A RESULT RELATED TO BRESSOUD POLYNOMIALS

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Abstract. In this paper I will first review the combinatorial interpretation of the Bressoud polynomials, then I will present a result which may be related to cyclic sieving phenomenon of the combinatorial objects we considered.

1. Introduction

First we introduce some concepts related to partitions.

Definition 1.1. Let Π be a partition whose Ferrers graph has a node in the i-th row and j-th column; we call this node the (i, j)th node. We define the hook difference at the (i, j)th node to be the number of nodes in the ith row of Π minus the number of nodes in the jth column of Π. We also define the diagonal number of the (i, j)th node to be $i - j$.

To illustrate, let's see an example.

Example 1.2. Let Π be 6+5+3+1, figure 1 shows the hook differences of nodes of Π while figure 2 shows the diagonal number of nodes of Π.

Figure 1. Hook Difference

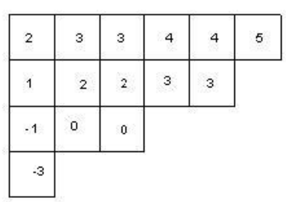
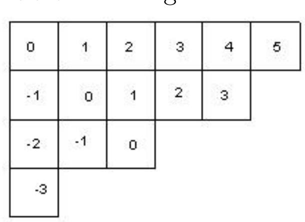


Figure 2. Diagonal Number



Key words and phrases. Bressoud polynomials, cyclic sieving phenomenon.

For positive integers m, n, we consider partitions with at most m parts each $\leq n$ with some property.

Definition 1.3. Let a, b be positive integers. Define $p_{K,i}(n, m; a, b; n)$ to be the number of partitions of n into at most m parts each $\leq n$ such that the hook differences on diagonal $1-b$ are $\geq -i+b+1$ and on diagonal $a-1$ are $\leq K-i-a-1$. Define $D_{K,i}(n, m; a, b; n)$ to be the related generating function, or

$$D_{K,i}(n, m; a, b; n) = \sum_{n\geq 0} p_{K,i}(n, m; a, b; n)q^n. \tag{1.1}$$

We can now state the main result in [1].

Theorem 1.4. *We have*

$$D_{K,i}(n, m; a, b; n) = \sum_{\mu=-\infty}^{\infty} q^{\mu(K\mu+i)(a+b)-Kb\mu} \begin{bmatrix} n+m \\ n-K\mu \end{bmatrix}_q - \sum_{\mu=-\infty}^{\infty} q^{\mu(K\mu-i)(a+b)-Kb\mu+bi} \begin{bmatrix} n+m \\ n-K\mu+i \end{bmatrix}_q \tag{1.2}$$

Proof. See [1]. □

Motivated by Theorem 1.4, we define the Bressoud polynomial $br(n, m, K, i, a, b, q)$ as

$$br(n, m, K, i, a, b, q) = \sum_{\mu=-\infty}^{\infty} q^{\mu(K\mu+i)(a+b)-Kb\mu} \begin{bmatrix} n+m \\ n-K\mu \end{bmatrix}_q - \sum_{\mu=-\infty}^{\infty} q^{\mu(K\mu-i)(a+b)-Kb\mu+bi} \begin{bmatrix} n+m \\ n-K\mu+i \end{bmatrix}_q \tag{1.3}$$

2. Main result

Since Bressoud polynomials are generating functions of some combinatorial objects, naturally, we may consider the cyclic sieving phenomenon (CSP) of certain objects. (Readers not familiar with CSP may refer to [2].) However, finding some good operation to perform CSP seems hard. But we do find a numerical result (originally suggested by Dennis Stanton and generalized by the author) which may suggest some evidence.

Note that when $K = 2i$, or $(K, i) = (2t, t)$ for some $t \in \mathbb{Z}$, the double sum becomes a single sum

$$br(n, m, 2t, t, a, b, q) = \sum_{s=-\infty}^{\infty} (-1)^s q^{\frac{t(a+b)s^2}{2} + \frac{t(a-b)s}{2}} \begin{bmatrix} n+m \\ n-ts \end{bmatrix}_q \tag{2.1}$$

Now we can state the main

Theorem 2.1. *We have the following two identities:*

(i) When ta, tb are integers and $ta - tb$ is not divisible by 2, then

$$br(2n, 2m, 2t, t, a, b, -1) = br(n, m, 2t, t, -, -, 1), \tag{2.2}$$

here '-' means that the value does not depend on these two parameters, as can be seen from the definition of Bressoud polynomials.

(ii) When ta, tb are even integers, then

$$br(2n, 2m, 2t, t, a, b, -1) = br(n, m, t, \frac{t}{2}, -, -, 1) \tag{2.3}$$

Proof. We need a

Lemma 2.2. *Suppose q is an r-th root of unity, m, n are positive integers such that $m = rm_1 + m', n = rn_1 + n', m_1, n_1 \in \mathbb{Z}, 0 \le m', n' \le r - 1$. Then*

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \binom{m_1}{n_1} \begin{bmatrix} m' \\ n' \end{bmatrix}_q. \tag{2.4}$$

Proof. Recall that

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{\prod_{i=1}^{m}(1-q^i)}{\prod_{i=1}^{n}(1-q^i)\prod_{i=1}^{m-n}(1-q^i)}. \tag{2.5}$$

Then we substitute q equals to an r-th root of unity into (2.5), we divide the proof into 2 cases.

Case 1. $m' < n'$. Then the multiplicity of 0 in the numerator is m_1 while the multiplicity of 0 in the denominator is $m_1 - 1$. Hence $\begin{bmatrix} m \\ n \end{bmatrix}_q = 0$ in this case. Note that when $m' < n'$, $\begin{bmatrix} m' \\ n' \end{bmatrix}_q = 0$. Hence in this case the lemma is proved.

Case 2. $m' \ge n'$. Then

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \lim_{q' \to q} \frac{\prod_{i=1}^{m}(1-q'^i)}{\prod_{i=1}^{n}(1-q'^i)\prod_{i=1}^{m-n}(1-q'^i)} = \lim_{q' \to q} \frac{\prod_{j=1}^{m_1}(1-q'^{rj})}{\prod_{j=1}^{n_1}(1-q'^{rj})\prod_{j=1}^{m_1-n_1}(1-q'^{rj})}$$
$$\frac{\prod_{1\le i\le m, r\nmid i}(1-q^i)}{\prod_{1\le i\le n, r\nmid i}(1-q^i)\prod_{1\le i\le m-n, r\nmid i}(1-q^i)} = \binom{m_1}{n_1}\begin{bmatrix} m' \\ n' \end{bmatrix}_q$$

Hence in this case the lemma is also proved. $\square$

Suppose c, d are non-negative integers. By previous Lemma, when d is odd,

$$\begin{bmatrix} 2c \\ d \end{bmatrix}_{-1} = 0, \tag{2.6}$$

when d is even,

$$\begin{bmatrix} 2c \\ d \end{bmatrix}_{-1} = \binom{c}{\frac{d}{2}}. \tag{2.7}$$

By (2.1),

$$br(2n, 2m, t, a, b, -1) = \sum_{s=-\infty}^{\infty} (-1)^s q^{\frac{t(a+b)s^2}{2} + \frac{t(a-b)s}{2}} \begin{bmatrix} 2n+2m \\ 2n-ts \end{bmatrix}_{-1}, \tag{2.8}$$

Combine (2.6), (2.7), when ta, tb are integers and $ta - tb$ is odd, only even s in (2.8) contributes to the sum, when s is even, write $s = 2s'$ and using (2.7), also combine that 2 is not divisible by $ta - tb$, we obtain

$$br(2n, 2m, 2t, t, a, b, -1) = \sum_{s=-\infty}^{\infty} (-1)^s q^{\frac{t(a+b)s^2}{2} + \frac{t(a-b)s}{2}} \begin{bmatrix} 2n+2m \\ 2n-ts \end{bmatrix}_{-1}$$
$$= \sum_{s'=-\infty}^{\infty} (-1)^{2t(a+b)s'^2 + t(a-b)s'} \binom{n+m}{n-ts'} = \sum_{s'=-\infty}^{\infty} (-1)^{s'} \binom{n+m}{n-ts'} = br(n, m, 2t, t, -, -, 1)$$

Similarly, when ta, tb are even integers

$$br(2n, 2m, 2t, t, a, b, -1) = \sum_{s=-\infty}^{\infty} (-1)^s (-1)^{\frac{t(a+b)s^2}{2} + \frac{t(a-b)s}{2}} \begin{bmatrix} 2n+2m \\ 2n-ts \end{bmatrix}_{-1}$$

$$= \sum_{s=-\infty}^{\infty} (-1)^s \binom{n+m}{n-\frac{ts}{2}} = br(n, m, t, \frac{t}{2}, -, -, 1)$$

The proof is now complete. □

References

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