Hecke Algebras, Representations, and Character Tables of Monoids

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Abstract

The representation theory of finite groups is well studied; however, the representation theory of finite monoids is not. We use the representation theory of the group of units in order to understand the representations of monoids. We extend well-known results for groups to analogous results for monoids. In particular, we prove an analog of the Borel-Matsumoto theorem and Froebnius Reciprocity for subgroups of the group of units in a monoid using Godelle's definition of convolution for monoids. We introduce a new description of the symplectic rook monoid and provide an embedding of it into the better-known rook monoid. We then investigate the specific nature of the symplectic rook monoid. The irreducible representations are indexed both by partitions of at most n and by pairs of partitions whose sum is exactly n. We use combinatorial techniques based on this to examine its character table and to develop branching rules for decomposing its irreducible representations as representations of the group of units. Using results from Solomon about the structure of the character table of the rook monoid, we determine a new way of producing the character table for the Iwahori-Hecke algebra of the rook monoid. In the spirit of Solomon and utilizing techniques from Geck and Pfeiffer, we provide a description of the character table of the symplectic rook monoid. We then extend this to the Iwahori-Hecke algebra of the symplectic rook monoid.

1 Introduction

1.1 Motivation

Let G be a split reductive group over a field F with Borel subgroup B. Recall that G has the Bruhat decomposition:

$$G = \bigsqcup_{w \in W} B\underline{w}B,$$

where W, the Weyl group, is a finite Coxeter group.

Now, we embed G into $M_n(F)$, the monoid of $n \times n$ matrices, and take the Zariski closure of G. In the process, we get a *reductive monoid* M sharing many important structural properties with G. For instance, M has an extended Bruhat decomposition:

$$M = \bigsqcup_{r \in R} B\underline{r}B,$$

where R, the Renner monoid of M, has factorization $R = WE(\overline{T})$, where $E(\overline{T})$ is a set of idempotents along the closure of the maximal torus of G.

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By this construction, we generate two monoids associated with G through whose structure we can study G's representation theory. We generate a larger *reductive* monoid $M = \overline{G}$ and a smaller, finite *Renner* monoid R that governs the structure much of the structure of M.

We begin with a case study. Let $G = GL_n(F)$. Recall that G has Weyl group $W = S_n$. Embed G into $M_n(F)$. Since G is Zariski-dense, $M = \overline{G} = M_n(F)$. We now consider the Renner monoid of M. We have that $R = R_n$, where

$$R_n := \{\text{Matrices which represent at most } n \text{ non-attacking rooks on an } n \times n \text{ chessboard}\}$$

where a rook is represented by a 1 and an empty space is represented by a 0. As an example,

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in R_3(\mathbb{R}), \qquad N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \notin R_3(\mathbb{R})$$

Formally, let R be the set of all one-to-one maps σ with domain and range $I(\sigma), J(\sigma) \subseteq \{1, \ldots, n\}$. Then we can define $\sigma\tau : I(\sigma) \cap \tau^{-1}(J(\sigma)) \to J(\tau)$ by $i(\sigma\tau) = (i\sigma)\tau \in J(\tau)$ for $i \in \{1, \ldots, n\}$, where the action considered is a right action. This forms a monoid isomorphic to a submonoid of $M_n(F)$ such that

$$R \ni \sigma \mapsto \sum_{i \in I(\sigma)} E_{i,i\sigma}$$

with $E_{i,j}$ being the $n \times n$ matrix of 1's and 0's with a single 1 at entry (i,j). It is easy to see that the the size of the rook monoid is:

$$\sum_{i=0}^{n} \binom{n}{i}^2 i! \tag{1}$$

For further definitions, see Solomon [8]. Turning to applications, monoids are abundant in computer programming, and even implemented in languages like Haskell. Gondran and Minoux [5] provide an extensive listing of applications of monoids in data analysis and searching. Monoids (respectively reductive moniods) are extensions of groups (respectively reductive groups) and can shed new light on old groups that we know well. Solomon does so with his analysis of irreducible representations of S_n [9] by looking at those of the rook monoid.

1.2 Results

In this paper, we extend some well known results for groups to analogous results for monoids and then investigate the specific nature of the symplectic rook monoid, along with its character table and rules for decomposing its irreducible representations. In section 2, we prove the Borel-Matsumoto theorem for monoids. In section 3 we introduce the symplectic rook monoid and provide an embedding of it into the rook monoid. In section 4, we provide the character table for the symplectic rook monoid, RSp_{2n}, when $n = 3$, as calculated according to results from Li, Li, Cao [7]. In section 5, we recall results from Solomon [9] about the structure of the character table of the rook monoid. Then using techniques from Geck and Pfeiffer's text on the characters of finite Coxeter Groups and Iwahori-Hecke algebras [3], we provide an analogous decomposition of the character table of the symplectic rook monoid.

1.3 Essential Definitions

- Let M be a monoid with group of units $G(M)$. We call M algebraic if it is a Zariski-closed subset of $Mat_n(F)$ for some $n \in \mathbb{Z}$ and F a field. We call M reductive if M is an irreducible algebraic monoid with $G(M)$ a reductive group under the usual definition. If M is reductive, M has the Renner decomposition

$$M = \bigsqcup_{r \in R} B\underline{r}B \tag{2}$$

where B is a Borel subgroup of $G(M)$, R is the Renner monoid of M, and $\underline{r}$ is a choice of lift of $r \in R$.

- The symplectic rook monoid and symplectic Renner monoid will be taken to be synonymous due to them being isomorphic as shown in [7]. We define the symplectic rook monoid RSp_{2n} in terms of the admissible sets definition, again as below.
- Let $\mathcal{N} = \{1, 2, ..., 2n-1, 2n\}$. As in [7], let $\theta : \mathcal{N} \to \mathcal{N}$ be the involution given by $\theta(t) = 2n + 1 - t$. A subset P of $\mathcal{N}$ is admissible if, for all $t \in \mathcal{N}$, then $\theta(t) \notin \mathcal{N}$. Define RSp_{2n} to be the monoid of all injective partial transformations on $\mathcal{N}$ sending admissible sets to admissible sets. The B_n Weyl group embeds as $G(RSp_{2n})$, the group of full injective transformations on $\mathcal{N}$.
- Let $\mathcal{Q}_n = SP_n \sqcup \bigcup_{r=0}^{n} S_r$ be the union of signed partitions of size n and the union of partitions of r for all $0 \leq r \leq n$. This is an indexing set for the collection of irreducible representations of the symplectic rook monoid, as well as its conjugacy classes. Furthermore, it has a standard ordering as described on p. 847 of [7].

2 The Borel-Matsumoto Theorem for Finite Monoids

Let M be a finite monoid, $G(M)$ the group of units of M, and K a subgroup of $G(M)$, and F a field of characteristic not dividing $|K|$.

For $\phi, \psi : M \to F$, define $\phi * \psi$ as in Godelle [4] by

$$\phi * \psi(m) = \sum_{yz=m} \phi(y)\psi(z) \tag{3}$$

Similarly, for (π, V) a representation of M and ϕ as above define $\pi(\phi)$ by

$$\pi(\phi)v = \sum_{x \in M} \phi(x)\pi(x)v \tag{4}$$

Proposition 1. *For $\phi, \psi \in \mathcal{H}$, $\pi(\phi * \psi) = \pi(\phi) \circ \pi(\psi)$.*

Proof. Consider $\pi(\phi) \circ \pi(\psi)$. We have the following:

$$\begin{aligned}
(\pi(\phi) \circ \phi(\psi))v &= \sum_{x \in M} \phi(x)\pi(x) \sum_{y \in M} \psi(y)\pi(y)v) && (5)\\
&= \sum_{x,y \in M} \phi(x)\psi(y)\pi(x)\pi(y)v && (6)\\
&= \sum_{x,y \in M} \phi(x)\psi(y)\pi(xy)v && (7)\\
&= \sum_{z \in M} \sum_{xy=z} \phi(x)\psi(y)\pi(z)v && (8)\\
&= \sum_{z \in M} (\phi * \psi)(z)\pi(z)v && (9)\\
&= \pi(\phi * \psi)v && (10)
\end{aligned}$$

Thus $\pi(\phi) \circ \pi(\psi) = \pi(\phi * \psi)$. □

Let $\mathcal{H}$ be the F-algebra of functions from M to F under addition and convolution. Define, for $v \in V$,

$$\mathcal{H}v = \{\pi(\phi)v \mid \phi \in \mathcal{H}\}. \tag{11}$$

Define an action of M on $\mathcal{H}v$ by $m \cdot (\pi(\phi)v) = \pi(m)\pi(\phi)v$. Notice that, since for $f_m : M \to F$ defined by $f_m(m) = 1, f_m(x) = 0$ for $x \neq m$, $\pi(f_m)v = \pi(m)v$, $\mathcal{H}v$ is closed under action by M. Thus, it is a subrepresentation.

Similarly, let $\mathcal{H}_K$ be the F-algebra of functions from M to F under convolution that are constant on double-cosets of K; i.e. ϕ such that $\phi(m) = \phi(k_1 m k_2)$ for all $k_1, k_2 \in K$. Also let

$$V^k = \{v \in V \mid \pi(k)v = v \quad \forall k \in K\}$$

as in Bump.

Theorem 2.1. *Let (π, V) be an irreducible representation of M with $V^K \neq \{0\}$. Then V^K is irreducible as an $\mathcal{H}_K$-module.*

Proof. We follow Bump's proof of the group case closely [1]. We claim that, for all nonzero $u \in V^K$, that $\mathcal{H}_K u := \{\pi(\phi)u | \phi \in \mathcal{H}_K\}$ equals V^K. In other words, we wish to show that, for all $v \in V^K$ there exists $\phi \in \mathcal{H}_K$ such that $\pi(\phi)u = v$.

Since (π, V) is an irreducible representation of M, there are no proper non-trivial subrepresentations in V. Because there is an M-action on $\mathcal{H}u \neq \{0\}$, then $\mathcal{H}u = V$. Thus there exists $\psi \in \mathcal{H}$ such that $\pi(\psi)u = v$.

Define $\phi \in \mathcal{H}$ by, for $x \in M$

$$\phi(x) = \frac{1}{|K|^2} \sum_{k_1, k_2 \in K} \psi(k_1 x k_2) \tag{12}$$

Since ϕ must be invariant over left and right cosets of K, ϕ lies in $\mathcal{H}_K$. Now consider the following:

$$\pi(\phi)u = \frac{1}{|K|^2} \sum_{k_1, k_2 \in K} \sum_{x \in M} \psi(k_1 x k_1)\pi(x)u \tag{13}$$

Notice that $x \mapsto k_1^{-1} x k_2^{-1}$ is a bijection from M to M, as it has an inverse $x \mapsto k_1 x k_2$. Thus we can make the following change of variables:

$$\pi(\phi)u = \frac{1}{|K|^2} \sum_{k_1, k_2 \in K} \sum_{x \in M} \psi(x)\pi(k_1^{-1} x k_2^{-1})u \tag{14}$$

$$= \frac{1}{|K|^2} \sum_{k_1, k_2 \in K} \sum_{x \in M} \psi(x)\pi(k_1)^{-1}\pi(x)\pi(k_2)^{-1}u. \tag{15}$$

Since $u \in V^K$, we have that $\pi(k_2)^{-1}u = u$. Thus,

$$\pi(\phi)u = \frac{1}{|K|} \sum_{k_1 \in K} \sum_{x \in M} \psi(x)\pi(k_1)^{-1}\pi(x)u \tag{16}$$

$$= \frac{1}{|K|} \sum_{k_1 \in K} \pi(k_1)^{-1} \sum_{x \in M} \psi(x)\pi(x)u \tag{17}$$

$$= \frac{1}{|K|} \sum_{k_1 \in K} \pi(k_1)^{-1}\pi(\psi)u. \tag{18}$$

Since $\pi(\psi)u = v$ and $v \in V^K$,

$$\pi(\phi)u = \frac{1}{|K|} \sum_{k_1 \in K} \pi(k_1)^{-1}v = v. \tag{19}$$

Thus, for all $v \in V^K$ there exists $\phi \in \mathcal{H}_K$ such that $\pi(\phi)u = v$. Thus, V^K is irreducible as an $\mathcal{H}_K$-module. $\square$

Denote, for (π, V) a representation of M, let $(\pi|_G, V)$ be the restricted representation of $G(M)$ defined by $\pi|_G(g) = \pi(g)$ for $g \in G(M)$. Define the contragredient representation of $G(M)$ $(\hat{\pi}|_G, \hat{V})$ by $\langle \pi|_G(g)v, \hat{v}\rangle = \langle v, \hat{\pi}|_G(g^{-1})\hat{v}\rangle$ for all $g \in G(M)$.

Lemma 1. *Let l: $V^K \to F$ be a linear functional. Then there exists $\hat{v} \in \hat{V}^K$ such that for all $v \in V^K$, l(v) = $\langle v, \hat{v}\rangle$. [1]*

Proof. Let $\hat{v}_0$ be a linear functional on V that restricts to l on V^K.

Define $\hat{v} = \frac{1}{|K|}\sum_{k\in K} \hat{\pi}|_G(k)\hat{v}_0$. For $v \in V^K$, then, we have the following equalities:

$$\langle v, \hat{v}\rangle = \frac{1}{|K|}\sum_{k\in K}\langle v, \hat{\pi}|_G(k)\hat{v}_0\rangle \tag{20}$$

$$= \frac{1}{|K|}\sum_{k\in K}\langle \pi|_G(k)^{-1}v, \hat{v}_0\rangle \tag{21}$$

$$= \frac{1}{|K|}\sum_{k\in K}\langle \pi(k)^{-1}v, \hat{v}_0\rangle \tag{22}$$

$$= \frac{1}{|K|}\sum_{k\in K}\langle v, \hat{v}_0\rangle \tag{23}$$

$$= l(v) \tag{24}$$

□

Lemma 2. *If $V^K \neq 0$ then $\hat{V}^K \neq 0$. [1]*

Lemma 3. *Let R be an algebra over F, and N_1, N_2 simple R-modules that are finite-dimensional as vector spaces over F. If there exist linear functionals $L_i : N_i \to F$ and $n_i \in N_i$ such that $L_i(n_i) \neq 0$ and $L_1(rn_1) = L_2(rn_2)$ for all $r \in R$, then $N_1 \cong N_2$ as R-modules. [1]*

We particularly care about the case when two representations (π_i, V_i) share matrix coefficients $\langle \pi_i(m)v, \hat{v}_0\rangle$ for all $m \in M$.

Lemma 4. *Let (π, V) and (σ, W) be two irreducible representations of M with nonzero matrix coefficients $\langle \pi(m)v, \hat{v}_0\rangle = \langle \sigma(m)w, \hat{w}_0\rangle$ for some v, v_0, w, w_0, and all $m \in M$. Then $(\pi, V) \cong (\sigma, W)$.*

Proof. Define actions of F[M] on V and W by letting $mv = \pi(m)v$ and $mw = \sigma(m)w$ for all $v \in V, w \in W$, and $m \in M$ respectively and then extending by linearity. Thus V and W become F[M]-modules. Because the representations are each irreducible, V and W are simple as F[M]-modules. Since $\langle mv, \hat{v}_0\rangle = \langle mw, \hat{w}_0\rangle$ for all $m \in M$ are two equal linear functionals on V and W, then $V \cong W$ as F[M]-modules by Lemma 4. Equivalently, $(\pi, V) \cong (\sigma, W)$. □

Now we prove the second half of the Borel-Matsumoto Theorem.

Theorem 2.2. *If (π, V) and (σ, W) are two irreducible representations of M with V^K and W^K nonzero and isomorphic as $\mathcal{H}_K$-modules, then $(\pi, V) \cong (\sigma, W)$.*

Proof. Let $\lambda : V^K \to W^K$ be an isomorphism of $\mathcal{H}_K$-modules and $l : W^K \to F$ be a linear functional not equal to zero. Then by Bump's Lemma 2, there exist $\hat{v} \in \hat{V}^K$ and $\hat{w} \in \hat{W}^K$ such that $(l \circ \lambda)(v) = \langle v, \hat{v}\rangle$ and $l(w) = \langle w, \hat{w}\rangle$ for all $v \in \hat{V}^K, w \in \hat{W}^K$. Furthermore, there exist $w_0 \in W^K, v_0 \in V^K$ such that $\langle w_0, w\rangle \neq 0$ since l is nontrivial and $v_0 = \lambda^{-1}(w_0)$ since λ is an isomorphism.

Then for $\phi \in \mathcal{H}_K$, we have that

$$\langle \sigma(\phi)w_0, \hat{w}\rangle = \langle \sigma(\phi)\lambda(v_0), \hat{w}\rangle = \langle \lambda(\pi(\phi)v_0), \hat{w}\rangle = (l \circ \lambda)(\pi(\phi)v_0) = \langle \pi(\phi)v_0, \hat{v}\rangle. \tag{25}$$

We show that equation 23 holds for all $\phi \in \mathcal{H}$ as well as $\mathcal{H}_K$. For $\phi \in \mathcal{H}$, define $\phi_K \in \mathcal{H}_K$ by

$$\phi_K(x) = \frac{1}{|K|^2} \sum_{k_1,k_2 \in K} \phi(k_1 x k_2) \tag{26}$$

for all $x \in M$. By equation 23, then $\langle \pi(\phi_K)v_0, \hat{v}\rangle = \langle \sigma(\phi_K)w_0, \hat{w}\rangle$. Furthermore, we have that

$$\begin{aligned}
\langle \pi(\phi_K)v_0, \hat{v}\rangle &= \langle \frac{1}{|K|^2} \sum_{k_1,k_2 \in K} \sum_{x \in M} \phi(k_1 x k_2)\pi(x)v_0, \hat{v}\rangle & (27)\\
&= \frac{1}{|K|^2} \langle \sum_{k_1,k_2 \in K} \sum_{x \in M} \phi(x)\pi(k_1)^{-1}\pi(x)\pi(k_2)^{-1}v_0, \hat{v}\rangle & (28)\\
&= \frac{1}{|K|^2} \langle \sum_{k_1,k_2 \in K} \pi(k_1)^{-1} \circ (\sum_{x \in M} \phi(x)\pi(x)) \circ \pi(k_2)^{-1}v_0, \hat{v}\rangle & (29)\\
&= \frac{1}{|K|^2} \langle \sum_{k_1,k_2 \in K} \pi(k_1)^{-1}\pi(\phi)\pi(k_2)^{-1}v_0, \hat{v}\rangle & (30)\\
&= \frac{1}{|K|^2} \sum_{k_1,k_2 \in K} \langle \pi(k_1)^{-1}\pi(\phi)\pi(k_2)^{-1}v_0, \hat{v}\rangle & (31)\\
&= \frac{1}{|K|^2} \sum_{k_1,k_2 \in K} \langle \pi|_G(k_1)^{-1}\pi(\phi)\pi|_G(k_2)^{-1}v_0, \hat{v}\rangle & (32)\\
&= \frac{1}{|K|^2} \sum_{k_1,k_2 \in K} \langle \pi(\phi)\pi|_G(k_2)^{-1}v_0, \hat{\pi}|_G(k_1)\hat{v}\rangle. & (33)\\
&= \frac{1}{|K|^2} \sum_{k_1,k_2 \in K} \langle \pi(\phi)v_0, \hat{v}\rangle & (34)\\
&= \langle \pi(\phi)v_0, \hat{v}\rangle. & (35)
\end{aligned}$$

since $v_0 \in V^K$ and $\hat{v} \in \hat{V}^K$.

Thus $\langle \pi(\phi_K)v_0, \hat{v}\rangle = \langle \pi(\phi)v_0, \hat{v}\rangle$ for all $\phi \in \mathcal{H}$. Similarly, $\langle \sigma(\phi_K)w_0, \hat{w}\rangle = \langle \sigma(\phi)w_0, \hat{w}\rangle$. With this information, then, we have that $\langle \pi(\phi_K)v_0, \hat{v}\rangle = \langle \sigma(\phi_K)w_0, \hat{w}\rangle$ implies that $\langle \pi(\phi)v_0, \hat{v}\rangle = \langle \sigma(\phi)w_0, \hat{w}\rangle$.

Let $\phi_m \in \mathcal{H}$ for all $m \in M$ be the function that sends all x in M with $x \neq m$ to 0 and m to 1. Then $\pi(\phi_m)v = \pi(m)v$ and $\sigma(\phi_m)w = \sigma(m)w$.

Thus, we have that $\langle \pi(m)v_0, \hat{v}\rangle = \langle \sigma(m)w_0, \hat{w}\rangle$ for all $m \in M$. By Lemma 5, then, (π, V) and (σ, W) are equivalent. $\square$

Let M be a finite monoid, $G(M)$ its group of units, N a submonoid of M, $G(N)$ its group of units, and (π, V) a representation of M. Define the vector space $\mathrm{Ind}_N^M V$ as follows:

$$\mathrm{Ind}_N^M V = \{f : M \to V \mid f(nm) = \pi(n)f(m) \quad \forall n \in N,\ m \in M\} \tag{36}$$

Define $(\pi^M, \mathrm{Ind}_N^M V)$ by $\pi^M(m)f(x) = f(xm)$ for all m.

Lemma 5. *The pair* $(\pi^M, \mathrm{Ind}_N^M V)$ *is a representation of M.*

Proof. First, we check that $\mathrm{Ind}_N^M V$ is closed under the action of $\pi^M(m)$. Trivially, if $f(nx) = \pi(n)f(x)$ then $\pi^M(m)f(nx) = f(nxm) = \pi(n)f(xm)$ for all $m \in M, n \in N$.

We check that $\pi^M(m)$ is linear for all m.

$$\forall z \in F,\ \forall f, g \in \mathrm{Ind}_N^M \quad z\pi^M(m)f(x) = zf(xm) = \pi^M(m)(zf)(x)$$

$$\pi^M(m)(f+g)(x) = (f+g)(xm) = \pi^M(m)(f)(x) + \pi^M(m)(g)(x)$$

Now, we check that π^M is a homomorphism of monoids. Let $m, x, y \in M$. Then $\pi^M(mx)f(y) = f(ymx) = \pi^M(x)f(ym) = \pi^M(m)\pi^M(x)f(y)$. Finally, $\pi^M(1)f(x) = f(x)$, implying that π^M maps the identity to the identity. Clearly, then, $\pi^M(mx) = \pi^M(m)\pi^M(x)$, and $(\pi^M, \mathtt{Ind}_N^M)$ is a representation of M. $\square$

Thus we can call $(\pi^M, \mathtt{Ind}_N^M V)$ the induced representation of M.

We include, for completeness, a proof of Frobenius Reciprocity for monoids:

Theorem 2.3. *If* (π, V) *is a representation of N, a submonoid of M, and* (σ, W) *a representation of M, then* $\mathrm{Hom}_M(W, \mathtt{Ind}_N^M V) \cong \mathrm{Hom}_N(W, V)$ *as vector spaces.*

Proof. For $\phi \in \mathrm{Hom}_M(W, \mathtt{Ind}_N^M V)$, define $F : \mathrm{Hom}_M(W, \mathtt{Ind}_N^M V) \to \mathrm{Hom}_N(W, V)$ by $F(\phi)$, such that $F(\phi)(w) = \phi(w)(1)$, 1 being the identity element of M. We first show that $F(\phi)$ is linear. Because ϕ is linear,

$$F(\phi)(w + w_0) = \phi(w + w_0)(1) = \phi(w)(1) + \phi(w_0)(1) = F(\phi)(w) + F(\phi)(w_0)$$

and for $z \in F$,

$$F(\phi)(zw) = \phi(zw)(1) = z\phi(w)(1) = zF(\phi)(w)$$

We now claim that $F(\phi)$ is a morphism of N-modules For $n \in N$,

$$\begin{aligned} F(\phi)(\sigma(n)w) &= \phi(\sigma(n)w)(1) = \pi^M(n)\phi(\sigma(1)w)(1) \\ &= \phi(w)(n) = \pi(n)\phi(w)(1) = \pi(n)F(\phi)(w) \end{aligned}$$

Thus $F(\phi)$ is an N-module homomorphism from W to V. Since

$$F(\phi + \psi)(w) = (\phi + \psi)(w)(1) = \phi(w)(1) + \psi(w)(1) = F(\phi)(w) + F(\psi)(w)$$

and $F(z \cdot \phi)(w) = (z\phi)(w)(1) = zF(\phi)(w)$, then F is a vector space homomorphism. For $\tau \in \mathrm{Hom}_N(W, V)$, let $G : \mathrm{Hom}_N(W, V) \to \mathrm{Hom}_M(W, \mathtt{Ind}_N^M V)$ such that

$$(G(\tau)(w))(m) = G(\tau)(w)(m) = \tau(\sigma(m)w)$$

then, $\tau(\sigma(nm)w) = \tau(\sigma(n)\sigma(m)w) = \pi(n)\tau(\sigma(m)w)$, so $G(\tau)(w)$ is in $\mathtt{Ind}_N^M$. We check that $G(\tau)(-)(m)$ is linear. This follows from the definition:

$$\begin{aligned} G(\tau)(w + w_0)(m) &= \tau(\sigma(m)(w + w_0)) = \tau(\sigma(m)w + \sigma(m)w_0) \\ &= \tau(\sigma(m)w) + \tau(\sigma(m)w_0) = G(\tau)(w)(m) + G(\tau)(w_0)(m) \end{aligned}$$

and for $z \in F$, we have

$$G(\tau)(zw)(m) = \tau(\sigma(m)(zw)) = z\tau(\sigma(m)(w))$$

Next, we check that $G(\tau)$ respects M. We have that for $x \in M$,

$$\begin{aligned} G(\tau)(\sigma(x)w)(m) &= \tau(\sigma(m)\sigma(x)w) = \tau(\sigma(mx)w) \\ &= (\pi^M(x) \circ \tau)(\sigma(m)w) = \pi^M(x)(G(\tau)(w)(m)) \end{aligned}$$

Thus $G(\tau) \in \mathrm{Hom}_M(W, \mathtt{Ind}_N^M V)$. Finally, we check that G itself is linear:

$$\begin{aligned} G(\tau + \eta)(w)(m) &= (\tau + \eta)(\sigma(m)w) \\ &= \tau(\sigma(m)w) + \eta(\sigma(m)w) = G(\tau)(w)(m) + G(\eta)(w)(m) \end{aligned}$$

and for $k \in K$, $\quad G(k\tau)(w)(m) = k(\tau(\sigma(w)m)) = k \cdot G(\tau)(w)(m)$. Thus G is a homomorphism of vector spaces.

Now, we show that F and G are inverses. First, we check the mapping $G \circ F : \mathrm{Hom}_M(W, \mathrm{Ind}_N^M) \to \mathrm{Hom}_M(W, \mathrm{Ind}_N^M)$. Let $\phi \in \mathrm{Hom}_M(W, \mathrm{Ind}_N^M)$. Then $G \circ F(\phi)$ works as follows. Since $F(\phi)$ is the map sending w to $\phi(w)(1)$,

$$\begin{aligned}(G \circ F)(\phi)(w)(m) &= G(F(\phi))(w)(m) = F(\phi)(\sigma(m)w) \\ &= F(\phi)(\sigma(1 * m)w) = \pi^M(m)F(\phi)(w) \\ &= \pi^M(m)\phi(w)(1) = \phi(w)(m)\end{aligned}$$

by definition of the induced representation. Since $(G \circ F)(\phi)(w)(m) = \phi(w)(m)$, $G \circ F$ is the identity morphism on $\mathrm{Hom}_M(W, \mathrm{Ind}_N^W)$.

Next, we check $F \circ G : \mathrm{Hom}_N(V, W) \to \mathrm{Hom}_N(V, W)$. Let $\tau \in \mathrm{Hom}_N(V, W)$. Then

$$\begin{aligned}(F \circ G)(\tau)(w)(n) &= F(G(\tau))(w)(n) \\ &= G(\tau)(w)(1 \cdot n) = \tau(\pi(n)w) = \sigma(n)\tau(w) = \tau(w)(n)\end{aligned}$$

Thus $F \circ G$ is the identity morphism on $\mathrm{Hom}_N(V, W)$. Since we have that both $G \circ F$ and $F \circ G$ are identity morphisms on their respective domains, they are inverses. Thus, we have that $\mathrm{Hom}_M(W, \mathrm{Ind}_N^M) \cong \mathrm{Hom}_N(W, V)$ as vector spaces over F.

□

3 Matrix Presentation for the Symplectic Renner Monoid

In this section, we give a classification for the symplectic rook monoid as a replacement for the previously established classification (see Li, Li, and Cao [7] p. 843, Corollary 2.3):

$$\mathcal{RS}p_n = \{A \in \mathcal{R}_n \mid APA^t = A^tPA = 0 \text{ or } P\}$$

for

$$P = \begin{pmatrix} 0 & J_m \\ -J_m & 0 \end{pmatrix}, \qquad J_m = \begin{pmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ & \dots & \dots & \\ 1 & 0 & \dots & 0 \end{pmatrix}$$

where $n = 2m$, J_m is the $m \times m$ anti-diagonal matrix of 1's, and $\mathcal{R}_n$ is the rook monoid, a submonoid of Mat_n. Yet note that for $m = 1$, the symplectic Renner monoid is the entirety of the rook monoid (see p. 842 of [7]),

$$\mathcal{RS}p_2 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

However note that

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad P = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \Longrightarrow APA^t = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

indicating that the description given in the corollary is not accurate, and indeed, there is no immediate proof of this proposition in the same paper. Proposition 2.4 of the same paper gives the following classification

$$\mathcal{R} = \{A \in \mathcal{R}_n \mid APA^t = A^tPA = 0\} \cup W$$

we will take [7]'s definition of the symplectic Weyl group as the collection of "injective partial transformations of **n** that map all admissible sets of **n** to admissible sets" ([7], p.841). With the knowledge that $\{1, \dots, n\}$ is an admissable set, we see that elements of the Weyl group are full rank, and also elements of the rook monoid, so that the Weyl group is a submonoid of $S_n \subseteq \mathcal{R}_n$. With this, we establish:

Theorem 3.1.

$$\mathcal{R}\mathcal{S}p_n \cong \{A \in \mathcal{R}_n \mid A^t J_n A = A J_n A^t = 0 \quad \textbf{\textit{or}} \quad A^t J_n A = A J_n A^t = J_n\}$$

Proof. We first show that

$$M \in \{A \in \mathcal{R}_n \mid A^t P A = A P A^t = 0\} \iff M \in \{A \in \mathcal{R}_n \mid A^t J_n A = A J_n A^t = 0\}$$

Indeed, note that $J_n A^t = (b_{ij})$ is still an element of R_n by nature of having at most one entry in every row and column that is non-zero (and hence $= 1$), so that the condition that

$$A J_n A^t = 0 \iff \quad \forall i, j, \quad (A J_n A^t)_{ij} = c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = 0$$

means that there is at most one non-zero summand in the evaluation of c_{ij}, forcing that summand to be 0. Clearly, if $d_{ij} := (APA^t)_{ij}$, then $c_{ij} = \pm d_{ij}$, so that $c_{ij} = 0 \iff d_{ij} = 0$. Thus $A J_n A^t = 0 \iff APA^t = 0$, and the non-full rank components of these sets coincide.

It suffices to show that

$$W = \{A \in \mathcal{R}_n \mid A^t J_n A = A J_n A^t = J_n\}$$

Let $\bar{i} = n + 1 - i$. For $A \in W$, we have that

$$A(k) = i_k \implies A(\bar{k}) = \overline{i_k}$$

for if not, then using our first definition of the symplectic rook monoid, $\{k, A^{-1}(\overline{i_k})\}$ would be an admissible set mapped to a non-admissible set. Yet note that

$$A^t J_n A = A J_n A^t = J_n \iff A J_n = J_n A \iff J_n A J_n = A$$

having used the fact that $A^t = A^{-1}$ and $J_n^{-1} = J_n$, but $J_n A J_n = A$ is exactly the condition that $A(k) = i_k \iff A(\bar{k}) = \overline{i_k}$, as J_n is the permutation corresponding to $(1\ \bar{1})(2\ \bar{2}) \cdots (m\ \bar{m})$ and hence conjugating a permutation matrix by J_n makes it so that $k \mapsto \overline{A(\bar{k})}$, thus

$$A = J_n A J_n \iff A(k) = \overline{A(\bar{k})} \iff \overline{A(k)} = A(\bar{k})$$

so every $A \in W$ satisfies $A = J_n A J_n$. Similarly, matrices $A \in \mathcal{R}_n$ satisfying $A = J_n A J_n$ are full rank and map admissible sets to admissible sets. For if not, then there would exist

$$k,\ s \neq k, \bar{k} \ \text{ s.t. } \ A(k) = i_k, \quad A(s) = \overline{i_k}$$

contradicting the condition that $J_n A J_n = A$. $\square$

With this presentation in mind, we obtain the same formula for the size of this monoid:

$$|RSp_{2n}| = 2^n n! + \sum_{i=0}^{n} \binom{n}{i}^2 2^i \tag{37}$$

Remark This correction should not invalidate the rest of [7]'s results, as the authors make use of the symplectic Renner monoid in terms of admissible sets. That being said, this correction is useful for readers who would want to calculate the elements of the symplectic rook monoid as a submonoid of the rook monoid. Moreover, it has been difficult for the authors of this paper to find other characterizations of the symplectic rook monoid that describe the symplectic weyl group as a submonoid of $\mathcal{R}_n$, so the authors hope that this presentation will be computationally practical.

4 Example: Character Table for RSp6

Below we present the character table for the symplectic rook monoid, as calculated via the method of Li, Li, Cao [7]. The matrix is presented in the transpose form as their convention dictates, with different row and column labellings.

$$
C(\mathsf{RSp}_6) = \begin{array}{c|ccccccccccccccccc|}
 & W1 & W2 & W3 & W4 & W5 & W6 & W7 & W8 & W9 & W10 & (1,1,1) & (2,1) & (1,1,1) & (2) & (1,1) & (1) & (0) \\
\hline
c1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
c2 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
c3 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
c4 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
c5 & 2 & 2 & 0 & 0 & -2 & 0 & 1 & -1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
c6 & 2 & 2 & 0 & 0 & 2 & 0 & -1 & -1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
c7 & 3 & -1 & -1 & 1 & -1 & -1 & 0 & 0 & 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
c8 & 3 & -1 & 1 & 1 & 1 & -1 & 0 & 0 & -1 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
c9 & 3 & -1 & 1 & -1 & -1 & 1 & 0 & 0 & -1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
c10 & 3 & -1 & -1 & -1 & 1 & 1 & 0 & 0 & 1 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(1,1,1) & 8 & 0 & 0 & 0 & 0 & 4 & 0 & 2 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
(2,1) & 16 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\
(3) & 8 & 0 & 0 & 0 & 0 & -4 & 0 & 2 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\
(1,1) & 12 & 0 & 2 & 0 & 4 & 2 & 0 & 0 & 0 & 0 & 3 & 1 & 0 & 1 & 1 & 0 & 0 \\
(2) & 12 & 0 & -2 & 0 & 4 & -2 & 0 & 0 & 0 & 0 & 3 & -1 & 0 & 1 & -1 & 0 & 0 \\
(1) & 6 & 2 & 0 & 0 & 4 & 2 & 0 & 0 & 2 & 0 & 3 & 1 & 0 & 2 & 0 & 1 & 0 \\
(0) & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}
$$

where C_i stands for the ith irreducible character of indexed by a signed partition, (λ, μ), such that $|\lambda|+|\mu| = n$ and W_i stands for the ith conjugacy class of the Weyl group, with the same indexing convention. By [7], we know that the Munn classes and irreps of RSp_{2n} are both indexed by elements of Ω_n. In this case, the conjugacy class representatives for the Weyl group are

$$
\{W_1, W_2, W_3\} = \{ \begin{pmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0&1 \\ 0&0&0&0&1&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&1&0&0&0&0 \\ 1&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0&1 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 1&0&0&0&0&0 \end{pmatrix} \}
$$

$$
\{W_4, W_5, W_6\} = \{ \begin{pmatrix} 0&0&0&0&0&1 \\ 0&0&0&1&0&0 \\ 0&1&0&0&0&0 \\ 0&0&0&0&1&0 \\ 0&0&1&0&0&0 \\ 1&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0&1 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 1&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \end{pmatrix} \}
$$

$$
\{W_7, W_8, W_9\} = \{ \begin{pmatrix} 0&0&0&0&1&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&0&1 \\ 1&0&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&1&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&1&0 \\ 0&0&0&1&0&0 \\ 1&0&0&0&0&0 \\ 0&0&0&0&0&1 \\ 0&0&1&0&0&0 \\ 0&1&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&1&0 \\ 1&0&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&0&1 \\ 0&1&0&0&0&0 \end{pmatrix} \}
$$

$$W_{10} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\{W_1, W_2, W_3, W_4, W_5, W_6, W_7, W_8, W_9, W_{10}\} =$

$$\{((1)^3 \mid \emptyset),\ ((1) \mid (1)^2),\ ((2) \mid (1)),\ (\emptyset \mid (1,2)),\ ((1)^2 \mid (1)),\ ((2,1) \mid \emptyset),\ (\emptyset \mid (3)),\ ((1) \mid (2)),\ (\emptyset \mid (1)^3)\}$$

where the correspondence between $2n \times 2n$ matrices and signed partitions (also known as pairs of partitions) of n is explained in the subsequent section. Taking the transpose of the above yields

$$M := C(RSp_6) = \begin{pmatrix}
1 & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 3 & 3 & 8 & 16 & 8 & 12 & 12 & 6 & 1 \\
1 & 1 & 1 & 1 & 2 & 2 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\
1 & 1 & -1 & -1 & 0 & 0 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 2 & -2 & 0 & 1 \\
1 & -1 & -1 & 1 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & -1 & 1 & -1 & -2 & 2 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 4 & 4 & 4 & 1 \\
1 & -1 & -1 & 1 & 0 & 0 & -1 & -1 & 1 & 1 & 4 & 0 & -4 & 2 & -2 & 2 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 2 & -2 & 2 & 0 & 0 & 0 & 1 \\
1 & 1 & -1 & -1 & 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\
1 & -1 & 1 & -1 & -2 & 2 & 3 & -3 & 3 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 3 & 3 & 3 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & -1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

Note that we can write $M = YB = AY$ where Y is the block diagonal matrix of the Weyl Group, B_3, and then S_3, S_2, S_1, S_0. From here,

$$Y = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 0 & 0 & -2 & 0 & 1 & -1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 0 & 0 & 2 & 0 & -1 & -1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & -1 & -1 & 1 & -1 & -1 & 0 & 0 & 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & -1 & 1 & 1 & 1 & -1 & 0 & 0 & -1 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & -1 & 1 & -1 & -1 & 1 & 0 & 0 & -1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & -1 & -1 & -1 & 1 & 1 & 0 & 0 & 1 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

and

$$B = Y^{-1}M = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

$$A = MY^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 12 & 0 & 6 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 4 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 2 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 3 & 0 & 3 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

5 Finding the A matrix of $C(RSp_{2n})$

Let $M = C(RSp_{2n})$ as in the previous example. Let $\lambda \vdash r$ for $0 \leq r \leq n$, let σ be a representative of the Munn class indexed by $\alpha \in \mathcal{Q}_n$, let $\sigma_K \in S_r$ be the "conjugate" of σ, and let $C(r, \sigma) := C(r)$, all as on p. 849 of [7], except note that here we make the dependence of $C(r)$ on σ more explicit by writing $C(r, \sigma)$. The value of $\chi^*_\lambda(\sigma)$ is independent of which representative of the Munn class is chosen by theorem 4.3 of [7], and it corresponds the matrix entry $M_{\lambda,\alpha}$ using our index notation

$$(M)_{\lambda,\alpha} = \chi^*_\lambda(\sigma) = \sum_{K \in C(r,\sigma)} \chi_\lambda(\sigma_K)$$

Partition $C(r,\sigma)$ into a disjoint union as follows

$$\begin{aligned}C(r,\sigma) &= \bigsqcup_{\mu\vdash r}\{K\in C(r,\sigma)\mid K=\{i_1,\dots,i_{m(1)},j_1,\dots,j_{m(2)},\dots,l_1,\dots,l_{m(e)}\}\\ &\text{s.t. } \mu=(m(1),m(2),\dots,m(e))\}\\ &=\bigsqcup_{\mu\vdash r} S_{\mu,\sigma}\end{aligned}$$

i.e. we organize the K by the cycle types of S_r that they imitate (again, see p. 849 of Li, Li, and Cao for reference). Given a fixed $\mu\vdash r$, we know that

$$\forall K\in S_{\mu,\sigma},\qquad \sigma_\mu := \sigma_K = (1\cdots m(1))\ (m(1)+1\cdots m(1)+m(2))\cdots\left(\left[\sum_{i=1}^{e-1} m(i)\right]+1\cdots\left[\sum_{i=1}^{e} m(i)\right]\right)$$

The independence of σ_μ is immediate from having the same cycle type and Corollary 4.1 of [7]. From this, we can write

$$(M)_{\lambda,\alpha}=\sum_{K\in C(r,\sigma)}\chi_\lambda(\sigma_K)=\sum_{\mu\vdash r}\sum_{K\in S_{\mu,\sigma}}\chi_\lambda(\sigma_K)=\sum_{\mu\vdash r}|S_{\mu,\sigma}|\,\chi_\lambda(\sigma_K)$$

From here, we set

$$A_{\alpha,\mu}=\begin{cases}|S_{\mu,\alpha}| & |\mu|=|\alpha|\\ 0 & |\mu|\neq|\alpha|\end{cases}$$

where we adopt the convention that the rows of A are indexed by conjugacy classes and columns are indexed by the irreducible representations of RSp_{2n}. To see that this is the correct definition of A, note that

$$(AY)_{\lambda,\delta}=\sum_{\beta\in\mathcal{Q}_n}A_{\lambda,\beta}Y_{\beta,\delta}=\sum_{\beta\text{ s.t. }|\beta|=|\lambda|}|S_{\beta,\lambda}|Y_{\beta,\delta}$$

Because $|\beta|\neq|\delta|\implies Y_{\beta,\delta}=0$. Let G be the corresponding group in which the Munn Class indexed by λ and δ lies, and let σ be an element of that Munn class with $r=|\lambda|=|\delta|$. Then

$$(AY)_{\lambda,\delta}=\sum_{\beta\in G}|S_{\beta,\lambda}|Y_{\beta,\delta}=\sum_{\mu\vdash r}|S_{\mu,\sigma}|\chi_\lambda(\sigma_K)=(M)_{\lambda,(\sigma)}=(M)_{\lambda,\delta}$$

having noted that $(\sigma)=\delta$ as Munn classes.

In the particular case that both λ and δ are partitions of $r\le n$, then any element $\sigma\in C_\lambda$, the Munn class corresponding to partition λ, has rank less than or equal to n. In particular, let M_{σ_K} represent the $r\times r$ matrix corresponding to σ_K as an element of S_r. Then from our knowledge of the conjugacy classes of RSp_{2n}, we know that

$$N=\begin{pmatrix}M_{\sigma_K} & 0\\ 0 & 0\end{pmatrix}\in C_\lambda\subseteq RSp_{2n}$$

is a valid representative, and hence any set, $S\subseteq I^\circ(N)$ with $|S|=r$ consisting of some cycles of N° (see p. 849 of Li, Li, Cao) *will automatically be admissable.* In particular, this means that

$$|S_{\beta,\lambda}|=\binom{\beta}{\lambda}=\prod_{i=1}^{s}\binom{\beta_i}{\lambda_i}$$

which is the same A-matrix entry as in Solomon's "Representations of the Rook Monoid," p. 321 [9]. This shows that the character table of the symplectic rook monoid has the following form

$$C(RSp_{2n})=\begin{pmatrix}C(B_n) & *\\ 0 & C(R_n)\end{pmatrix}$$

where $C(M)$ denotes the character table of the inputted monoid as a block matrix in our larger matrix. Moreover,

$$A = \begin{pmatrix} \mathrm{Id} & U \\ 0 & T \end{pmatrix}$$

where $T_{\beta,\lambda} = \binom{\beta}{\lambda} = |S_{\beta,\lambda}|$ for the β, λ described above. U is given by the values of $|S_{\mu,\lambda}|$ for $\mu \in SP_m$ and $\lambda \in \cup_{r=0}^{m} P_r$ under the standard ordering of $\mathcal{Q}_n$.

5.1 Determining the U-block

To determine U, we consider elements of the Weyl group as described in Geck and Pfeiffer [3] propositions 1.4.1 (p. 21), 3.4.2 (p. $92-93$), and 3.4.7 (p. $96-97$). In particular, we can determine U by looking at the representatives

$$w_{\alpha,\beta} = b^-_{m_1,e_1} \cdots b^-_{m_l,e_l} \cdot b^+_{m_{l+1},e_{l+1}}, \cdots b^+_{m_r,e_r}$$

where $\beta = (e_1, \dots, e_l)$ and $\alpha = (e_{l+1}, \dots, e_r)$ consist of a decreasing partition and increasing partition of some r and $n-r$ respectively, so that $|\alpha| + |\beta| = n$. We have that

$$b^+_{m,e} = s_{m+1}s_{m+2} \cdots s_{m+e-1} \in W_n$$

$$b^-_{m,e} = t_m s_{m+1}s_{m+2} \cdots s_{m+e-1} \in W_n$$

where s_i is the matrix corresponding to the permutation $(i,\ i+1) \in S_n$, also realized as a matrix group, and t_m for $0 \le m \le n-1$ is the identity matrix in $\mathrm{Mat}(n)$ except $t_{m+1,m+1} = -1$. We note that

$$\begin{aligned} b^+_{m,e} = s_{m+1} \dots s_{m+e-1} &\cong (m+1,\ m+2)_n \cdots (m+e-1,\ m+e)_n \in S_n \\ &\to (m+1,\ m+2,\ \cdots\ m+e)_{2n} \quad (\overline{m+1},\ \overline{m+2},\ \cdots\ \overline{m+e})_{2n} \in S_{2n} \end{aligned}$$

$$\begin{aligned} b^-_{m,e} = t_m s_{m+1} \dots s_{m+e-1} &\cong t_m(m+1,\ m+2)_n \cdots (m+e-1,\ m+e)_n \\ &\to (m+1,\ m+2, \cdots m+e-1, m+e,\ \overline{m+1},\ \overline{m+2} \cdots \overline{m+e})_{2n} \in S_{2n} \end{aligned}$$

Here we've included the correspondence between signed $n \times n$ matrices, which are demarcated by $(\cdots)_n$ and unsigned matrices of size $2n \times 2n$, demarcated by $(\cdots)_{2n}$. This correspondence is as follows: given a signed matrix M_σ, then for the permutation matrix contained in $\mathrm{Mat}(2n)$, call it M', we have

$$(M_\sigma)_{ij} = 1 \implies (M')_{ij} = 1 = (M')_{\bar{i}\bar{j}} = 1$$

$$(M_\sigma)_{ij} = -1 \implies (M')_{\bar{i}j} = 1 = (M')_{i\bar{j}}$$

With this, the image of $b^+_{m,e}$ in $\mathrm{Mat}(2n)$ is justified. The image of $b^-_{m,e}$ can be explained as follows: consider the image of $b^-_{m,e}$ and that of $b^+_{m,e}$ as matrices in $\mathrm{Mat}(2n)$, denoted by M^- and M^+. These matrices are very similar, in particular, rows $m+2$ through $m+e-2$ coincide. However $M^+_{m+1,m+e} = 1 = M^+_{\overline{m+1},\overline{m+e}}$, while $M^-_{\overline{m+1},m+e} = 1 = M^-_{m+1,\overline{m+e}}$, which comes from the effect of multiplying by t_m. This means that instead of mapping $m+e$ to $m+1$ under the action of $b^-_{m,e}$, we have that $m+e$ is mapped to $\overline{m+1}$. From there, the cyclic permutation of $\overline{m+1} \to \overline{m+2} \to \cdots \to \overline{m+e}$ occurs normally, as

$$M^-_{\overline{m+2},\overline{m+1}} = \cdots = M^-_{\overline{m+e},\overline{m+e-1}} = M^+_{\overline{m+2},\overline{m+1}} = \cdots = M^+_{\overline{m+e},\overline{m+e-1}} = 1$$

holds. However $b^-_{m,e}\overline{m+e} = m+1$, as $M^-_{m+1,\overline{m+e}} = 1$, finishing the matrix description. As an example, consider

$$\begin{aligned} b^-_{0,3} &= t_0 s_1 s_2 = t_0(1\ 2)(2\ 3) = t_0(1\ 2\ 3) \\ &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \end{aligned}$$

$$\mapsto \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \to (1\ 2\ 3\ 6\ 5\ 4) = (1\ 2\ 3\ \overline{1}\ \overline{2}\ \overline{3})$$

With this, we note that the representatives $\{w_{\alpha,\beta}\}$ consist of blocks of *disjoint* signed permutations, i.e. each block affects some distinct subset of n. In particular, this is because the subset in question is $\{m_i+1,\dots,m_i+e_i\}$ for a given b_{m_i,e_i}. Now consider the set

$$C(r) = C(r,\sigma) = \{K \subseteq I^\circ(\sigma) \mid K \text{ admissable consists of all the elements of some cycles of } \sigma^\circ \text{ with } |K| = r\ \}$$

for which we sum over in

$$\chi^*_\lambda(\sigma) = \sum_{K\in C(r,\sigma)} \chi_\lambda(\sigma_K)$$

from pp. $848-849$ of [7]. We note that no admissable set in $C(r)$ can contain a cycle induced by a block $b^-_{m,e}$, as such a cycle contains both $m+i$ and $\overline{m+i}$ for $1 \le i \le e$. Thus the cycles from which an admissible $K \in C(r)$ can be created are those coming from the $b^+_{m,e}$ blocks.

Now recall the $S_{\mu,\alpha}$ notation. Let $\{\mu_i\}$ denote the number of i-cycles in the partition μ. Similarly, let $\{\alpha_i\}$ denote the number of admissible i-cycles produced by $w_{\alpha,\beta}$, which is equal to the number of $\{e_k\}_{k>l}$ such that $e_k = i$. Then

$$|S_{\mu,\sigma}| = \prod_{i=1}^{2n} \binom{\alpha_i}{\mu_i} \cdot 2^{\mu_i} = 2^{\sum_i \mu_i} \cdot \prod_{i=1}^{2n} \binom{\alpha_i}{\mu_i}$$

because if there are α_i i-cycles, then we have $\binom{\alpha_i}{\mu_i} \cdot 2^{\mu_i}$ ways of choosing them, as we could consider choosing i cycles from the collection of cycles acting on $\{1,\dots,n\}$ or from the disjoint collection of cycles acting $\{n+1,\dots,2n\} = \{\overline{n},\dots,\overline{1}\}$. Note that such cycles come in pairs as shown by the image of $b^+_{m,e}$ in $Mat(2n)$, and so for every chosen i cycle, we can choose that cycle or its admissible conjugate. This determines the values of $|S_{\mu,\sigma}| = |S_{\mu,\alpha}|$ and hence the value of U, completing the description of the A matrix .

6 Restricting Monoid Representations to Group Representations

Among many sources, Steinberg [10] tells us that the character table of any finite inverse semigroup is block upper-triangular. Using this, Solomon decomposes the character table of R_n into the product of a block diagonal matrix and a much simpler block-upper-triangular matrix [9]. In type A_n, he finds matrices A and B such that the character table $M = AY = YB$ where $Y = \text{diag}(X_n,\dots,X_0)$.

To explain his result, we first define

Definition 6.1. *Given groups G, H and corresponding representations V_G and V_H, we define the box tensor representation $V_G \boxtimes V_H$ to be the representation of $G \times H$ with the action $(g,h)\cdot(v_1 \boxtimes v_2) = gv_1 \boxtimes hv_2$.*

We can now describe Solomon's restriction. He shows that given χ^* an irreducible representation of R_n corresponding to a partition of k, the restriction $\chi^* |_{S_n} = \text{Ind}^{S_n}_{S_k\times S_{n-k}}(\chi \boxtimes \eta_{n-k})$, where η_{n-k} is the trivial representation. We show in general

Theorem 6.2. *Let W_n be a Weyl group of type A_n, B_n, C_n, or D_n, with corresponding Renner monoids RW_n. Let χ be a character S_r, and χ^* the associated character of W_n. Then*

$$\chi^*|_{W_n} = \text{Ind}^{W_n}_{S_k\times W_{n-k}}(\chi \boxtimes \eta_{n-k})$$

Proof. By setting $e = e_r$ we have $W^*(e)$ is generated by roots $\{\alpha \in \Delta, s_\alpha e = e s_\alpha \neq e\}$. Thus, $W^*(e)$ is generated by elements which involve the first r indices. We know what the Coxeter diagrams for these groups look like, and the only generators which involve the first r indices are the transpositions $(k\,k+1)$ for $k \leq r$. However, $(r\;r{+}1)e_r \neq e_r(r\;r{+}1)$, so $W^*(e) = \langle (1\;2), \cdots, (r{-}1\;r)\rangle = S_r$. We now apply the general character formula of Li, Li, and Cao. Given an irreducible character χ of $W^*(e)$, we have that

$$\chi^*(\sigma) = \sum_{K \in \mathcal{F}(e), K\sigma = K} \chi(\mu_K \sigma \mu_K^-)$$

where $\mathcal{F}(e) = w \cdot [r], w \in W$, for $[r] = \{1, \ldots, r\}$.

Because $K \in \mathcal{F}(e)$, $|K| = r$ and there exists a Weyl group element w that restricts to μ_k, so that $\mu_k \sigma \mu_k^-$ is the restriction of $w\sigma w^{-1}$ to $[r]$. Therefore, $\chi(\sigma) = (\chi \otimes \eta_{n-r})(w\sigma w^{-1})$, so that

$$\chi^*(\sigma) = \sum (\chi \boxtimes \eta_{n-r})(w\sigma w^{-1})$$

summing over all $|K| = r$ with $w_K \sigma w_K^{-1} \in S_r \times W_{n-r}$. The elements w_K are a set of coset representatives $W_n/(S_r \times W_{n-r})$, which implies

$$\chi^*(\sigma) = \sum (\chi \boxtimes \eta_{n-r})(w\sigma w^{-1}) = \mathrm{Ind}_{S_r \times W_{n-r}}^{W_n} (\chi \boxtimes \eta_{n-r})(\sigma)$$

as desired.

□

6.1 Characters of the Type A Renner Monoid

The above gives us an explicit combinatorial interpretation in the type A case. For an irreducible representation of S_k indexed by $\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_j \geq 1)$ where $\sum_{i=1}^{j} \lambda_i = k$, the induced representation of λ in R_n is

$$\chi^* = \bigoplus_{\lambda \subseteq \mu} \chi_\mu \tag{38}$$

where μ is an irreducible representation of S_n and μ differs from λ by a horizontal strip. This is the Pieri rule in the type A case, and it allows us to more easily compute the restriction of irreducible representations of the Rook monoid to S_n. Further, we use this as motivation in the type B case.

6.2 Type B Case

We'll now determine an analogue of the Pieri rule for B_n. We first highlight the two facts from Geck and Pfeiffer.

Proposition 2 (Geck and Pfeiffer Page 178, Lemma 6.1.3)**.** *Let $n \geq 1$ and $k, l \geq 0$ be integers such that $n = k + l$. Let (λ_1, λ_2) and (μ_1, μ_2) be pairs of partitions with $|\lambda_1| + |\lambda_2| = k$ and $|\mu_1| + |\mu_2| = l$. Then, using the diagonal embedding $W_k \times W_l \subseteq W_n$, we have*

$$\mathrm{Ind}_{B_k \times B_l}^{B_n} (\chi_{a_1,a_2} \boxtimes \chi_{b_1,b_2}) = \sum_{(\nu_1,\nu_2)} c_{a_1,b_1}^{\nu_1} c_{a_2,b_2}^{\nu_2} \chi_{\nu_1,\nu_2}$$

where the sum runs over all pairs of partitions (ν_1, ν_2) for which $|\nu_i| = |\lambda_i| + |\nu_i|$ for $i = 1, 2$.

Proposition 3 (Geck and Pfeiffer Page 179, Lemma 6.1.4)**.** *Let $n \geq 1$ and consider the subgroup $S_n \subset B_n$. Let $\nu \vdash n$ and $\chi_\nu \in \mathrm{Irr}(S_n)$ be the corresponding irreducible character. Then*

$$\mathrm{Ind}_{S_n}^{B_n} = \sum_{\lambda,\mu} c_{\lambda\mu}^{\nu} \chi_{(\lambda,\mu)}$$

We can now derive a more explicit formula for $\mathrm{Ind}_{S_k\times B_l}^{B_n}(\chi_\nu \boxtimes \eta_l)$ for a fixed partition $\nu \vdash k$. Using transitivity of induction, we have

$$\begin{aligned}
\mathrm{Ind}_{S_k\times B_l}^{B_n}(\chi_\nu \boxtimes \eta_l) &= \mathrm{Ind}_{B_k\times B_l}^{B_n}\mathrm{Ind}_{S_k\times B_l}^{B_k\times B_l}(\chi_\nu \boxtimes \eta_l)\\
&= \mathrm{Ind}_{B_k\times B_l}^{B_n}\left(\sum_{\lambda,\mu} c_{\lambda,\mu}^{\nu}\chi_{\lambda,\mu}\boxtimes \eta_l\right)\\
&= \sum_{\lambda,\mu} c_{\lambda,\mu}^{\nu}\,\mathrm{Ind}_{B_k\times B_l}^{B_n}(\chi_{\lambda,\mu}\boxtimes \chi_{[l],\emptyset})\\
&= \sum_{\substack{\lambda,\mu\\ \lambda+\mu\vdash k}} c_{\lambda,\mu}^{\nu} \sum_{\substack{\nu_1,\nu_2\\ |\nu_i|=|\lambda_i|+|\mu_i|\\ \nu_1+\nu_2\vdash n}} c_{\lambda,[l]}^{\nu_1}c_{\mu,\emptyset}^{\nu_2}\chi_{\nu_1,\nu_2}
\end{aligned}$$

Note that $c_{\mu,}^{\nu_2} = 0$ unless $\mu = \nu_2$ in which case it is equal to 1. This sum reduces to

$$\begin{aligned}
\mathrm{Ind}_{S_k\times B_l}^{B_n}(\chi_\nu \boxtimes \eta_l) &= \sum_{\substack{\lambda,\mu\\ \lambda+\mu\vdash k}} c_{\lambda,\mu}^{\nu} \sum_{\substack{\nu_1\\ \nu_1+\mu\vdash n}} c_{\lambda,[l]}^{\nu_1}\chi_{\nu_1,\mu}\\
&= \sum_{\substack{\gamma,\mu\\ \gamma+\mu\vdash n}}\left(\sum_{\substack{\lambda,\ \lambda+\mu\vdash k\\ \gamma-\lambda \text{ horiz. strip}\\ \text{of size } l}} c_{\lambda,\mu}^{\nu}c_{\lambda,[l]}^{\gamma}\right)\chi_{\gamma,\mu}\\
&= \sum_{\substack{\gamma,\mu\\ \gamma+\mu\vdash n}}\left(\sum_{\substack{\lambda\\ \gamma-\lambda \text{ horiz strip}\\ \text{of size } l}} c_{\lambda,\mu}^{\nu}\right)\chi_{\gamma,\mu}
\end{aligned}$$

The line comes from swapping the order of summation and noting that $c_{\lambda,[l]}^{\gamma}$ is an order type-A Pieri coefficient, which is 1 if $\gamma - \lambda$ is a horizontal strip and 0 otherwise by Geck and Pfeiffer p. 182 [3]. These explicit formulas yield a nicer way to determine the character table.

6.3 A and B matrices in type A case

Solomon showed that the character table, M, of the rook monoid can be written as

$$M = AY = YB \tag{39}$$

where Y is the block diagonal matrix, where the blocks are the character tables of S_n down to S_0. It turns out that the B matrix is simple to compute and is completely determined by the Pieri rule. We will show in the Hecke algebra case, $\mathcal{H}(R_n)$, that if M is the Hecke algebra character table and Y is the block diagonal matrix where the blocks are the character tables of $\mathcal{H}(S_n)$ down to $\mathcal{H}(S_0)$, then $M = YB$ where B is the same matrix as in the regular monoid case. The A matrix can be described using binomial coefficients. However, from the smallest cases, the A matrix is dependent upon q in the Hecke Algebra case, thus we do not compute this here.

6.4 B matrix for $\mathcal{H}(R_n)$

Let $\mathcal{H}(R_n)$ be the Hecke algebra of the Rook monoid. Let M_n be the character table of $\mathcal{H}(R_n)$ and let H_n be the character table of the Hecke algebra $\mathcal{H}(S_n)$. Let Y_n be the block diagonal matrix whose blocks are H_i for $i = 0, \ldots, n$ where H_n is the top left block and H_0 is the bottom right block.

Theorem 6.3. *The character table, M_n, can be decomposed as $M_n = Y_n B_n$ where B_n is the B matrix computed in Solomon [9].*

Proof. In [2], the authors determine the character table is

$$M_n = \begin{bmatrix} H_n & * \\ 0 & M_{n-1} \end{bmatrix} \tag{40}$$

Thus, by induction, assume $M_{n-1} = Y_{n-1} B_{n-1}$. Then it is clear that

$$B_n = \begin{bmatrix} Id & P \\ 0 & B_{n-1} \end{bmatrix} \tag{41}$$

Thus, we need to show that P is determined in exactly the same way as the upper right part of the matrix in [9]. Note that we already know what this P would be if we let $q = 1$. Further, let λ index an irreducible representation of $\mathcal{H}(R_n)$, then it is clear that the matrix P encodes the restriction of V^λ to $\mathcal{H}(S_n)$.

In particular,

$$V^\lambda \downarrow \mathcal{H}(S_n) = \bigoplus_{\mu \vdash n} \alpha_\mu W^\mu \tag{42}$$

where $\alpha_\mu \in \mathbb{N}$ and $\downarrow$ denotes the restriction of a representation to the indicated subalgebra/submonoid. We see that P encodes these α_μ, and the α_μ are independent of q. Thus, we can set $q = 1$, and our P matrix is determined by the Pieri rule for S_n. As a result, $B_n = B$. $\square$

Here we see that computing the character table of $\mathcal{H}(R_n)$ is reduced to knowing the character tables of the group case, and the character table of $q = 1$ case. We now extend this to the Hecke algebra $\mathcal{H}(RSp_{2n})$.

6.5 B Matrix in type B

Using inspiration of Solomon, we find the B matrix associated to RSp_{2n} and the B matrix of the Hecke algebra $\mathcal{H}(RSp_{2n})$. Note that the A matrix was computed in section 5. We show that the same B matrix determines the character table for $\mathcal{H}(RSp_{2n})$. However, the A matrix is not the same when we extend to the Hecke algebra.

Theorem 6.4. *The B matrix of RSp_{2n} is the same for $\mathcal{H}(RSp_{2n})$.*

Proof. Note that we have an isomorphic copy of $\mathcal{H}(R_n)$ sitting inside of $\mathcal{H}(RSp2n)$. From the generators in [4] of $\mathcal{H}(RSp_{2n})$, we see a subset is isomorphic to the generaotrs of $\mathcal{H}(R_n)$. These generators correspond to rows in the character table indexed by partitions of k for $k \in \{0, \ldots, n\}$. Thus the irreducible representations of $\mathcal{H}(RSp_{2n})$ restrict to irreducible representations of $\mathcal{H}(R_n)$, or in other words, the irreducible representations of $\mathcal{H}(R_n)$ extend to irreducible representations of $\mathcal{H}(RSp_{2n})$. In this case, we have the character table of $\mathcal{H}(RSp_{2n})$ is

$$M_{2n} = \begin{bmatrix} \mathcal{H}(B_n) & P \\ 0 & \mathcal{H}(R_n) \end{bmatrix} \tag{43}$$

Thus, we clearly see the B matrix of $\mathcal{H}(RSp_{2n})$ is

$$B = \begin{bmatrix} \mathcal{I}d & * \\ 0 & B_n \end{bmatrix} \tag{44}$$

where B_n was determined in the type A case. However, just like in the type A case, we know exactly how to compute P. In particular, the section of P in the matrix are sums of elements from $\mathcal{H}(Sp_n)$. We determined that these must be integers because if λ indexes an irreducible representation of $\mathcal{H}(RSp_{2n})$, then we can compute its restriction to $\mathcal{H}(Sp_n)$ as

$$V^\lambda \downarrow \mathcal{H}(B_n, B) = \bigoplus_{\mu+\gamma \vdash n} \alpha_{\mu,\gamma} V^{\mu,\gamma} \tag{45}$$

As in the previous section, we know that $\alpha_{\mu,\gamma}$ are all positive integers or 0. In particular, they do not depend on q. Since we know what these values are when $q = 1$, we also know these values for all q. Hence, the $*$ section of our B matrix is determined by our B_n Pieri rules. Thus, the B matrix for $\mathcal{H}(RSp_{2n})$ is the same as the B matrix for RSp_{2n}. $\square$

We now justify the form of M_{2n}:

Lemma 6. *The lower right section of the character table of $\mathcal{H}(RSp_{2n})$ is the character table of $\mathcal{H}(R_n)$*

Proof. Suppose V^λ is an irreducible representation of $\mathcal{H}(RSp_{2n})$. Further, suppose V^λ corresponds to a representation that is not 0 on $\mathcal{H}(R_n)$, i.e. according to our index, it is a column of the right half of our matrix. Then, since $\mathcal{H}(R_n) \subseteq \mathcal{H}(RSp_{2n})$, we can compute the restriction of V^λ to $\mathcal{H}(R_n)$. If μ_i index the irreducible representations of $\mathcal{H}(R_n)$ for $i = 1, \dots, k$, then

$$V^\lambda \downarrow \mathcal{H}(R_n, B) = \bigoplus_{i=1}^{k} \alpha_i W^{\mu_i} \tag{46}$$

where, $\alpha_{mu_i} \geq 0$. Let χ_λ be the character of λ when restricted to $\mathcal{H}(R_n)$, and let χ_{μ_i} be the character of μ_i in $\mathcal{H}(R_n)$. From the computation of our character table in the RSp_{2n} case, we know that when $q = 1$, $\chi_\lambda = \chi_{\mu_j}$ for some j. Thus, we know

$$\chi_\lambda(q) = \sum_{i=1}^{k} \alpha_i \chi_{\mu_i}(q) \xrightarrow{q=1} \chi_\lambda = \sum_{i=1}^{k} \alpha_i \chi_{\mu_i} = \chi_{\mu_j} \tag{47}$$

Thus, this implies that $\alpha_j = 1$ and $\alpha_i = 0$ for $i \neq j$. In particular, this tells us the lower half of the character table of $\mathcal{H}(RSp_{2n}, B)$ is just the character table of $\mathcal{H}(R_n, B)$. $\square$

We note that it turns out the A matrix is dependent upon q when we extend our monoid to the hecke algebra case.

Example 6.5.1. *The following is the character table of $\mathcal{H}(R_2)$.*

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ q & -1 & q-1 & q \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & q-1 & q \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ q & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{48}$$

What we can see here is that the A matrix (the left matrix to the right of the equals sign) is dependent upon q. Further, this is why we do not investigate the A matrix any further for $\mathcal{H}(R_n)$. Since the lower right part of the character table of $\mathcal{H}(RSp_{2n})$ contains $\mathcal{H}(R_n)$, we see the A matrix here would also depend on q. Thus, we believe the B matrix is more useful as there is no q dependence.

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