mathcal{E}(B_n/D_{2n})$ is rank-unimodal and rank-symmetric. Furthermore, in the case that $n = p, 2p$ for p a prime, the action of D_{2n} on B_n is CCT.

7.1. **Groups of order n.** In this subsection, we show that for any group G with $|G| = n$ and a transitive action $\phi : G \times B_n \to B_n$, then

$$|(\mathcal{E}(B_n)/G)_i| = \binom{n-1}{i}.$$

Note that by identifying $G \cong [n]$ as sets, any such action is isomorphic to the action induced by left multiplication

$$\psi : G \times G \to G, (g, h) \mapsto g \cdot h.$$

Lemma 7.1. *For G a group with $|G| = n$, and a transitive action $\phi : G \times [n] \to [n]$, it follows that for all $x \in G, \mathrm{Stab}(x) = \{e\}$.*

Proof. This statement follows immediately from the Orbit Stabilizer theorem. Namely, for $x \in [n]$ the Orbit Stabilizer theorem implies $|G| = |Stab(x)||Gx|$. Then, since G is transitive, $|Gx| = |G|$ and so $|Stab(x)| = 1$. Since it is always true that $e \in Stab(y)$, it follows for all $x \in G, \mathrm{Stab}(x) = \{e\}$. □

Proposition 7.2. *Let G be a group with $|G| = n$, and a transitive action $\phi : G \times [n] \to [n]$, inducing an action $\phi : G \times \mathcal{E}(B_n) \to \mathcal{E}(B_n)$. For any element $(x, y) \in \mathcal{E}(B_n), \mathrm{Stab}(x, y) = \{e\}$.*

Proof. Note that the action ϕ induces an action of G on B_n, which in turn induces an action of G on $\mathcal{E}(B_n)$. Then, by definition $\mathrm{Stab}(x, y) = \mathrm{Stab}(x) \cap \mathrm{Stab}(y) = \mathrm{Stab}(x) \cap \mathrm{Stab}(y \setminus x)$. However, $y \setminus x$ is by assumption a 1 element set, because $|y| = |x| + 1, x \subset y$. By Lemma 7.1, $\mathrm{Stab}(y \setminus x) = \{e\}$. Hence, $\mathrm{Stab}(x, y) = \mathrm{Stab}(x) \cap \mathrm{Stab}(y \setminus x) = \mathrm{Stab}(x) \cap \{e\} = \{e\}$ for all $(x, y) \in \mathcal{E}(B_n)$, as claimed. □

Lemma 7.3. *Let G be a group with $|G| = n$ and a transitive action $\psi : G \times [n] \to [n]$, inducing an action $\phi : G \times \mathcal{E}(B_n) \to \mathcal{E}(B_n)$. Then, $|(\mathcal{E}(B_n)/G)_i| = \binom{n-1}{i}$.*

Proof. There are $\binom{n}{i}$ elements $x \in (B_n)_i$. For each $x \in (B_n)_i$ there are $n - i$ elements $y \in (B_n)_{i+1}$ so that $x \subset y$. Hence, there are a total of $\binom{n}{i} \cdot (n-i)$ elements $(x, y) \in \mathcal{E}^r(B_n)_i$. By Proposition 7.2, we have $|\mathrm{Stab}(x, y)| = 1$. Therefore, by the Orbit Stabilizer Theorem, it follows $|G(x, y)| = n$. From this,

$$|\mathcal{E}(B_n)_i/G| = |\mathcal{E}(B_n)_i|/|G| = \frac{\binom{n}{i} \cdot (n-i)}{n} = \binom{n-1}{i}.$$

□

7.2. **Quotient by the Cyclic Group.** In this subsection, let $G = C_n$ be the cyclic group of order n and let G act on $[n]$ identifying the $[n]$ with the vertices of a regular n-gon, and having C_n act by cyclically permuting these vertices. Then, let G act on B_n by the induced action. We show the poset $\mathcal{E}(B_n/G)$ is rank-unimodal. From Lemma 7.3, we know that for $G = C_n$, the size of the ith rank of the poset $\mathcal{E}(B_n)/C_n$ is

$$|(\mathcal{E}(B_n)/C_n)_i| = \binom{n-1}{i}.$$

We will relate $|(\mathcal{E}(B_n/C_n))_i|$ to $|(\mathcal{E}(B_n)/C_n)_i|$ to obtain the unimodality.

Remark 7.4. Since the action of G on $[n]$ by rotation of the vertices of an n-gon can also be viewed as the action of G on itself by left multiplication, by Proposition 4.22, we know this action of G on B_n is not CCT for $n > 2$.

Notation 7.5. We fix an arbitrary n to start with, set

$$Q(n) = \mathcal{E}(B_n)/C_n, \quad P(n) = \mathcal{E}(B_n/C_n)$$

and let

$$q(i,n) = |Q(n)_i|, \quad p(i,n) = |P(n)_i|.$$

When there is no ambiguity of which n we are referring to, we denote $Q(n), P(n)$ respectively by Q, P, and write $q_i = q(i,n), p_i = p(i,n)$.

The quotient B_n/C_n is the well-studied necklace poset. The elements in the poset are represented by n filled or empty beads ordered cyclically. Label the positions of the beads in the necklace by $1, 2, ..., n$. More specifically, since each element $x \in B_n$ is represented by a sequence of empty or filled beads, an element $G(x) \in B_n/C_n$ is represented by such a sequence up to rotational equivalence. Similarly, an element $(Gx, Gy) \in \left(\mathcal{E}(B_n)/C_n\right)_i$ where $x \lessdot y$ can be regarded as a necklace where $i+1$ beads out of n are filled (which represent y), and 1 of the filled bead is distinct from all others (so that y together with this one bead represents x).

Notation 7.6. We fix $\sigma_0 = (1\, 2\, ...\, n)$, the generator of the group C_n.

Notation 7.7. For the remainder of this section, we abuse notation by writing (Gx, Gy) in place of $(Gx, Gy) \cap \mathcal{E}(B_n)$ as sets, and hence, viewing $(Gx, Gy) = (Gx, Gy) \cap \mathcal{E}(B_n) \subset \mathcal{E}(B_n)$. Namely,

$$(Gx, Gy) = \{(\sigma x, \tau y) \in \mathcal{E}(B_n) : \sigma, \tau \in G;\ \sigma x \lessdot \tau y\}.$$

Similarly, we view

$$G(x,y) = \{(gx, gy) \in \mathcal{E}(B_n) : g \in G\}$$

Definition 7.8. Let $\sigma \in G$. We call each subset $s \subset x \in B_n$ that is fixed by σ a *full cycle* (under the action of σ). Let $\mathcal{S}$ be the set of all full cycles in x. We call $x \setminus (\cup_{s \in \mathcal{S}} s)$ a *tail cycle* (under the action of σ).

Lemma 7.9. *Suppose that $\sigma_0^r x \lessdot y$ for some (x,y), where $x \lessdot y$ and $r < n/2$. Also suppose that there is no $g \in G$ such that $gx = \sigma_0^r(x)$ and $gy = y$. Then r is the only integer $1 \leq r < n/2$ such that $\sigma_0^r(x) \lessdot y$.*

Proof. By assumption, x has a tail cycle under the action of σ_0^r, for otherwise x is fixed by σ_0^r, because then $g = \mathrm{id}$ satisfies $gx = \sigma_0^r x, gy = y$. Additionally, y must have some tail cycles under the action of σ_0^r, for otherwise y is fixed under σ_0^r and we can take $g = \sigma_0^r$. Call γ the tail cycle of y, so γ consists of some filled beads in the necklace, one of which is the element $a = y \backslash x$. Let σ be a rotation such that $\sigma x \lessdot y$. Then σ has to take a full cycle of x (under σ_0^r) to another full cycle (under σ_0^r), and it is clear that $\sigma(\gamma \backslash a) \lessdot \gamma$. Recall that $\sigma_0^r(\gamma \backslash a) \lessdot \gamma$ by assumption, so the elements in γ are r-positions away from each other (in the necklace representation), in other words, we can write $\gamma = \{a, a-r, a-2r, ...a-lr\}$, but $a + r \notin \gamma$, since γ is a tail cycle. Assume for contradiction that there is some other $\sigma' \neq \sigma_0^r$ such that $\sigma'(\gamma \backslash a) \lessdot \gamma$, then there exists $m > 1$ for which $\sigma'(a - mr) = a$, namely the rotation sends $a - mr \mapsto a$. But then, since $m > 1$, $a - (m-1)r \in x$ and $a - (m-1)r \neq a$. Hence $(a - (m-1)r) \mapsto a + r \in y$, a contradiction. $\square$

Lemma 7.10. *Let $G = C_n$, and $q_i = q(i,n), p_i = p(i,n)$ as usual, then*

$$q_i - p_i = |\{G(x,y) : \exists\ \sigma_0^r x \lessdot y \text{ where } r < n/2, \text{ but } \nexists\, g \text{ s.t. } gx = \sigma_0^r x, gy = y\}|.$$

Proof. By definition, q_i counts number of the disjoint subsets $G(x,y)$ in $\mathcal{E}(B_n)$ and p_i counts the number of disjoint subsets (Gx, Gy). Hence the difference of $q_i - p_i$ is counting the difference of the number of G-orbits $G(x,y)$ and the number of $(Gx, Gy) \in \mathcal{E}(B_n/C_n)$. Note that as subsets of $\mathcal{E}(B_n)$, $G(x,y) \subset (Gx, Gy)$, so the problem of counting $q_i - p_i$ is the same as that of counting the number of distinct G-orbits $G(x,y)$ with non-identity $\sigma \in G$ such that $\sigma x < y$.

As mentioned above, it is clear that the G-orbits correspond to the (rotational equivalence classes of) fillings of the necklaces, where we fill $i+1$ elements with 1 of them being marked as the distinct element. Lemma 7.9 tells us that each subset $(Gx, Gy) \subset \mathcal{E}(B_n)$, contains at most two G-orbits of the form $G(x, y)$ (also regarded as subsets of $\mathcal{E}(B_n)$). The number of (Gx, Gy) containing exactly two G orbits is precisely the number of pairs of G-orbits $(G(x, y), G(\sigma x, y))$ (where $x \lessdot y$ and $\sigma x \lessdot y$) with $G(x, y) \neq G(\sigma x, y)$. The number of such pairs is then the number of $G(x, y)$ such that $\exists r, r < \frac{n}{2}, G(\sigma_0^r x, y) \neq G(x, y), \sigma_0^r x \lessdot y$. We restrict $r < n/2$ to avoid double counting pairs. This proves the Lemma. $\square$

For the unimodality of p_i, we first consider a special case where $n = p$ is a prime, for which we have a simple formula for p_i based on the formula for q_i.

Lemma 7.11. *Let $G = C_p$ where p is a prime, $p \neq 2$ then $q_i - p_i = (p-1)/2$ for $1 \leq i \leq p-2$.*

Proof. By Lemma 7.10, we need to prove that the number of G-orbits $G(x, y)$ such that there is some $r < n/2$ with $\sigma_0^r x \lessdot y$ but there does not exist $g \in \operatorname{stab}(y)$ such that $gx = \sigma_0^r x$ is $(p-1)/2$.

We consider the action of G on B_p. Note that p being a prime guarantees that the action of any nontrivial $\sigma \in G$ has no fixed points, since σ is always an p-cycle in its cycle decomposition. Now suppose $(x, y) \in \mathcal{E}(B_p)_i$ is an element such that $\sigma x = y$ for some $\sigma \neq e$. Then, there is no $g \in C_p$ such that $gx = \sigma x$ and $gy = y$, since $gx = \sigma x \Rightarrow g = \sigma$ and $gy = y \Rightarrow g = e$. Therefore, in the case where p is a prime, we have:

$$q_i - p_i = |\{G(x, y) : \exists\ \sigma_0^r x \lessdot y \text{ where } 1 \leq r < n/2\}|.$$

We claim that for each r such that $1 \leq r < p/2$, there is precisely one G orbit $G(x, y)$ such that $\sigma_0^r x \lessdot y$.

First consider $r = 1$, so $\sigma = \sigma_0$. Suppose there is $(x, y) \in \mathcal{E}(B_n)$ such that $x \lessdot y$ and $\sigma x \lessdot y$, i.e., we take out the distinct filled bead from y in the necklace, and rotate y by σ_0, then the remaining i filled beads remain in σy. It is clear that the only way this can happen is if we have $i+1$ consecutive filled beads, which gives the same G-orbit as $G(x, y)$.

The same construction applies to all cases where $1 \leq r < p/2$, so for each $r = 1, 2, ..., (p-1)/2$, there is a precisely one orbit $G(x_r, y_r)$ such that $x_r \lessdot y_r$ and $\sigma_0^r x < y$. This proves the lemma. $\square$

Corollary 7.12. *Let $G = C_n$ where n is a prime, and p_i as defined above. Then the sequence p_i is unimodal.*

Proof. By the Lemma 7.11 and Lemma 7.3, we can explicitly compute the p_i to obtain that $p_i \leq p_{i+1}$ for $0 < i < n/2$. By the symmetry of p_i, we only need to show $p_0 \leq p_1$, but $p_0 = 1$, so the claim holds. $\square$

A similar method can be used to bound $q_i - p_i$ for n not necessarily prime. Let $\lambda_i = q_i - p_i$, our next goal in the section is to show that, for sufficiently large n, the difference $\lambda_{i+1} - \lambda_i$ is bounded above by $q_{i+1} - q_i$. This is shown in Lemma 7.17.

Note that this suffices to show that the p_i are unimodal, since this would imply

$$p_{i+1} - p_i = (q_{i+1} - \lambda_{i+1}) - (q_i - \lambda_i) = (q_{i+1} - q_i) - (\lambda_{i+1} - \lambda_i) \geq 0.$$

Definition 7.13. Fix $r < n/2$ to be an integer and let $G(x, y)$ be a G-orbit of edges. We call $G(x, y)$ an *isolated pair* if $\sigma_0^r x \lessdot y,$ but there does not exist $g \in C_n$ such that $gx = \sigma_0^r x,$ and $gy = y$.

First we prove several sub-lemmas.

Definition 7.14. For the next couple proofs, we will use the notation

$$S_1 = \{G(x, y) : \exists\ \sigma_0^r x \lessdot y \text{ where } r < n/2, \text{ but } \nexists\, g \text{ s.t. } gx = \sigma_0^r x, gy = y\}$$

and

$$S_2 = \{G(x, y) : \exists\ \sigma_0^r x \lessdot y \text{ where } r < n/2, \text{ and } y \text{ has a tail cycle}\}.$$

Lemma 7.15. *$\lambda_i = p_i - q_i$ is equal to the number of $G(x,y) \in \epsilon(B_n)/G$ such that there exists $r < n/2$ with $\sigma_0^r x \lessdot y$ and y has a tail cycle under the action of σ_0^r.*

Proof. From Lemma 7.10 we know that, $\lambda_i = q_i - p_i = |S_1|$. We want to show that $S_1 = S_2$. It is clear that $S_2 \subset S_1$, so it remains to show that $S_1 \subset S_2$.

For each $r < n/2$ such that $(r,n) = 1$, there is precisely one such G-orbit for each i, since in this case σ_0^r is a full n-cycle and the argument in Lemma 7.11 applies. In this case, suppose that $\sigma_0^r x \lessdot y$ then y is automatically a tail cycle.

Now consider $r < n/2$ such that $(r,n) > 1$. In this case, σ_0^r fixes elements of the form $\{s, s+r, s+2r, ...,\}$ for any starting point s, which are proper subsets of $[n]$. For $y \in B_n$, one may write y as the union of full cycles and at most one tail cycle under the action of σ_0^r. Now, if $y \in B_n$ has no tail cycles under the action of σ_0^r then $(\sigma_0^r x, \sigma_0^r y) = (\sigma_0^r x, y) \in G(x,y)$. Hence $G(x,y) \notin S_1$, and this proves the lemma. □

Lemma 7.16. *We may bound*

$$\lambda_{i+1} - \lambda_i \leq |\{G(x,y) \in S_2(i+1) : \text{ the tail cycle in } y \text{ has at most } 2 \text{ elements}\}|.$$

Proof. By the previous lemma, $\lambda_i = q_i - p_i$ is equal to the number of $G(x,y)$ such that there exists r with $x \lessdot y, \sigma_0^r x \lessdot y$ and y has a tail cycle under the action of σ_0^r where $r < n/2$. In other words, $\lambda_i = |S_2(i)|$. Therefore, $\lambda_{i+1} - \lambda_i = |S_2(i+1)| - |S_2(i)|$.

Now for each i we define two subsets $S_3(i) \subset S_2(i)$ and $S_4(i) \subset S_2(i)$ as follows:

$$S_3(i) = \{G(x,y) \in S_2(i) : \text{ the tail cycle in } y \text{ has only 2 elements}\};$$

and

$$S_4(i) = \{G(x,y) \in S_2(i) : \text{ the tail cycle in } y \text{ has at least 3 elements}\}.$$

It is clear that S_3 and S_4 are disjoint subsets of S_2 and that $S_2 = S_3 \cup S_4$. In particular, we know that $|S_2(i+1)| = |S_3(i+1)| + |S_4(i+1)|$. Thus

$$\lambda_{i+1} - \lambda_i = |S_3(i+1)| + |S_4(i+1)| - |S_2(i)|.$$

Our goal is to show that $|S_4(i+1)| - |S_2(i)| \leq 0$ and then the lemma follows.

We create an injective map ϕ from $S_4(i+1)$ to $S_2(i)$. For each orbit $G(x,y) \in S_4(i+1)$ with $\sigma_0^r x \lessdot y$ for some $r < n/2$, by construction we know that there is a tail cycle in y which consists of at least two elements in $[n]$. Let the tail cycle be $w := \{s, s+r, s+2r, \ldots, s+lr\}$ for some $l \geq 1$, and $x = y \backslash \{s+lr\}$. We define $\phi(G(x,y)) = G(x', y')$ where $x' = x \backslash \{s\}$ and $y' = y \backslash \{s\}$. It is clear that $x' \lessdot y'$ and $\sigma_0^r x' \lessdot y'$, and that y' has a tail cycle, namely $w' = \{s+r, ..., s+lr\}$. This shows that $\phi : S_4(i+1) \to S_2(i)$ is a well defined map. The map is easily seen to be injective, since on the image $\phi(S_4(i+1))$ there is a well defined right sided inverse ϕ^{-1}, by adding an element in a similar fashion to both x', y' where $G(x', y') \in \mathrm{Im}\phi$. We remark that the map ϕ is not necessarily surjective. □

Lemma 7.17. *Let $G = C_n$ and $q(i,m)$ be as defined above, then*

$$\lambda_{i+1} - \lambda_i \leq \sum_{\substack{k|(n,i) \\ 3 \leq k}} k \cdot q\left(\frac{i}{k}, \frac{n}{k}\right).$$

Proof. By Lemma 7.16 we need to show that $|S_3(i+1)| \leq \sum_{\substack{d|(n,i) \\ 3 \leq d}} d \cdot q\left(\frac{i}{d}, \frac{n}{d}\right)$.

For any $G(x,y) \in S_3(i+1)$, y consists of the union of several full cycles and a tail cycle of size 2. The full cycles in y are d rotationally symmetric for some $d|n$ and $d|i$ since $|y| = i+2$. We count the number of such $G(x,y)$ according to the number d. For each $d|(i,n)$, we define

$$V_d = \{G(x,y) \in S_3(i+1) : \text{ the full cycles in } y \text{ is } d\text{-rotationally symmetric}\}.$$

Note that $V_2 = \emptyset$, for otherwise for any $G(x,y) \in V_2$, y would have only full cycles where each cycle consists of 2 elements. Therefore,

$$|S_3(i+1)| \leq \sum_{\substack{d|(i,n)\\ 3\leq d}} |V_d|,$$

since $S_3(i+1) = \bigcup_{\substack{d|(i,n)\\ 3\leq d}} V_d$ though the union of the right hand side is in general not disjoint.

Now we give an upper bound for $|V_d|$ for each $d|(n,i)$. Namely, we show that $|V_d| \leq d \cdot q(i/d, n/d)$. It is clear that this suffices to prove the lemma.

Note that for any $G(x,y) \in V_d$, there exists some r such that $\sigma_0^r x \lessdot y$, and from construction we know that $d|r$. For a given $d|(i,n)$, we consider the number of possible G-orbits Gx such that there is some y with $G(x,y) \in V_d$. By definition we know that such x consists of full cycles and a tail cycle of 1 element under σ_0^r for any r such that $d|r$, and each full cycle has d elements. Let $\{s, s+d, ..., s+(n-d)\}$ be a full cycle in x, the number of Gx is then the number of the possible ways to choose $i/d - 1$ elements between s and $s+d$ to form the starting points of other full cycles and 1 additional element to form the unique tail cycle. This is precisely count the number of $G(a,b) \in \mathcal{E}(B_{n/d})_{i/d}$, which is $q(i/d, n/d)$ by definition. For any such x as described, there is at most $d-1$ different y such that $G(x,y) \in V_d$, since the rotation r has to be a multiple of d. This finishes the proof of the lemma. □

Theorem 7.18. *The poset $\mathcal{E}(B_n/C_n)$ is rank-symmetric and rank-unimodal.*

Proof. We need to show that for $G = C_n$, the p_i are unimodal. So, we only need to bound $\sum_{\substack{k|(n,i)\\ 3\leq k<n/2}} k \cdot q(\frac{i}{k}, \frac{n}{k})$. It is clear that for sufficiently large n, say for $n \geq 9$,

$$q((i)/k, n/k) = \binom{n/k - 1}{i/k - 1} \leq \binom{\lceil n/3 - 1\rceil}{\lceil i/3 - 1\rceil}.$$

Since $k \leq i$, and $i < n/2$, we coarsely bound the sum by

$$\sum_{\substack{k|(n,i)\\ 3\leq k<n/2}} k \cdot \binom{\lceil n/3 - 1\rceil}{\lceil i/3 - 1\rceil} \leq \left(\frac{n}{2}\right)^2 \binom{\lceil n/3 - 1\rceil}{\lceil i/3 - 1\rceil}.$$

We want to show that this is smaller than the difference of

$$q_i - q_{i-1} = \frac{(n-1)!}{(i)!(n-i)!} - \frac{(n-1)!}{(i-1)!(n-i+1)!} = \binom{n}{i}\frac{n-2i}{n} \geq \binom{n}{i}\frac{2}{n}.$$

Namely, we want to show that, for sufficiently large n,

$$\left(\frac{n}{2}\right)^3 \cdot \binom{\lceil n/3 - 1\rceil}{\lceil i/3 - 1\rceil} \leq \binom{n}{i}.$$

This bound works for n sufficiently large. For smaller n the claim can be (and has been) checked.

Now, by Lemma 7.17, we know that

$$p_{i+1} - p_i = (q_{i+1} - q_i) - (\lambda_{i+1} - \lambda_i) \geq 0,$$

which proves that the p_i are increasing for $i < n/2$. By symmetry, the p_i are unimodal. □

7.3. **Quotient by the Dihedral Group.** In this subsection, we show a similar result for the Dihedral groups that act naturally on B_n. Namely, for $G = D_{2n}$, the dihedral group of order $2n$, the poset $\mathcal{E}(B_n/G)$ is rank-symmetric and rank-unimodal.

Notation 7.19. In this section, we fix an arbitrary n again, set

$$Q = \mathcal{E}(B_n)/D_{2n}, \quad P = \mathcal{E}(B_n/D_{2n})$$

and let

$$q_i = |\big(\mathcal{E}(B_n)/D_{2n}\big)_i|, \quad p_i = |\big(\mathcal{E}(B_n/D_{2n})\big)_i|.$$

Proposition 1.6. *Let p be prime. The following actions are CCT.*

(1) *The action of S_n on B_n.*
(2) *The action of D_{2p} on B_p.*
(3) *The action of D_{4p} on B_{2p}.*

Proof. We have already seen that part (1) holds trivially. We prove part (2), and the proof of part (3) is similar.

We claim that given $G(x,y) \in \mathcal{E}(B_p)$ such that $\sigma x \lessdot y$ where $\sigma \in D_{2p}$, there exists some $\tau \in D_{2p}$ such that $\tau x = \sigma x$ and $\tau y = y$.

The action of D_{2p} on B_p is induced by the action of D_{2p} on $[p]$ where $[p]$ is identified with vertices of some regular p-polygon. Note that D_{2p} is generated by the cyclic subgroup C_p (corresponding to rotations of the polygon) and an arbitrary reflection r. In particular, any element in D_{2p} is either some reflection r by one of the lines of symmetry of the polygon, or some rotation σ_0^d where σ_0 is the generator $\sigma_0 = (12\cdots p)$ as defined previously in this section, and d is some integer. Hence we only need to show the claim when $\sigma = r$ or $\sigma = \sigma_0^d$. It is clear that the claim holds for $\sigma = r$: if $x \lessdot y$ and $r \cdot x \lessdot y$, then it is easy to see that $r \cdot y = y$. Now suppose $\sigma_0^d \cdot x \lessdot y$ for some $G(x,y) \in \mathcal{E}(B_p)_i$. Following the proof of 7.11, we know that (x,y) is of form $x = \{s, s+d, ..., s+(i-1)d\}$ for some starting point $s \in [n]$ and $y = \{s, s+d, ..., s+(i-1)d, s+i\cdot d\}$. Now let r_0 be the reflection that sends $x \mapsto (2s + i\cdot d) - x$ for all $x \in [n]$, reducing mod n whenever necessary. Then r_0 sends $s \mapsto s + i\cdot d, s+d \mapsto s+(i-1)d, \ldots$. It is clear that $r_0 x = \sigma_0^d x$ and r_0 fixes y by construction. This proves the claim, which by definition shows that the action D_{2p} on B_p is CCT. $\square$

Remark 7.20. Let G be as in Proposition 1.6 We remark that, in addition to unimodality, Theorem 1.5 implies that the poset $\mathcal{E}(B_n/G)$ is Peck.

Remark 7.21. It is easy to see that if $n \neq p, n \neq 2p, n > 8$ for any prime p, then the action of D_{2n} on B_n is not CCT. We give an example of a non-CCT pair. Assume $n \neq p, 2p$ in its factorization, then $n = mk$ for some $m \geq k \geq 3$. Let us consider elements x, y, z where z consists of a full cycle of k elements and a tail of 2 elements. For example, $z = \{1, m+1, 2m+1, ..., (k-1)m+1, 2, m+2\}$. Let $x = z \backslash \{m+2\}$ and $y = z\backslash\{2\}$. We immediately have that $x, y \lessdot z$, and $x \in D_{2n} y$ since x is sent to y by the permutation $\sigma_0^m \in D_{2n}$. It is also clear that there is no $g \in D_{2n}$ translating x to y while fixing z, from the asymmetry of the element z. Therefore, the action of D_{2n} on B_n as described is CCT if and only if $n = p$ or $n = 2p$ for some prime p.

Hence, a complete list of n for which D_{2n} is *CCT* is given by $n = p, n = 2p, n = 1, n = 8$ where p varies over all primes.

Remark 7.22. An alternate proof of Lemma 7.23 follows directly from Polya's Theorem 6.6, by counting the number of ways to color n elements with three colors, so that there are i of the first color, 1 of the second color, and $n - i - 1$ of the third color.

Lemma 7.23. *We have an explicit formula for $q_i = |\big(\mathcal{E}(B_n)/D_{2n}\big)_i|$:*

$$q_i = \frac{1}{2}\Big(\binom{n-1}{i} + \frac{1}{2}[(-1)^{n(i+1)} + 1] \cdot \binom{\lceil \frac{n}{2} \rceil - 1}{\lceil \frac{i+1}{2} \rceil - 1}\Big)$$

Proof. Consider an element $(x,y) \in \mathcal{E}(B_n)$ where $x \lessdot y$. Note that any element τ that fixes both x and y has to fix the difference $y \backslash x$, which is a one-element set. Since $G = D_{2n}$, we know that $1 \leq |\operatorname{Stab}(x,y)| \leq 2$. Let μ_1 be the number of G-orbits with the trivial stabilizer and μ_2 be the number of G-orbits with the stabilizer of size 2, which contains the identity and a reflection. By the Orbit Stabilizer Theorem, each orbit with the trivial stabilizer is of size $|D_{2n}|/1 = 2n$, while all other orbits are of size $|D_{2n}|/2 = n$. Therefore

$$\mu_1 \cdot 2n + \mu_2 \cdot n = |\{(x,y)\}| = \binom{n}{i+1} \cdot \binom{i+1}{1} = n\binom{n-1}{i}.$$

Our goal is to calculate $q_i = \mu_1 + \mu_2$, so it remains to calculate μ_2, which counts the number of G-orbits such that the reflection also fixes x. Without loss of generality, we may assume that $y \backslash x = \{1\}$.

There are two cases:

(1) If n is odd and $i+1$ is even, then it is clear that no reflections fix x for any x, so $\mu_2 = 0$.
(2) For all other cases, there are precisely $\lceil \frac{n}{2} - 1 \rceil$ places to insert $\lceil \frac{i+1}{2} - 1 \rceil$ elements of $[n]$ to form x.

This gives the desired formula $\mu_2 = \dfrac{1}{2}[(-1)^{n(i+2)} + 1] \cdot \dbinom{\lceil \frac{n}{2} \rceil - 1}{\lceil \frac{i+1}{2} \rceil - 1}$. Therefore,

$$q_i = \frac{1}{2}\left(\binom{n-1}{i} + \frac{1}{2}[(-1)^{n(i+2)} + 1] \cdot \binom{\lceil \frac{n}{2} \rceil - 1}{\lceil \frac{i+1}{2} \rceil - 1}\right)$$

□

Corollary 7.24. *The poset $\mathcal{E}(B_n/D_{2n})$ is rank-symmetric and rank-unimodal.*

Proof. The proof is similar to the proof of Lemma 7.17 and Corollary 7.18. In fact, we can bound the difference of $\lambda_{i+1} - \lambda_i$ by precisely the same bound used in Lemma 7.17, since the difference we get here (where $G = D_{2n}$) is smaller than the previous difference (where $G = C_n$). Then, a similar proof shows that the difference of $q_{i+1} - q_i$ obtained in Lemma 7.23 is significantly larger than the upper bound. □

8. The object $\mathcal{E}(B_n)$.

In this section, we explicitly compute the raising operators corresponding to both $\mathcal{H}(B_n), \mathcal{E}(B_n)$ and explicitly show that $\mathcal{E}(B_n)$ is unitary Peck by showing certain raising maps are invertible.

8.1. **Computation of Order Raising Maps.**

In this subsection, we give explicit formulas for compositions of the order raising maps on $\mathcal{H}(B_n)$ and $\mathcal{E}(B_n)$

Notation 8.1. For this section, we will let M be the Lefschetz map on $\mathcal{E}(B_n)$, as defined in 2.3.

Notation 8.2. For the remainder of this section, we fix n, take i with $0 \le i < n/2$, and implicitly suppose $|a| = n-i-1, |b| = n-i, |x| = i, |y| = i+1$. Let $k = n - 2i - 1$, that is, $k = |a| - |x|$. Additionally, whenever we write an expression of the form (x, y) or (a, b), it is assumed that $x \lessdot y, a \lessdot b$. We shall also implicitly sum over all $x \subset y$ in what follows.

Proposition 8.3. *Defining L to be the Lefschetz map $\mathcal{H}(B_n) \to \mathcal{H}(B_n)$, we have L^{n-2i-1} : $\mathcal{H}^1(B_n)_i \to \mathcal{H}^1(B_n)_{n-i-1}$, and explicitly*

$$L^{n-2i-1}(x,y) = k! \sum_{\substack{y \not\subset a,\\ x \subset a,\\ y \subset b}} (a,b).$$

Proof. Note that the conditions $y \not\subset a, x \subset a, y \subset b$ are equivalent to $(a,b) >_{\mathcal{H}(B_n)} (x,y)$. Clearly, if $(a,b) \not>_{\mathcal{H}(B_n)} (x,y)$, then the coefficient of (a,b) in $L^{n-2i-1}(x,y)$ is 0. So, to complete the proof, it suffices to show that if $(a,b) >_{\mathcal{H}(B_n)} (x,y)$ then the coefficient of (a,b) in $L^{n-2i-1}(x,y)$ is $k!$. However, this coefficient is precisely the number of sequences

$$(x,y) = (x_0,y_0) \lessdot_{\mathcal{H}(B_n)} (x_1,y_1) \lessdot_{\mathcal{H}(B_n)} \cdots \lessdot_{\mathcal{H}(B_n)} (x_k,y_k) = (a,b).$$

By definition of $\mathcal{H}$, we must have that $y_k \setminus x_k = y \setminus x$ for all k. Therefore, the number of such sequences is equal to the number of sequences $x = x_0 \lessdot_{B_n} x_1 \lessdot_{B_n} \cdots \lessdot_{B_n} x_k = a$, since choosing the x_i determine y_i because $y_i = x_i \cup (y \setminus x)$. Finally, the number of such sequences

$$x = x_0 \lessdot_{B_n} x_1 \lessdot_{B_n} \cdots \lessdot_{B_n} x_k = a,$$

is equivalent to the number of ways to add the elements in $a \setminus x$ to x. This is because each sequence

$$x = x_0 \lessdot_{B_n} x_1 \lessdot_{B_n} \cdots \lessdot_{B_n} x_k = a$$

is determined uniquely by the singletons $x_{i+1} \setminus x_i, 1 \leq i \leq k$. Since in total we are adding k elements to x in order to obtain a, there are $k!$ ways to do this. Therefore, coefficient of (a, b) in $L^{n-2i-1}(x, y)$ is $k!$ □

Proposition 8.4. *The map M satisfies*

$$M^{n-2i-1}(x, y) = (2^k - 1)(k - 1)! \sum_{y \subset a} (a, b) + k! \sum_{\substack{y \not\subset a, \\ x \subset a, \\ y \subset b}} (a, b)$$

Proof. For a particular $(x, y), (a, b)$, if neither $y \subset a$ holds nor all three of $y \not\subset a, x \subset a$, and $y \subset b$ hold, then $(a, b) \not\geq_{\mathcal{E}(B_n)} (x, y)$, and so the coefficient of (a, b) in $M^{n-2i-1}(x, y)$ would then be 0.

Clearly, we cannot have both $y \subset a$ and $y \not\subset a, x \subset a, y \subset b$, hold at the same time. So, suppose $y \not\subset a, x \subset a, y \subset b$. This implies that $b \setminus a = y \setminus x$. Then, the coefficient of (a, b) in $M^{n-2i-1}(x, y)$ is precisely the number of sequences

$$(x, y) = (x_0, y_0) \lessdot_{\mathcal{E}(B_n)} (x_1, y_1) \lessdot_{\mathcal{E}(B_n)} \cdots \lessdot_{\mathcal{E}(B_n)} (x_k, y_k) = (a, b) \tag{8.1}$$

However, since $y \setminus x = b \setminus a$, it must be that $y_k \setminus x_k = y \setminus x$ as well. Therefore, the number of such sequences is equal to $k!$, as was shown in the proof of Proposition 8.3.

To complete the proof, we need to show that if $y \subset a$ then the coefficient of (a, b) in $M^{n-2i-1}(x, y)$ is $(2^k - 1)(k - 1)!$. Equivalently, we need to show that the number of sequences as in (8.1) is $(2^k - 1)(k - 1)!$.

For the moment, fix j and consider the set of all sequences of the form in Equation (8.1) such that $(b \setminus a) \cup y_{j-1} = y_j$ First, let us show the number of such sequences with this j fixed is $(k - 1)!2^j$. To do this, start by considering the number of sequences $y_0, \ldots, y_k$ such that $y = y_0 \lessdot_{B_n} y_1 \lessdot_{B_n} \cdots \lessdot_{B_n} y_k = b$. Since we enforce $y_j = y_{j-1} \cup (b \setminus a)$, the number of such sequences is precisely the number of ways to order the elements of $a \setminus y$. Since $|a| - |y| = k - 1$, there are $(k - 1)!$ such ways. Additionally, for $l < j - 1$ we must have $x_{l+1} = x_l \cup (y_{l+1} \setminus y_l)$ or $x_{l+1} = x_l \cup (y_l \setminus x_l)$.

Either of these is possible at every step. Additionally, for $l \geq j - 1$, we must have $x_{l+1} = x_l \cup (y_{l+1} \setminus y_l)$. So for any fixed sequence of y_k with so that j is minimal with $(b \setminus a) \in y_j$, there are precisely 2^{j-1} possible sequences

$$x = x_0 \lessdot_{B_n} x_1 \lessdot_{B_n} \cdots \lessdot_{B_n} x_k = a$$

so that

$$(x, y) = (x_0, y_0) \lessdot_{\mathcal{E}(B_n)} (x_1, y_1) \lessdot_{\mathcal{E}(B_n)} \cdots \lessdot_{\mathcal{E}(B_n)} (x_k, y_k) = (a, b).$$

So, in total, there are $2^{j-1}(k - 1)!$ sequences of the form in (8.1) with $(b \setminus a) \cup y_{j-1} = y_j$. Now, there clearly must be exactly one $j, 1 \leq j \leq k$ such that this is the case. Therefore, the coefficient we are looking for is

$$\sum_{j=1}^{k} 2^{j-1}(k - 1)! = (2^k - 1)(k - 1)!,$$

as claimed.

□

8.2. **Proof that $\mathcal{E}(B_n)$ is unitary Peck.** In this subsection, we will show the rows of M form a basis by showing we can make a change of basis to a map which takes M^{n-2i-1} to L^{n-2i-1}. Since we know L^{n-2i-1} is an isomorphism, it will follow that M^{n-2i-1} is as well.

Notation 8.5. Let $\beta = \frac{2^k-1}{k}$. Denote

$$v_{(a,b)} = \beta \sum_{y\subset a}(x,y) + \sum_{y\subset b, x\subset a, y\not\subset a}(x,y).$$

Note that $v_{(a,b)}$ are simply the rows of M^{n-2i-1}, each divided by the constant $k!$.

Notation 8.6. For any set s of size at least $n-i$,

$$z_s = \frac{1}{\beta\binom{|s|-i-1}{n-2i-2}(|s|-n+i+1)+\binom{|s|-i-1}{n-2i-1}} \sum_{b\subset s, a\subset b} v_{(a,b)}$$

Lemma 8.7. *For any set s of size at least $n-i$, we have $z_s = \sum_{y\subset s}(x,y)$. In particular, $\sum_{y\subset s}(x,y)$ lies in the span of the $v_{(a,b)}$*

Proof. We have

$$\begin{aligned}
&\left(\beta\binom{|s|-i-1}{n-2i-2}(|s|-n+i+1)+\binom{|s|-i-1}{n-2i-1}\right) z_s \\
&= \sum_{b\subset s, a\subset b} v_{(a,b)} \\
&= \sum_{b\subset s, a\subset b}\left(\beta\sum_{y\subset a}(x,y) + \sum_{y\subset b, x\subset a, y\not\subset a}(x,y)\right) \\
&= \sum_{b\subset s, a\subset b}\beta\sum_{y\subset a}(x,y) + \sum_{b\subset s, a\subset b}\sum_{y\subset b, x\subset a, y\not\subset a}(x,y). \\
&= \beta\binom{|s|-i-1}{n-2i-2}(|s|-n+i+1)\sum_{y\subset s}(x,y) + \sum_{b\subset s, a\subset b}\sum_{y\subset b, x\subset a, y\not\subset a}(x,y). \\
&= \beta\binom{|s|-i-1}{n-2i-2}(|s|-n+i+1)\sum_{y\subset s}(x,y) + \binom{|s|-i-1}{n-2i-1}\sum_{y\subset s}(x,y) \\
&= \left(\beta\binom{|s|-i-1}{n-2i-2}(|s|-n+i+1)+\binom{|s|-i-1}{n-2i-1}\right)\sum_{y\subset s}(x,y).
\end{aligned}$$

In going between the fourth line and the fifth line, one needs to count the number of y satisfying $y\subset a\subset b\subset s$. If we fix s, a there are $(|s|-n+i+1)$ choices for the element b, since $b = a\cup\{s\}$ for $s\notin a$. Then, we need to count the number of a with $y\subset a\subset s$. This is exactly $\binom{|s|-i-1}{n-2i-2}$. Hence, the total of number of such y is the product $\binom{|s|-i-1}{n-2i-2}(|s|-n+i+1)$.

In going from the fifth line to the sixth line, for y, s fixed, we count the number of b with $y\subset b\subset s$. This is exactly $\binom{|s|-i-1}{n-2i-1}$. Then, for each of these possibilities, if we additionally fix x, the value of a such that $x\subset a, y\not\subset a$ is uniquely determined by $a = b\setminus(y\setminus x)$. □

Notation 8.8. For any set s of size at least $n-i$, let $w_s = \sum_{a\subset s, t\notin s} v_{(a,a\cup\{t\})}$.

Lemma 8.9. *We have*

$$w_s = \sum_{t\notin s}\left(\beta\binom{|s|-i-1}{n-2i-2}\sum_{y\subset s}(x,y) + \binom{|s|-i}{n-2i-1}\sum_{x\subset s}(x, x\cup\{t\})\right).$$

Proof.

$$\begin{aligned}
w_s &= \sum_{a \subset s, t \notin s} v_{(a, a\cup\{t\})} \\
&= \sum_{a \subset s, t \notin s} \left(\beta \sum_{y \subset a} (x,y) + \sum_{y \subset a \cup \{t\}, x \subset a, y \not\subset a} (x,y) \right) \\
&= \sum_{t \notin s} \left(\sum_{a \subset s} \beta \sum_{y \subset a} (x,y) + \sum_{a \subset s} \sum_{y \subset a \cup \{t\}, x \subset a, y \not\subset a} (x,y) \right) \\
&= \sum_{t \notin s} \left(\beta \binom{|s|-i-1}{n-2i-2} \sum_{y \subset s} (x,y) + \sum_{a \subset s} \sum_{y \subset a \cup \{t\}, x \subset a, y \not\subset a} (x,y) \right) \\
&= \sum_{t \notin s} \left(\beta \binom{|s|-i-1}{n-2i-2} \sum_{y \subset s} (x,y) + \binom{|s|-i}{n-2i-1} \sum_{x \subset s} (x, x \cup \{t\}) \right)
\end{aligned}$$

The equalities between lines three and four, and four and five hold for similar reasons as the equalities between lines four and five, and five and six in 8.7 ☐

Notation 8.10. For any set s with $|s| \geq n-i$, let $u_s = \frac{w_s - (n-|s|)\beta\binom{|s|-i-1}{n-2i-2} z_s}{\binom{|s|-i}{n-2i-1}} + z_s$.

Lemma 8.11. *We have $u_s = \sum_{x \subset s} (x,y)$. In particular, $\sum_{x \subset s}(x,y)$ lies in the span of $v_{(a,b)}$.*

Proof. By 8.9 and 8.7 we have

$$\begin{aligned}
u_s &= \frac{w_s - (n-|s|)\beta\binom{|s|-i-1}{n-2i-2} z_s}{\binom{|s|-i}{n-2i-1}} + z_s \\
&= \frac{\sum_{t \notin s} \left(\beta \binom{|s|-i-1}{n-2i-2} \sum_{y \subset s}(x,y) + \binom{|s|-i}{n-2i-1} \sum_{x \subset s} (x, x\cup\{t\}) \right)}{\binom{|s|-i}{n-2i-1}} \\
&\qquad - \frac{(n-|s|)\beta\binom{|s|-i-1}{n-2i-2} \sum_{y \subset s}(x,y)}{\binom{|s|-i}{n-2i-1}} + \sum_{y \subset s}(x,y) \\
&= \frac{\sum_{t \notin s} \left(\binom{|s|-i}{n-2i-1} \sum_{x \subset s}(x, x \cup \{t\}) \right)}{\binom{|s|-i}{n-2i-1}} + \sum_{y \subset s}(x,y) \\
&= \sum_{t \notin s} \sum_{x \subset s} (x, x \cup \{t\}) + \sum_{y \subset s} (x,y) \\
&= \sum_{x \subset s} (x,y)
\end{aligned}$$

The penultimate line is equal to the ultimate line because any element (x,y) with $x \subset s$ must either have $y \subset s$ or else $y \subset s \cup \{t\}$ for $t \notin s$. The two terms on the penultimate line cover precisely these two cases. ☐

Notation 8.12. For any $s \subset [n]$, with $|s| \geq n-i$, define $h_s = z_{[n]} - u_s$.

Lemma 8.13. *With h_s as defined above, $h_s = \sum_{x \not\subset s}(x,y)$*

Proof. Using 8.7 and 8.11

$$\begin{aligned} h_s &= z_{[n]} - u_s \\ &= \sum_{x \subset y} (x,y) - \sum_{x \subset s, x \subset y} (x,y) \\ &= \sum_{x \not\subset s} (x,y) \end{aligned}$$

□

Notation 8.14. For I with $I \subset [n]$, define $l_I = \sum_{I \subset x} (x,y)$.

Notation 8.15. For $I \subset [n]$ let the complement of I be denoted by $I^c = \{i \in [n] \colon i \notin I\}$.

Lemma 8.16. *Let I be fixed, $i \geq |I| \geq 2$. If l_J lies in the span of $v_{(a,b)}$ for all $J \subset I$, then so does l_I.*

Proof. Let $I \subset [n]$. Let $A_i = \{x \colon i \in x\}$. Then, $\cap_{i \in I} A_i = \{x \colon I \subset x\}$. And, in general, $\cap_{k \in J} A_k = \{x : J \subset x\}$. Using the principle of inclusion exclusion, we can write

$$\sum_{x \in \cap_{i \in I} A_i} (x,y) = \sum_{x \in \cup_{k \in I} A_k} (x,y) + \sum_{J \subset I, J \neq I, J \neq \emptyset} (-1)^{|I|-|J|+1} \sum_{x \in \cap_{k \in J} A_k} (x,y). \tag{8.2}$$

Our aim is to show the left hand side of Equation (8.2) lies in the span of $v_{(a,b)}$ so it suffices to show all terms on the right hand of Equation (8.2) side lie in the span of $v_{(a,b)}$. Observe that $\{x \in \cup_{k \in I} A_k\} = \{x \colon x \not\subset I^c\}$, and so $\sum_{x \in \cup_{k \in I} A_k} (x,y) = h_{I^c}$, by Lemma 8.13. Furthermore, we are assuming $\sum_{x \in \cap_{k \in J} A_k} (x,y)$, lie in the span of $v_{(a,b)}$, for $J \subset I$. Hence, the right hand side of Equation (8.2) lies in the span of $v_{(a,b)}$, and it follows that $l_I = \sum_{x \in \cap_{i \in I} A_i} (x,y)$ does as well.

□

Lemma 8.17. *For all $J \subset [n]$, with $|J| \leq i$, we have l_J lies in the span of $v_{(a,b)}$.*

Proof. We prove this by induction on the size of I. First, we check the two base cases: when $|I| = 0, |I| = 1$. In the case $I = \emptyset$, we know $l_\emptyset$ lies in the span of $v_{(a,b)}$ because $z_{[n]} = l_\emptyset$, as $z_{[n]} = \sum_{x \subset y} (x,y) = l_\emptyset$, from Lemma 8.7. In the case $|I| = 1$, we have $l_I = h_{I^c}$, because if $I = \{i\}$, then $\{x \colon x \not\subset I^c\} = \{x \colon i \in x\} = \{x \colon I \subset x\}$.

This completes the base case. So, to complete the proof, it suffices to show the induction step: that if we know this Lemma holds for all $|s| > j$ then it holds for $|s| = j$. This is exactly what Lemma 8.16 proves. □

Lemma 8.18. *For $|x| = i$, we have $l_x = \sum_{y \supset x} (x,y)$, which lies in the span of $v_{(a,b)}$.*

Proof. This follows immediately from the definition of l_x, since the only element of size i containing x is x itself. We know l_x lies in the span of $v_{(a,b)}$ by Lemma 8.17. □

Notation 8.19. Define $m_a = \sum_{x \subset a} l_x$.

Lemma 8.20. *With m_a as defined above, we have $m_a = \sum_{x \subset a} (x,y)$.*

Proof. By 8.18, we obtain

$$m_a = \sum_{x \subset a} l_x = \sum_{x \subset a} \sum_{y \supset x} (x,y) = \sum_{x \subset a} (x,y)$$

□

Notation 8.21. Denote $r_a = \sum_{b, b \subset a} v_{(a,b)}$.

Lemma 8.22. *With r_a as defined above, we have*

$$r_a = ((i+1)\beta - 1) \sum_{y \subset a} (x,y) + \sum_{x \subset a} (x,y).$$

Proof.

$$\begin{aligned}
r_a &= \sum_{b \supset a} v_{(a,b)} \\
&= \sum_{b \supset a} \left(\beta \sum_{y \subset a} (x,y) + \sum_{\substack{y \subset b, \\ x \subset a, \\ y \not\subset a}} (x,y). \right) \\
&= \sum_{b \supset a} \beta \sum_{y \subset a} (x,y) + \sum_{b \supset a} \sum_{\substack{y \subset b, \\ x \subset a, \\ y \not\subset a}} (x,y). \\
&= (i+1)\beta \sum_{y \subset a} (x,y) + \sum_{b \supset a} \sum_{\substack{y \subset b, \\ x \subset a, \\ y \not\subset a}} (x,y). \\
&= (i+1)\beta \sum_{y \subset a} (x,y) + \sum_{x \subset a, y \not\subset a} (x,y). \\
&= ((i+1)\beta - 1) \sum_{y \subset a} (x,y) + \left(\sum_{x \subset a, y \not\subset a} (x,y) + \sum_{y \subset a} (x,y) \right). \\
&= ((i+1)\beta - 1) \sum_{y \subset a} (x,y) + \sum_{x \subset a} (x,y).
\end{aligned}$$

□

Notation 8.23. Assuming we do not have $(i+1)\beta = 1$, define $t_a = \frac{r_a - m_a}{(i+1)\beta - 1}$

Lemma 8.24. *With t_a as defined above, $t_a = \sum_{y \subset a} (x,y)$*

Proof. By 8.22 and 8.20, we have

$$\begin{aligned}
t_a &= \frac{r_a - m_a}{(i+1)\beta - 1} \\
&= \frac{((i+1)\beta - 1) \sum_{y \subset a} (x,y) + \sum_{x \subset a} (x,y) - \sum_{x \subset a} (x,y)}{(i+1)\beta - 1} \\
&= \frac{((i+1)\beta - 1) \sum_{y \subset a} (x,y)}{(i+1)\beta - 1} \\
&= \sum_{y \subset a} (x,y)
\end{aligned}$$

□

Notation 8.25. Assuming $\beta(i+1) \neq 1$, let $q_{a,b} = v_{(a,b)} - \beta t_a$.

Lemma 8.26. *We have $q_{a,b} = \sum_{\substack{y \subset b, \\ x \subset a, \\ y \not\subset a}} (x,y)$.*

Proof. Using 8.24

$$\begin{aligned}
q_{a,b} &= v_{(a,b)} - \beta t_a \\
&= \beta \sum_{y \subset a} (x,y) + \sum_{\substack{y \subset b,\\ x\subset a,\\ y \not\subset a}} (x,y) - \beta \sum_{y\subset a} (x,y) \\
&= \sum_{\substack{y \subset b,\\ x\subset a,\\ y \not\subset a}} (x,y)
\end{aligned}$$

□

Theorem 8.27. *For $n > 2$, the matrix M^{n-2i-1} is invertible.*

Proof. We saw that if $\beta(i+1) \neq 1$, we have that $q_{a,b} = \sum_{\substack{y \subset b,\\ x\subset a,\\ y \not\subset a}} (x,y)$ lie in the span of $v_{(a,b)}$. It is always the case that $\beta \geq 1, i \geq 0$. The only time in which $\beta = 1, i = 0$ is when $n = 2$. Therefore, $q_{a,b}$ are always defined for $n > 2$. However, $q_{a,b}$ are exactly the rows of L^{n-2i-1} as defined in Proposition 8.3. Using Corollary 3.26, we know $\mathcal{H}(B_n)$ is unitary Peck, and therefore the rows of L^{n-2i-1} are independent. Therefore, the rows of M^{n-2i-1} span an independent set in a vector space of the same dimension. So, the rows of M^{n-2i-1} are independent. Hence, M^{n-2i-1} is indeed an isomorphism, when $n > 2$. □

Corollary 8.28. *The poset $\mathcal{E}(B_n)$ is unitary Peck for $n > 2$.*

Proof. By definition, a poset $P, \mathrm{rk}(P) = t$ is unitary peck if and only if the Lefschetz map $L : P \to P, x \mapsto \sum_{y \gtrdot x} y$, satisfies $L^{t-2i} : P_i \to P_{t-i}$ is an isomorphism. In the case $P = \mathcal{E}(B_n), \mathrm{rk}(P) = n - 1$, we have that M is the Lefschetz map and by Theorem 8.27, $M^{n-1-2i} : \mathcal{E}(B_n)_i \to \mathcal{E}(B_n)_{n-1-i}$ is indeed an isomorphism. Hence, $\mathcal{E}(B_n)$ is unitary Peck. □

9. Odds, Ends, and Failed Attempts

In this section, we include various remarks, as well as failed attempts, which don't particularly belong in other sections.

9.1. **A q-analog.** In this subsection we consider a q-analog of the Boolean algebra and prove Lemma 9.4, which is a q-analog to Lemma 7.3. This subsection is a start toward developing a q-analog of the unimodality of the necklace poset.

Let q be a prime and $B_n(q)$ be the poset of all $\mathbb{F}_q$-subspaces of $V_n(q) := (\mathbb{F}_q)^n$, graded naturally by dimensions.

Let $C_n(q)$ be the multiplicative subgroup of the finite field $\mathbb{F}_{q^n}$, so $C_n(q)$ acts $\mathbb{F}_q$-linearly on $\mathbb{F}_{q^n}$ by multiplication, which is isomorphic to $V_n(q)$ as an $\mathbb{F}_q$ vector space. The action of $C_n(q)$ on $V_n(q)$ induces a group action on the poset $B_n(q)$. The poset $B_n(q)/C_n(q)$ is the q-analog of the necklace poset.

The following Lemma is a standard result. For a proof of the lemma, we refer the reader to, for example, [7].

Lemma 9.1. *The number $|B_n(q)_i|$, that is to say, the number of i dimensional subspace in $V_n(q)$, is $\binom{n}{i}_q$*

Notation 9.2. Define the graded poset $T = \mathcal{E}\big(B_n(q)\big)/C_n(q)$ and let $t_i = |T_i|$.

Lemma 9.3. *For any $V_x \in B_n(q)_i, V_y \in B_n(q)_{i+1}$, so that $(V_x, V_y) \in \mathcal{E}\big(B_n(q)\big)/C_n(q)$, then $Stab_{C_n(q)}(V_x, V_y) \cong \mathbb{F}_q^\times$. In particular, $|Stab_{C_n(q)}(V_x, V_y)| = q - 1$.*

Proof. We claim that if $\tau \in C_n(q)$ such that $\tau V_x = V_x$ and $\tau V_y = V_y$, then $\tau \in \mathbb{F}_q^\times$. First, it is clear that any scalar $c \in \mathbb{F}_q^\times$ fixes any $V \in B_n(q)$. Now assume $V_x \subset V_y$ and $\tau V_x = V_x, \tau V_y = V_y$ as in the claim. Then, in particular τ permutes all the elements in V_x, i.e., $\tau : V_x \to V_x$ is an isomorphism of $\mathbb{F}_q$-vector spaces. Pick a nonzero element $a \in V_x$, it is clear that

$$\tau^{q^i}(a) = \tau^{q^i} a = \tau a.$$

The latter equation implies that $\tau^{q^i} - \tau = 0$, so τ satisfies the polynomial $X^{q^i} - X \in \mathbb{F}_q[X]$. Similarly, since $\tau V_y = V_y$, τ is a root of the polynomial $X^{q^{i+1}} - X \in \mathbb{F}_q[X]$. Let $f(X)$ be a monic irreducible polynomial of τ over $\mathbb{F}_q$, then $\mathbb{F}_q(\tau)$ is a subfield of both $\mathbb{F}_{q^i}$ and $\mathbb{F}_{q^{i+1}}$, hence $\deg f(X)|i$ and $\deg f(X)|(i+1)$. This shows that $\deg f(X) = 1$, in another word, τ is a $\mathbb{F}_q$-scalar. $\square$

Lemma 9.4. *Let q be a prime, then*

$$|\big(\mathcal{E}\big(B_n(q)\big)/C_n(q)\big)_i| = \binom{n-1}{i}_q.$$

Proof. Let $V_x \in B_n(q)_i, V_y \in B_n(q)_{i+1}$ so that $(V_x, V_y) \in \mathcal{E}(B_n(q))$. By Lemma 9.3, for any $(V_x, V_y) \in \mathcal{E}^1(B_n(q))_i, |Stab_{C_n(q)}(V_x, V_y)| = q - 1$. By the Orbit Stabilizer Theorem, we know that the size of each orbit in $\mathcal{E}((B_n(q))_i$ under the action of $C_n(q)$ is

$$|C_n(q)|/(q-1) = \frac{q^n - 1}{q - 1} = (n)_q.$$

Now we can calculate the number q_i, which is the total number of elements in $(B_n(q))_i$ divided by the size of each orbit:

$$q_i = \frac{\binom{n}{i+1}_q \binom{i+1}{1}_q}{(n)_q} = \binom{n-1}{i}_q.$$

$\square$

9.2. **Quotients by Automorphism Groups of Rooted Trees.** In Subsection 4.2.1, we saw that the full automorphism groups of rooted trees are CCT. It is well known that the quotient $B_{ml}/(S_m \wr S_l)$ is a distributive lattice. Hence, we were curious whether B_{ml}/G is a distributive lattice as well, where G is the automorphism group of a rooted tree. The answer is a definitive no, as can be seen in the case $G = (S_2 \wr S_2) \wr S_2$, where G acts on a the binary rooted tree P with $|L(P)| = 8$, whose vertices are labeled $0, \ldots, 7$ from left to right. A picture of this tree can be seen in Figure 5. Letting G be the group of automorphisms of this tree, there is a natural action of G on B_n as described in Lemma 4.12. The Hasse diagram of the poset B_n/G is shown in Figure 9.2. In this case, it is easy to see that $G\{0,5,6\} \lessdot G\{0,1,5,6\}, G\{0,5,6\} \lessdot G\{0,1,2,5\}, G\{0,1,5\} \lessdot G\{0,1,5,6\}, G\{0,1,5\} \lessdot G\{0,1,2,5\}$. Hence, $G\{0,5,6\}, G\{0,1,5\}$ do not have a well defined join, and so B_{ml}/G is not a lattice.

9.3. **A Failed Attempt for Showing $\mathcal{E}(B_n/G)$ is unimodal.** In this section, we describe one path we were pursuing in order to show $\mathcal{E}(B_n/G)$ is unimodal. We were attempting to do this by trying to show there were injective order raising maps $U_i : \mathcal{E}(B_n/G)_i \to \mathcal{E}(B_n/G)_{i+1}$.

We now define several maps, so that we can draw a certain commuting diagram in Remark 9.6

Notation 9.5. Let $U_i : P_i \to P_{i+1}$ be the raising operator for the poset P. Then, we obtain an induced map

$$U_i \otimes U_{i+1} : P_i \otimes P_{i+1} \to P_{i+1} \otimes P_{i+1}, (x, y) \mapsto U(x) \otimes U(y).$$

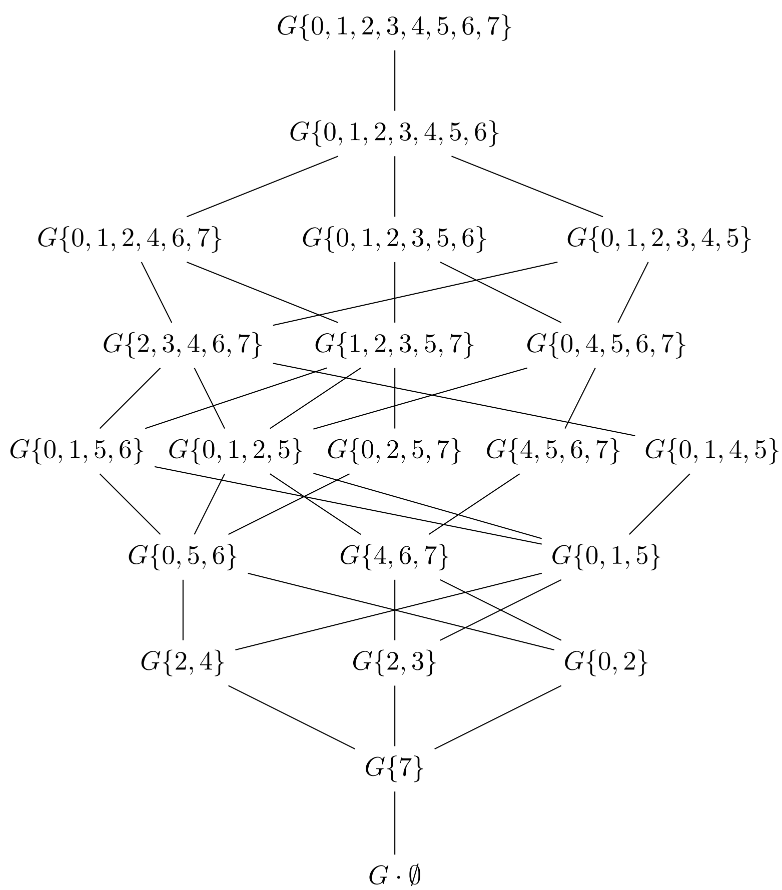


FIGURE 7. The Hasse Diagram of B_n/G where G is the group of rooted tree automorphisms on the tree from Figure 5.

We also have the natural inclusions

$$\begin{aligned} k_i &: \mathcal{E}(P)_i \to P_i \otimes P_{i+1}, \\ &(x,y) \mapsto (x,y) \\ k_i^{G\times G} &: \mathcal{E}(P/G)_i \to (P/G)_i \otimes (P/G)_{i+1}, \\ &(Gx,Gy) \mapsto (Gx,Gy), \end{aligned}$$

where we have $x \lessdot y$ and $Gx \lessdot Gy$. The maps above are defined on a basis, and are extended by linearity.

Next, we define the map

$$\begin{aligned} j_i : (P/G)_i \otimes (P/G)_{i+1} &\to P_i \otimes P_{i+1}, \\ (Gx,Gy) &\mapsto \left(\frac{1}{|G|}\sum_{g\in G} gx, \frac{1}{|G|}\sum_{h\in G} hy).\right) \end{aligned}$$

where x is an arbitrary representative of Gx and y is an arbitrary representative of Gy

Then, define the map

$$\begin{aligned} p_i : P_i \otimes P_{i+1} &\to (P/G)_i \otimes (P/G)_{i+1}, \\ (x,y) &\mapsto (Gx,Gy). \end{aligned}$$

Further, define the map

$$\begin{aligned}(U_i \otimes U_{i+1})^{G\times G} : P_i \otimes P_{i+1} &\to P_{i+1} \otimes P_{i+1}, \\ (Gx, Gy) &\mapsto p_{i+1} \circ (U_i \otimes U_{i+1}) \circ j_i(Gx, Gy).\end{aligned}$$

We also have the projections inclusions

$$\begin{aligned}\pi_i : P_i \otimes P_{i+1} &\to \mathcal{E}(P)_i, \\ (x, y) &\mapsto \begin{cases} (x, y), & \text{if } x \lessdot y \\ 0 & \text{otherwise} \end{cases} \\ \pi_i^{G\times G} : (P/G)_i \otimes (P/G)_{i+1} &\to \mathcal{E}(P/G)_i, \\ (Gx, Gy) &\mapsto \begin{cases} (Gx, Gy), & \text{if } Gx \lessdot Gy \\ 0 & \text{otherwise} \end{cases},\end{aligned}$$

where we have $x \lessdot y$ and $Gx \lessdot Gy$. The maps above are defined on a basis, and are extended by linearity.

Finally, denote

$$\begin{aligned}\mathcal{E}(U)_i : \mathcal{E}(P)_i &\to \mathcal{E}(P)_{i+1} \\ (x, y) &\mapsto k_i \circ (U \otimes U) \circ \pi_{i+1}(x, y) \\ \mathcal{E}(U)_i^{G\times G} : \mathcal{E}(P/G)_i &\to \mathcal{E}(P/G)_{i+1} \\ (Gx, Gy) &\mapsto k_i^{G\times G} \circ (U \otimes U)^{G\times G} \circ \pi_{i+1}^{G\times G}(Gx, Gy)\end{aligned}$$

where it is defined above on a basis and we extend to the whole space by linearity.

Remark 9.6. For $i < \frac{n}{2}$ we obtain the following (almost commuting, but $j_{i+1} \circ p_{i+1} \neq \text{id.}$) diagram

$$\begin{tikzcd}
\mathcal{E}(P)_i \arrow[r, hook, "k_i"] \arrow[rrr, hook, bend left, "\mathcal{E}(U)_i"] & P_i \otimes P_{i+1} \arrow[r, hook, "U_i \otimes U_{i+1}"] & P_{i+1} \otimes P_{i+2} \arrow[r, two heads, "\pi_{i+1}"] \arrow[d, shift right, two heads, "p_{i+1}"'] & \mathcal{E}(P)_{i+1} \\
\mathcal{E}(P/G)_i \arrow[r, hook, "k_i^{G\times G}"] \arrow[rrr, hook, bend right, "\mathcal{E}(U)_i^{G\times G}"'] & (P/G)_i \otimes (P/G)_{i+1} \arrow[u, hook, "j_i"'] \arrow[r, hook, "(U_i \otimes U_{i+1})^{G\times G}"] & (P/G)_{i+1} \otimes (P/G)_{i+2} \arrow[r, two heads, "\pi_{i+1}^{G\times G}"] \arrow[u, shift right, hook, "j_{i+1}"'] & \mathcal{E}(P/G)_{i+1}
\end{tikzcd}$$

Unfortunately, in general, with the above definitions of the maps,

$$\ker\left(\pi_{i+1}^{G\times G} \circ p_{i+1}\right) \cap \operatorname{Im}((U \otimes U) \circ j_i \circ k_i^{G\times G}) \not\subset \ker \pi_{i+1} \cap \operatorname{Im}((U \otimes U) \circ j_i \circ k_i^{G\times G}).$$

However, if we did have

$$\ker\left(\pi_{i+1}^{G\times G} \circ p_{i+1}\right) \cap \operatorname{Im}((U \otimes U) \circ j_i \circ k_i^{G\times G}) \subset \ker \pi_{i+1} \cap \operatorname{Im}((U \otimes U) \circ j_i \circ k_i^{G\times G}),$$

using the fact that $\mathcal{E}(U)_i, U_i, U_{i+1}$ are all injective, a fairly simple diagram chase would reveal $\mathcal{E}(U)_i^{G\times G}$ is injective. This, in turn, would imply $\mathcal{E}(B_n/G)_i$ is symmetric, unimodal, and Sperner. We have tried several variations on these exact maps, but were never quite able to obtain the desired $\ker\left(\pi_{i+1}^{G\times G} \circ p_{i+1}\right) \cap \operatorname{Im}((U \otimes U) \circ j_i \circ k_i^{G\times G}) \subset \ker \pi_{i+1} \cap \operatorname{Im}((U \otimes U) \circ j_i \circ k_i^{G\times G})$.

9.4. **Further Questions.**

9.4.1. *Category Theoretic Properties.* First, it is worth noting that the functor $\mathcal{E} : \mathcal{P} \to \mathcal{P}$ is neither essentially surjective nor injective on objects. Let P be the poset with points a, b, c, d and relations $a \lessdot c, b \lessdot c, a \lessdot d$. It is easy to see there is no poset A with $\mathcal{E}(A) = P$. This shows $\mathcal{E}$ is not essentially surjective. Next, let Q be the poset whose elements are e, f, g with $e \lessdot f, e \lessdot g$, and let R be the poset whose elements are h, i, j, k with $h \lessdot i, j \lessdot k$. It is clear that $\mathcal{E}(Q) \cong \mathcal{E}(R)$ are both a disjoint union of two points, but $Q \ncong R$. So, $\mathcal{E}$ is also not injective on objects.

However, there are many aspects of the functor $\mathcal{E}$, which are still unexplored. First, note that it is worth noting that $\mathcal{E}$ also determines a endofunctor on the category of posets (even those without a grading), which we now notate $\mathcal{Q}$, as well as an endofunctor on the category of graded posets $\mathcal{P}$. In general, the points of a poset P can be identified with the set $Hom_{\mathcal{Q}}(B_0, P)$, and the edges of P can be identified with the set $Hom_{\mathcal{Q}}(B_1, P)$. It is simple to see additionally that the points of $\mathcal{E}(P)$ are $Hom_{\mathcal{Q}}(B_0, \mathcal{E}(P)) \cong Hom_{\mathcal{Q}}(B_1, P)$. Additionally, the edges of $\mathcal{E}(P)$ are $Hom_{\mathcal{Q}}(B_1, \mathcal{E}(P)) \cong Hom_{\mathcal{Q}}(B_2, P)$. These relations may be a reasonable alternate way to think of the functor of edges, and are further suggestive of the existence of an adjoint functor.

Question 9.7. Does the functor $\mathcal{E}$ have a left or right adjoint?

9.4.2. *Unitary Peck Generalizations.* Additionally, there are several posets which we would like to be unitary Peck, as this would allow us to apply much of the theory developed in this paper. They would also make several of the generalizations of $\mathcal{E}$ potentially more interesting.

As mentioned in the introduction, a natural extension of Conjecture 1.3 would be an analogous result for q-analogs. We suspect that a proof similar to that of Corollary 8.28 may also prove Question 9.9.

Definition 9.8. Let $B_n(q)$ be the graded poset whose elements are linear subspaces $V \subset \mathbb{F}_q^n$ with $V \leq W$ if $V \subset W$.

Question 9.9. Is $\mathcal{E}(B_n(q))$ unitary peck?

9.4.3. *A q-analog of Conjecture 1.3.* In particular, if the answer to the previous Question 9.9 is affirmative, it would immediately hold that $\mathcal{E}(B_n(q))/G$ is Peck, and if G is CCT then $\mathcal{E}(B_n(q)/G)$ is Peck. Hence, we pose the following question:

Question 9.10. For G a group with a CCT action on $B_n(q)$ is $\mathcal{E}(B_n(q)/G)$ Peck?

Even more generally, we wonder if the q-analog of Conjecture 1.3 holds.

Question 9.11. For G a group acting on $B_n(q)$, is $\mathcal{E}(B_n(q)/G)$ Peck? If not, is $\mathcal{E}(B_n(q)/G)$ rank-unimodal?

9.4.4. *Additional CCT Actions.* We also found several additional interesting examples of CCT actions, and we are curious whether they generalize. Once such action is the linear automorphism of the n-cube. Using computers, we found that for $n \leq 3$, the linear automorphisms of the n-cube induces a CCT action on B_{2^n}. We wonder if this generalizes.

Question 9.12. Does the group of linear automorphism of an n-cube in $\mathbb{R}^n$, whose vertices lie at $(\pm 1, \ldots, \pm 1)$ induce a CCT action on B^{2^n}?

There is also the question of which exceptional regular polytopes induce CCT actions. We have shown it holds for the octahedron, the simplex (tetrahedron) in Subsubsection 4.2.2. We also checked using the computer that it holds for the cube. We wonder whether it holds for all exceptional regular polytopes.

Question 9.13. Do the remaining five exceptional regular polytopes (namely the dodecahedron and icosahedron in $\mathbb{R}^3$ as well as the 24-cell, 120-cell, and 600-cell polytopes in $\mathbb{R}^4$) induce CCT actions?

Also, we found using computers that the group of invertible linear maps on $\mathbb{F}_2^3$ acting on the the seven nonzero points of $\mathbb{F}_2^3$ induce an action on B_7 which is CCT. We wonder if this generalizes to other groups of invertible linear maps on finite fields.

Question 9.14. Is the action of $GL_n(\mathbb{F}_q)$ on B_{q^n-1} (induced by the action of $GL_n(\mathbb{F}_q)$ on $(\mathbb{F}_q^n)^\times$) CCT? What about the action of $PGL_n(\mathbb{F}_q)$ on $B_n(q)$? If not, what about the action of $PGL_n(\mathbb{F}_2)$ on $B_n(2)$?

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References

[1] Eric Babson and Dmitry N. Kozlov. Group actions on posets. *J. Algebra*, 285:439–450, 2005. 11

[2] Katherine Burggraf, Jesús de Loera, and Mohamed Omar. On volumes of permutation polytopes. arxiv: 1103.0039, 2011. 14

[3] Curtis Greene and Daniel J. Kleitman. Proof techniques in the theory of finite sets. In *Studies in combinatorics (MAA Stud. Math.)*, volume 17, pages 22–79. Math. Assoc. America, Washington, D.C., 1978. 10

[4] Igor Pak and Greta Panova. Unimodality via kronecker products. *Journal of Algebraic Combinatorics*, page To appear, 2013. 3, 17

[5] Richard P. Stanley. Weyl groups, the hard lefschetz theorem, and the sperner property. *SIAM J. on Algebraic and Discrete Methods*, 1:168–184, 1980. 4

[6] Richard P. Stanley. Quotients of peck posets. *Order*, 1:29–34, 1984. 4, 10

[7] Richard P. Stanley. *Enumerative Combinatorics*, volume 1. Cambridge University Press, 2011. 19, 37

[8] Richard P. Stanley. *Algebraic Combinatorics*. Springer, 2013. 20, 22

David Hemminger, Duke University
E-mail address: david.hemminger@duke.edu

Aaron Landesman, Harvard University
E-mail address: aaronlandesman@gmail.com

Zijian Yao, Brown University
E-mail address: zijian_yao@brown.edu