FORMULAS FOR BIRATIONAL R-MATRIX ACTION

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Abstract. We study the action of the symmetric group on an array of variables defined using the birational R-matrix. We solve the case of 1-shifts completely and propose a conjecture for the action of transpositions. We also provide a combinatorial interpretation for functions that arise in the formulas in terms of noncrossing paths on cylindric networks.

1. Introduction and Background

Lam and Pylyavskyy introduced the birational R-matrix in [LP08] to study matrix factorizations and total positivity in loop groups. It is related to the study of geometric crystals and its tropicalization is the combinatorial R-matrix of affine crystals. The birational R-matrix also has applications to discrete Painlevé dynamical systems.

Let $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$ be two sets of formal variables, where $n \geq 1$. For $1 \leq i \leq n$, let

$$\kappa_i(\mathbf{a}, \mathbf{b}) = \sum_{j=i}^{i+n-1} \prod_{k=i+1}^{j} b_k \prod_{k=j+1}^{i+n-1} a_k,$$

where the indices k are taken mod n. Then

$$\eta : (\mathbf{a}, \mathbf{b}) \mapsto (\mathbf{b}', \mathbf{a}')$$

where $\mathbf{a}' = (a'_1, \ldots, a'_n), \mathbf{b}' = (b'_1, \ldots, b'_n)$, and

$$a'_i = \frac{a_{i-1}\kappa_{i-1}(\mathbf{a}, \mathbf{b})}{\kappa_i(\mathbf{a}, \mathbf{b})}$$

$$b'_i = \frac{b_{i+1}\kappa_{i+1}(\mathbf{a}, \mathbf{b})}{\kappa_i(\mathbf{a}, \mathbf{b})}.$$

For example, for $n = 4$,

$$a'_2 = a_1 \frac{\kappa_1(\mathbf{a}, \mathbf{b})}{\kappa_2(\mathbf{a}, \mathbf{b})} = a_1 \frac{a_2a_3a_4 + b_2a_3a_4 + b_2b_3a_4 + b_2b_3b_4}{a_3a_4a_1 + b_3a_4a_1 + b_3b_4a_1 + b_3b_4b_1}.$$

Now for $1 \leq i < m$, let

$$\eta_i(\mathbf{x}_1, \ldots, \mathbf{x}_m) = (\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}, \eta(\mathbf{x}_i, \mathbf{x}_{i+1}), \mathbf{x}_{i+2}, \ldots, \mathbf{x}_m).$$

This is the birational R-matrix.

Theorem 1.1 ([LP08] Lemma 6.1, Theorem 6.3)**.** The birational R-matrix has the following properties:

- η is an involution: $\eta^2 = 1$;
- η satisfies the braid relations: for $1 \leq i < m$,

$$\eta_i\eta_{i+1}\eta_i(\mathbf{x}_1, \ldots, \mathbf{x}_m) = \eta_{i+1}\eta_i\eta_{i+1}(\mathbf{x}_1, \ldots, \mathbf{x}_m).$$

Let s_i denote the transposition that switches i and $i + 1$. Since the η_i's are involutions that satisfy the braid relations, this theorem implies that the birational R-matrix defines an action of the symmetric group by letting

$$s_i(\mathbf{x}_1, \ldots, \mathbf{x}_m) = \eta_i(\mathbf{x}_1, \ldots, \mathbf{x}_m).$$

To refer to specific variables after applying a permutation, we write $s(\mathbf{x}_1, \ldots, \mathbf{x}_m) = (s(\mathbf{x}_1), \ldots, s(\mathbf{x}_m))$ where $s(\mathbf{x}_i) = (s(x_i^{(1)}), \ldots, s(x_i^{(n)}))$. When indices are in parentheses, they are taken mod n.

Main Problem. For any $s \in S_m$, $1 \leq i \leq m$ and $1 \leq r \leq n$, we would like to write $s(x_i^{(r)})$ explicitly as a rational function of the original variables.

The definition only provides us with explicit formulas when $s = s_i$ for some i. Given $i < j$, a lemma of [LP10] writes down formulas for the action of transpositions of the form $s_{j-1}s_{j-2} \ldots s_i$ on $\mathbf{x}_j$ in

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terms of σ functions (to be defined later). We use this lemma to solve the case of $s_{j-1}s_{j-2}\dots s_i$ and $s_i s_{i+1}\dots s_{j-1}$ completely. We then introduce Ω functions and propose a conjectural identity that they satisfy, which would imply explicit formulas for permutations that are transpositions. We also record some preliminary observations beyond the permutations mentioned previously. Lastly, based on the work in [LP08] on the combinatorial interpretation of τ functions in terms of highway path families in cylindrical networks, we provide a combinatorial interpretation of the σ and $\bar{\sigma}$ functions and conjecture a combinatorial interpretation of the Ω functions.

2. Formulas

We rely heavily on σ and $\bar{\sigma}$ functions as the building blocks for our formulas. To define them, we first define τ functions. We follow Section 2.2 of [LP10].

Let n be a positive integer, k a nonnegative integer, and let $1 \le r \le n$. Then $\tau_k^{(r)}$ is defined as follows:

$$\tau_k^{(r)}(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m) = \sum_{1 \le i_i \le i_2 \le \cdots \le i_k \le n} x_{i_1}^{(r)} x_{i_2}^{(r-1)} \dots x_{i_k}^{(r-k+1)}$$

where no index appears more than $n-1$ times in the sum.

The σ and $\bar{\sigma}$ functions are defined using τ. We can think of them as the τ functions with the caveat that $\mathbf{x}_1$ or $\mathbf{x}_m$ variables are now allowed to appear more than $n-1$ times.

$$\sigma_k^{(r)}(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m) = \sum_{i=0}^{k} x_1^{(r)} x_1^{(r-1)} \dots x_1^{(r-i+1)} \tau_{k-i}^{(r-i)}(\mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_m),$$

$$\bar{\sigma}_k^{(r)}(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m) = \sum_{i=0}^{k} \tau_{k-i}^{(r)}(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{m-1}) x_m^{(r-k+i)} x_m^{(r-k+i-1)} \dots x_m^{(r-k)}.$$

Example 2.1. Let $n = 4$. Write

$$\mathbf{a} = (a_1, \dots, a_4), \mathbf{b} = (b_1, \dots, b_4), \mathbf{c} = (c_1, \dots, c_4)$$

in place of $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$. Then

$$\begin{aligned}
\tau_5^{(3)}(\mathbf{b}, \mathbf{c}) &= b_3 b_2 b_1 c_4 c_3 + b_3 b_2 c_1 c_4 c_3, \\
\sigma_6^{(4)}(\mathbf{a}, \mathbf{b}, \mathbf{c}) &= \sum_{i=0}^{6} a_4 \dots a_{4-i+1} \tau_{6-i}^{(4-i)}(\mathbf{b}, \mathbf{c}) \\
&= \tau_6^{(4)}(\mathbf{b}, \mathbf{c}) + a_4 \tau_5^{(3)}(\mathbf{b}, \mathbf{c}) + a_4 a_3 \tau_4^{(2)}(\mathbf{b}, \mathbf{c}) + a_4 a_3 a_2 \tau_3^{(1)}(\mathbf{b}, \mathbf{c}) \\
&\quad + a_4 a_3 a_2 a_1 \tau_2^{(4)}(\mathbf{b}, \mathbf{c}) + a_4 a_3 a_2 a_1 a_4 \tau_1^{(3)}(\mathbf{b}, \mathbf{c}) + a_4 a_3 a_2 a_1 a_4 a_3, \\
\bar{\sigma}_6^{(4)}(\mathbf{a}, \mathbf{b}, \mathbf{c}) &= \sum_{i=0}^{6} \tau_{6-i}^{(4)}(\mathbf{a}, \mathbf{b}) c_{i-2} c_{i-3} \dots c_3 \\
&= \tau_6^{(4)}(\mathbf{a}, \mathbf{b}) + \tau_5^{(4)}(\mathbf{a}, \mathbf{b}) c_3 + \tau_4^{(4)}(\mathbf{a}, \mathbf{b}) c_4 c_3 + \tau_3^{(4)}(\mathbf{a}, \mathbf{b}) c_1 c_4 c_3 \\
&\quad + \tau_2^{(4)}(\mathbf{a}, \mathbf{b}) c_2 c_1 c_4 c_3 + \tau_1^{(4)}(\mathbf{a}, \mathbf{b}) c_3 c_2 c_1 c_4 c_3 + c_4 c_3 c_2 c_1 c_4 c_3,
\end{aligned}$$

We state a fundamental identity of the σ and $\bar{\sigma}$ functions.

Lemma 2.2.

$$\sigma_{(n-1)(j-i)}^{(r)}(\mathbf{x}_i, \dots, \mathbf{x}_j) = \sum_{k=0}^{n-1} \left(\prod_{t=0}^{k-1} x_i^{(r-t)} \right) \sigma_{(n-1)(j-i-1)}^{(r-k)}(\mathbf{x}_i, \dots, \mathbf{x}_{j-1}) \left(\prod_{s=0}^{n-k-2} x_j^{(r-k+j-i-1-s)} \right),$$

$$\bar{\sigma}_{(n-1)(j-i)}^{(r)}(\mathbf{x}_i, \dots, \mathbf{x}_j) = \sum_{k=0}^{n-1} \left(\prod_{t=0}^{k-1} x_i^{(r-t)} \right) \bar{\sigma}_{(n-1)(j-i-1)}^{(r-k)}(\mathbf{x}_{i+1}, \dots, \mathbf{x}_j) \left(\prod_{s=0}^{n-k-2} x_j^{(r-k+j-i-1-s)} \right)$$

Proof. We sketch the proof of the first identity. The second identity is exactly dual.

We can group the terms of $\sigma^{(r)}_{(n-1)(j-i)}(\mathbf{x}_i,\ldots,\mathbf{x}_j)$ by the number of times $\mathbf{x}_j$ variables are used at the end. By definition of the σ functions, $\mathbf{x}_j$ can appear at most $n-1$ times.

$$\begin{aligned}\sigma^{(r)}_{(n-1)(j-i)}(\mathbf{x}_i,\ldots,\mathbf{x}_j) &= \sum_{k=0}^{n-1}\sigma^{(r-k)}_{(n-1)(j-i)-k}(\mathbf{x}_i,\ldots,\mathbf{x}_{j-1})\left(\prod_{s=0}^{n-k-2}x_j^{(r-k+j-i-1-s)}\right)\\ &= \sum_{k=0}^{n-1}\left(\prod_{t=0}^{k-1}x_i^{(r-t)}\right)\sigma^{(r-k)}_{(n-1)(j-i-1)}(\mathbf{x}_i,\ldots,\mathbf{x}_{j-1})\left(\prod_{s=0}^{n-k-2}x_j^{(r-k+j-i-1-s)}\right),\end{aligned}$$

Since all terms of $\sigma^{(r-k)}_{(n-1)(j-i)-k}(\mathbf{x}_i,\ldots,\mathbf{x}_{j-1})$ must use $\mathbf{x}_i$ at least $n-1-k$ times, the second equality holds by a change of summation index from k to $n-1-k$. ☐

2.1. **How 1-Shifts Act.** In this section, we state explicit formulas for the action of a permutation of the form $s_is_{i+1}\ldots s_{j-1}$ and $s_{j-1}s_{j-2}\ldots s_i$, where $1\le i<j\le m$. Such permutations are shifts by ±1, and we call them 1-shifts.

Theorem 2.3 ([LP10] Lemma 3.1)**.** Let $1\le i<j\le m$. Then

$$\kappa_r(s_{j-2}s_{j-3}\cdots s_i(\mathbf{x}_{j-1}),\mathbf{x}_j)=\frac{\sigma^{(r-j+i)}_{(n-1)(j-i)}(\mathbf{x}_i,\ldots,\mathbf{x}_j)}{\sigma^{(r-j+i)}_{(n-1)(j-i-1)}(\mathbf{x}_i,\ldots,\mathbf{x}_{j-1})}$$

and

$$s_{j-1}\ldots s_i(x_j^{(r)})=\frac{x_i^{(r-j+i)}\sigma^{(r-j+i-1)}_{(n-1)(j-i)}(\mathbf{x}_i,\ldots,\mathbf{x}_j)}{\sigma^{(r-j+i)}_{(n-1)(j-i)}(\mathbf{x}_i,\ldots,\mathbf{x}_j)}.$$

The following lemma is the dual of Theorem 2.3 and the proof exactly emulates the one in [LP10].

Theorem 2.4 (Dual of Theorem 2.3)**.** Let $1\le i<j\le m$. Then

$$\kappa_r(\mathbf{x}_i,s_{i+1}\ldots s_{j-1}(\mathbf{x}_{i+1}))=\frac{\bar\sigma^{(r-1)}_{(n-1)(j-i)}(\mathbf{x}_i,\ldots,\mathbf{x}_j)}{\bar\sigma^{(r)}_{(n-1)(j-i-1)}(\mathbf{x}_{i+1},\ldots,\mathbf{x}_j)}$$

and

$$s_i\ldots s_{j-1}(x_i^{(r)})=\frac{x_j^{(r+j-i)}\bar\sigma^{(r)}_{(n-1)(j-i)}(\mathbf{x}_i,\ldots,\mathbf{x}_j)}{\bar\sigma^{(r-1)}_{(n-1)(j-i)}(\mathbf{x}_i,\ldots,\mathbf{x}_j)}$$

Proof. We prove the two statements in parallel by induction on $j-i$. For $j-i=1$ they coincide with the formulae for the κ_r and the R-action of s_i. By the induction assumption,

$$s_{i+1}\ldots s_{j-1}(x_{i+1}^{(r)})=\frac{x_j^{(r+j-i-1)}\bar\sigma^{(r)}_{(n-1)(j-i-1)}(\mathbf{x}_{i+1},\ldots,\mathbf{x}_j)}{\bar\sigma^{(r-1)}_{(n-1)(j-i-1)}(\mathbf{x}_{i+1},\ldots,\mathbf{x}_j)}.$$

Therefore

$$\begin{aligned}\kappa_r(\mathbf{x}_i,s_{i+1}\ldots s_{j-1}(\mathbf{x}_{i+1})) &= \sum_{s=0}^{n-1}\left[\prod_{t=1}^{s}s_{i+1}\ldots s_{j-1}(x_{i+1}^{(r+t)})\right]x_i^{(r+s+1)}\cdots x_i^{(r+n-1)}\\ &= \sum_{s=0}^{n-1}\left[\prod_{t=1}^{s}\frac{x_j^{(r+t+j-i-1)}\bar\sigma^{(r+t)}_{(n-1)(j-i-1)}(\mathbf{x}_{i+1},\ldots,\mathbf{x}_j)}{\bar\sigma^{(r+t-1)}_{(n-1)(j-i-1)}(\mathbf{x}_{i+1},\ldots,\mathbf{x}_j)}\right]x_i^{(r+s+1)}\cdots x_i^{(r+n-1)}\\ &= \sum_{s=0}^{n-1}\left[\prod_{t=0}^{n-s-2}x_i^{(r+n-1-t)}\right]\frac{\bar\sigma^{(r+s)}_{(n-1)(j-i-1)}(\mathbf{x}_{i+1},\ldots,\mathbf{x}_j)}{\bar\sigma^{(r)}_{(n-1)(j-i-1)}(\mathbf{x}_{i+1},\ldots,\mathbf{x}_j)}\left[\prod_{t=0}^{s-1}x_j^{(r+s+j-i-1-t)}\right]\\ &= \frac{\bar\sigma^{(r-1)}_{(n-1)(j-i)}(\mathbf{x}_i,\ldots,\mathbf{x}_j)}{\bar\sigma^{(r)}_{(n-1)(j-i-1)}(\mathbf{x}_{i+1},\ldots,\mathbf{x}_j)}.\end{aligned}$$

The last equality holds by Lemma 2.2. Now we can also prove the second claim, since

$$\begin{aligned}
&s_i \dots s_{j-1}(x_i^{(r)}) \\
&= \frac{s_{i+1}\dots s_{j-1}(x_{i+1}^{(r+1)})\kappa_{r+1}(\mathbf{x}_i, s_{i+1}\dots s_{j-1}(\mathbf{x}_{i+1}))}{\kappa_r(\mathbf{x}_i, s_{i+1}\dots s_{j-1}(\mathbf{x}_{i+1}))} \\
&= \frac{x_j^{(r+j-i)}\bar{\sigma}^{(r+1)}_{(n-1)(j-i-1)}(\mathbf{x}_{i+1},\dots,\mathbf{x}_j)}{\bar{\sigma}^{(r)}_{(n-1)(j-i-1)}(\mathbf{x}_{i+1},\dots,\mathbf{x}_j)} \frac{\bar{\sigma}^{(r)}_{(n-1)(j-i)}(\mathbf{x}_i,\dots,\mathbf{x}_j)}{\bar{\sigma}^{(r+1)}_{(n-1)(j-i-1)}(\mathbf{x}_{i+1},\dots,\mathbf{x}_j)} \frac{\bar{\sigma}^{(r)}_{(n-1)(j-i-1)}(\mathbf{x}_{i+1},\dots,\mathbf{x}_j)}{\bar{\sigma}^{(r-1)}_{(n-1)(j-i)}(\mathbf{x}_i,\dots,\mathbf{x}_j)} \\
&= x_j^{(r+j-i)} \frac{\bar{\sigma}^{(r)}_{(n-1)(j-i)}(\mathbf{x}_i,\dots,\mathbf{x}_j)}{\bar{\sigma}^{(r-1)}_{(n-1)(j-i)}(\mathbf{x}_i,\dots,\mathbf{x}_j)}.
\end{aligned}$$

□

In this theorem, we write down what happens to all variables in columns $i,\dots,j-1$ after the application of a permutation of the form $s_{j-1}\dots s_i$, as well as what happens to all variables in columns $i+1,\dots,j$ after the application of a permutation of the form $s_i\dots s_{j-1}$.

Theorem 2.5. *Let $1 \le i < j \le m$. Then for $i \le k < j$,*

$$s_{j-1}\dots s_i(x_k^{(r)}) = \frac{x_{k+1}^{(r+1)}\sigma^{(r-k+i)}_{(n-1)(k+1-i)}(\mathbf{x}_i,\dots,\mathbf{x}_{k+1})\sigma^{(r-k+i-1)}_{(n-1)(k-i)}(\mathbf{x}_i,\dots,\mathbf{x}_k)}{\sigma^{(r-k+i-1)}_{(n-1)(k+1-i)}(\mathbf{x}_i,\dots,\mathbf{x}_{k+1})\sigma^{(r-k+i)}_{(n-1)(k-i)}(\mathbf{x}_i,\dots,\mathbf{x}_k)}.$$

Similarly, for $i < k \le j$,

$$s_i\dots s_{j-1}(x_k^{(r)}) = \frac{x_{k-1}^{(r-1)}\bar{\sigma}^{(r-2)}_{(n-1)(j-k+1)}(\mathbf{x}_{k-1},\dots,\mathbf{x}_j)\bar{\sigma}^{(r)}_{(n-1)(j-k)}(\mathbf{x}_k,\dots,\mathbf{x}_j)}{\bar{\sigma}^{(r-1)}_{(n-1)(j-k+1)}(\mathbf{x}_{k-1},\dots,\mathbf{x}_j)\bar{\sigma}^{(r-1)}_{(n-1)(j-k)}(\mathbf{x}_k,\dots,\mathbf{x}_j)}.$$

Proof. We prove the first part of the lemma. The second part is exactly dual.

Let $s = s_{k-1}s_{k-2}\cdots s_i$. By Theorem 2.3,

$$\kappa_r(s(\mathbf{x}_k), \mathbf{x}_{k+1}) = \frac{\sigma^{(r-k+i-1)}_{(n-1)(k+1-i)}(\mathbf{x}_i, \mathbf{x}_{i+1},\dots,\mathbf{x}_{k+1})}{\sigma^{(r-k+i-1)}_{(n-1)(k-i)}(\mathbf{x}_i, \mathbf{x}_{i+1},\dots,\mathbf{x}_k)}.$$

So

$$\begin{aligned}
s_{j-1}s_{j-2}\dots s_i(x_k^{(r)}) &= s_k s_{k-1}\cdots s_i(x_k^{(r)}) \\
&= s_k(s(x_k^{(r)})) \\
&= x_{k+1}^{(r+1)}\frac{\kappa_{r+1}(s(\mathbf{x}_k),\mathbf{x}_{k+1})}{\kappa_r(s(\mathbf{x}_k),\mathbf{x}_{k+1})} \\
&= x_{k+1}^{(r+1)}\frac{\sigma^{(r-k+i)}_{(n-1)(k+1-i)}(\mathbf{x}_i,\mathbf{x}_{i+1},\dots,\mathbf{x}_{k+1})\sigma^{(r-k+i-1)}_{(n-1)(k-i)}(\mathbf{x}_i,\mathbf{x}_{i+1},\dots,\mathbf{x}_k)}{\sigma^{(r-k+i-1)}_{(n-1)(k+1-i)}(\mathbf{x}_i,\mathbf{x}_{i+1},\dots,\mathbf{x}_{k+1})\sigma^{(r-k+i)}_{(n-1)(k-i)}(\mathbf{x}_i,\mathbf{x}_{i+1},\dots,\mathbf{x}_k)}.
\end{aligned}$$

as desired. □

2.2. **How Transpositions Act.** Let $1 \le i < j \le m$. In this section, we state a conjecture for how permutations of the form $s_i s_{i+1}\dots s_{j-2}s_{j-1}s_{j-2}\dots s_{i+1}s_i$ act. Note that this is exactly the transposition that switches i and j. Since

$$s_i s_{i+1}\dots s_{j-2}s_{j-1}s_{j-2}\dots s_{i+1}s_i(x_j^{(r)}) = s_{j-1}\dots s_i(x_j^{(r)}),$$

and

$$s_i s_{i+1}\dots s_{j-2}s_{j-1}s_{j-2}\dots s_{i+1}s_i(x_i^{(r)}) = s_i\dots s_{j-1}(x_i^{(r)}),$$

we know how a transposition that switches i and j acts on $\mathbf{x}_i$ and $\mathbf{x}_j$ by the discussion from the previous section. To investigate the action on intermediate variables, we introduce the Ω functions and state a conjectural identity of the Ω functions that would imply our proposed formula for how a transposition acts on intermediate variables. We prove this identity when $n = 2$.

Definition 2.6. For $i \le k \le j-1$, define

$${}^{(k)}\Omega^{(r)}_{(n-1)(j-i)}(\mathbf{x}_i,\dots,\mathbf{x}_j) = \sum_{\ell=0}^{n-1}\sigma^{(r)}_{(n-1)(k-i)+\ell}(\mathbf{x}_i,\dots,\mathbf{x}_k)\bar{\sigma}^{(r+k-i-\ell)}_{(n-1)(j-k)-\ell}(\mathbf{x}_{k+1},\dots,\mathbf{x}_j).$$

By the same argument as Lemma 2.2, when $k = i$ and when $k = j-1$, Ω specializes to $\bar{\sigma}$ and σ respectively.

Example 2.7. When $n = 2$,

$$\begin{aligned}
{}^{(k)}\Omega_{j-i} &= x_i\sigma_{k-i}(\mathbf{x}_i,\ldots,\mathbf{x}_k)x_{k+1}\overline{\sigma}_{j-k-2}(\mathbf{x}_{k+2},\ldots,\mathbf{x}_j)+\\
&\quad x_i\sigma_{k-i}(\mathbf{x}_i,\ldots,\mathbf{x}_k)\overline{\sigma}_{j-k-2}(\mathbf{x}_{k+2},\ldots,\mathbf{x}_j)x_j+\\
&\quad \sigma_{k-i}(\mathbf{x}_i,\ldots,\mathbf{x}_k)x_{k+1}\overline{\sigma}_{j-k-2}(\mathbf{x}_{k+2},\ldots,\mathbf{x}_j)x_j+\\
&\quad \sigma_{k-i}(\mathbf{x}_i,\ldots,\mathbf{x}_k)\overline{\sigma}_{j-k-2}(\mathbf{x}_{k+2},\ldots,\mathbf{x}_j)x_jx_j\\
&= \sigma_{k-i+1}(\mathbf{x}_1,\ldots,\mathbf{x}_k)\bar{\sigma}_{j-k-1}(\mathbf{x}_{k+1},\ldots,\mathbf{x}_j)+\\
&\quad \sigma_{k-i}(\mathbf{x}_1,\ldots,\mathbf{x}_k)\bar{\sigma}_{j-k}(\mathbf{x}_{k+1},\ldots,\mathbf{x}_j),
\end{aligned}$$

where the superscripts are all falling appropriately.

Conjecture 2.8. For $i < k \le j-1$, the following identity of ${}^{(k-1)}\Omega$ and ${}^{(k)}\Omega$ holds:

$$\begin{aligned}
&\left[\prod_{t=1}^{n-1}\sigma^{(r-k+i+t)}_{(n-1)(k-i)}(\mathbf{x}_i,\ldots,\mathbf{x}_k)\right]\ {}^{(k-1)}\Omega^{(r-k+i)}_{(n-1)(j-i)}(x_i,\ldots,x_j)\\
&= \sum_{s=0}^{n-1}\prod_{t=r+1}^{r+s}x_j^{(t+j-k)}\prod_{t=r+s+1}^{r+n-1}x_k^{(t+1)}\left[\prod_{\substack{0\le t\le n-1\\ t\neq s,s+1}}\sigma^{(r-k+i+t)}_{(n-1)(k-i)}(\mathbf{x}_i,\ldots,\mathbf{x}_k)\right]\\
&\qquad {}^{(k)}\Omega^{(r-k+i+s)}_{(n-1)(j-i)}(x_i,\ldots,x_j)\sigma^{(r-k+i+s+1)}_{(n-1)(k-i-1)}(\mathbf{x}_i,\ldots,\mathbf{x}_{k-1}).
\end{aligned}$$

For example, consider the case where $j = 4, i = 1, r = n = 4$, and $k = 3$. Denote $x_1^{(k)},\ldots,x_4^{(k)}$ by $a_k,\ldots,d_k$ respectively. Then the identity above is

$$\begin{aligned}
\sigma_6^{(3)}(a,b,c)\sigma_6^{(4)}(a,b,c)\sigma_6^{(1)}(a,b,c){}^{(2)}\Omega_9^{(2)}(a,b,c,d) &= c_2c_3c_4\sigma_9^{(2)}(a,b,c,d)\sigma_3^{(3)}(a,b)\sigma_6^{(4)}(a,b,c)\sigma_6^{(1)}(a,b,c)\\
&+ d_2c_3c_4\sigma_6^{(2)}(a,b,c)\sigma_9^{(3)}(a,b,c,d)\sigma_3^{(4)}(a,b)\sigma_6^{(1)}(a,b,c)\\
&+ d_2d_3c_4\sigma_6^{(2)}(a,b,c)\sigma_6^{(3)}(a,b,c)\sigma_9^{(4)}(a,b,c,d)\sigma_3^{(1)}(a,b)\\
&+ d_2d_3d_4\sigma_3^{(2)}(a,b)\sigma_6^{(3)}(a,b,c)\sigma_6^{(4)}(a,b,c)\sigma_9^{(1)}(a,b,c,d).
\end{aligned}$$

This identity is true in the case of $n = 2$.

Theorem 2.9. When $n = 2$, for $i < k \le j-1$, Conjecture 2.8 holds. Equivalently, we have the following identity of ${}^{(k-1)}\Omega$ and ${}^{(k)}\Omega$:

$$\begin{aligned}
\sigma^{(r-k+i+1)}_{k-i}(\mathbf{x}_i,\ldots,\mathbf{x}_k)\ {}^{(k-1)}\Omega^{(r-k+i)}_{j-i}(\mathbf{x}_i,\ldots,\mathbf{x}_j) &= x_k^{(r)}\ {}^{(k)}\Omega^{(r-k+i)}_{j-i}(\mathbf{x}_i,\ldots,\mathbf{x}_j)\sigma^{(r-k+i+1)}_{k-i-1}(\mathbf{x}_i,\ldots,\mathbf{x}_{k-1})\\
&+ x_j^{(r+1+j-k)}\ {}^{(k)}\Omega^{(r-k+i+1)}_{j-i}(\mathbf{x}_i,\ldots,\mathbf{x}_j)\sigma^{(r-k+i)}_{k-i-1}(\mathbf{x}_i,\ldots,\mathbf{x}_{k-1}).
\end{aligned}$$

Proof. It suffices to match up the terms on the left and right hand sides that contain the same number of $x_j's$. When $n = 2$,

$$\begin{aligned}
&{}^{(k)}\Omega^{(r)}_{j-i}(\mathbf{x}_i,\ldots,\mathbf{x}_j)\\
&= \sigma^{(r)}_{k-i}(\mathbf{x}_i,\ldots,\mathbf{x}_k)\bar{\sigma}^{(r-k+i)}_{j-k}(\mathbf{x}_{k+1},\ldots,\mathbf{x}_j)+\sigma^{(r)}_{k-i+1}(\mathbf{x}_i,\ldots,\mathbf{x}_k)\bar{\sigma}^{(r-k+i-1)}_{j-k-1}(\mathbf{x}_{k+1},\ldots,\mathbf{x}_j),
\end{aligned}$$

So

$$\begin{aligned}
&{}^{(k)}\Omega^{(r-k+i)}_{j-i}(\mathbf{x}_i,\ldots,\mathbf{x}_j)\\
&= \sigma^{(r-k+i)}_{k-i}(\mathbf{x}_i,\ldots,\mathbf{x}_k)\bar{\sigma}^{(r)}_{j-k}(\mathbf{x}_{k+1},\ldots,\mathbf{x}_j)+\sigma^{(r-k+i)}_{k-i+1}(\mathbf{x}_i,\ldots,\mathbf{x}_k)\bar{\sigma}^{(r-1)}_{j-k-1}(\mathbf{x}_{k+1},\ldots,\mathbf{x}_j),
\end{aligned}$$

and

$$\begin{aligned}
&{}^{(k)}\Omega^{(r-k+i+1)}_{j-i}(\mathbf{x}_i,\ldots,\mathbf{x}_j)\\
&= \sigma^{(r-k+i+1)}_{k-i}(\mathbf{x}_i,\ldots,\mathbf{x}_k)\bar{\sigma}^{(r+1)}_{j-k}(\mathbf{x}_{k+1},\ldots,\mathbf{x}_j)+\sigma^{(r-k+i+1)}_{k-i+1}(\mathbf{x}_i,\ldots,\mathbf{x}_k)\bar{\sigma}^{(r)}_{j-k-1}(\mathbf{x}_{k+1},\ldots,\mathbf{x}_j).
\end{aligned}$$

One can check that the product of q $x_j's$ in all terms that contain exactly q x_j's have the same indices. Therefore, we can focus the part of the left and right hand side that contains q x_j's in each term after dividing by the x_j's their terms contain. For the left hand side, the result is

$$\sigma_{k-i}^{(r-k+i+1)}(\mathbf{x}_i,\dots,\mathbf{x}_k)\sigma_{k-i-1}^{(r-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_{k-1})\tau_{j-k-q+1}^{(r-1)}(\mathbf{x}_k,\dots,\mathbf{x}_{j-1}) + \sigma_{k-i}^{(r-k+i+1)}(\mathbf{x}_i,\dots,\mathbf{x}_k)\sigma_{k-i}^{(r-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_{k-1})\tau_{j-k-q}^{(r)}(\mathbf{x}_k,\dots,\mathbf{x}_{j-1}).$$

For the right hand side, the result is

$$\begin{aligned}(1)\quad & x_k^{(r)}\sigma_{k-i-1}^{(r-k+i+1)}(\mathbf{x}_i,\dots,\mathbf{x}_{k-1})(\sigma_{k-i}^{(r-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_k)\tau_{j-k-q}^{(r)}(\mathbf{x}_{k+1},\dots,\mathbf{x}_{j-1}) \\ &\qquad + \sigma_{k-i+1}^{(r-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_k)\tau_{j-k-1-q}^{(r-1)}(\mathbf{x}_{k+1},\dots,\mathbf{x}_{j-1})) \\ &+ \sigma_{k-i-1}^{(r-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_{k-1})(\sigma_{k-i}^{(r-k+i+1)}(\mathbf{x}_i,\dots,\mathbf{x}_k)\tau_{j-k-q+1}^{(r+1)}(\mathbf{x}_{k+1},\dots,\mathbf{x}_{j-1}) \\ &\qquad + \sigma_{k-i+1}^{(r-k+i+1)}(\mathbf{x}_i,\dots,\mathbf{x}_k)\tau_{j-k-q}^{(r)}(\mathbf{x}_{k+1},\dots,\mathbf{x}_{j-1})).\end{aligned}$$

We can combine the two terms that contain $\tau_{j-k-q}^{(r)}(\mathbf{x}_{k+1},\dots,\mathbf{x}_j)$ as follows:

$$\begin{aligned}& x_k^{(r)}\sigma_{k-i-1}^{(r-k+i+1)}(\mathbf{x}_i,\dots,\mathbf{x}_{k-1})\sigma_{k-i}^{(r-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_k) + \sigma_{k-i-1}^{(r-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_{k-1})\sigma_{k-i+1}^{(r-k+i+1)}(\mathbf{x}_i,\dots,\mathbf{x}_k) \\ =\; & x_k^{(r)}\sigma_{k-i-1}^{(r-k+i+1)}(\mathbf{x}_i,\dots,\mathbf{x}_{k-1})\sigma_{k-i}^{(r-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_k) + \sigma_{k-i-1}^{(r-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_{k-1})x_i^{(r-k+i+1)}\sigma_{k-i}^{(r-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_k) \\ =\; & \sigma_{k-i}^{(r-k+i+1)}(\mathbf{x}_i,\dots,\mathbf{x}_k)\sigma_{k-i}^{(r-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_k).\end{aligned}$$

So (1) can be simplified as follows:

$$\begin{aligned}& \sigma_{k-i}^{(r-k+i+1)}(\mathbf{x}_i,\dots,\mathbf{x}_k)\sigma_{k-i}^{(r-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_k)\tau_{j-k-q}^{(r)}(\mathbf{x}_{k+1},\dots,\mathbf{x}_{j-1}) \\ &+ \sigma_{k-i-1}^{(r-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_{k-1})\sigma_{k-i}^{(r-k+i+1)}(\mathbf{x}_i,\dots,\mathbf{x}_k)\tau_{j-k-q+1}^{(r+1)}(\mathbf{x}_{k+1},\dots,\mathbf{x}_{j-1}) \\ &+ x_k^{(r)}\sigma_{k-i-1}^{(r-k+i+1)}(\mathbf{x}_i,\dots,\mathbf{x}_{k-1})\sigma_{k-i+1}^{(r-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_k)\tau_{j-k-1-q}^{(r-1)}(\mathbf{x}_{k+1},\dots,\mathbf{x}_{j-1}) \\ =\; & \sigma_{k-i}^{(r-k+i+1)}(\mathbf{x}_i,\dots,\mathbf{x}_k)(\sigma_{k-i}^{(r-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_k)\tau_{j-k-q}^{(r)}(\mathbf{x}_{k+1},\dots,\mathbf{x}_{j-1}) \\ &+ \sigma_{k-i-1}^{(r-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_{k-1})\tau_{j-k-q+1}^{(r+1)}(\mathbf{x}_{k+1},\dots,\mathbf{x}_{j-1}) \\ &+ x_k^{(r)}\sigma_{k-i-1}^{(r-k+i+1)}(\mathbf{x}_i,\dots,\mathbf{x}_{k-1})x_i^{(r-k+i)}\tau_{j-k-1-q}^{(r-1)}(\mathbf{x}_{k+1},\dots,\mathbf{x}_{j-1})) \\ =\; & \sigma_{k-i}^{(r-k+i+1)}(\mathbf{x}_i,\dots,\mathbf{x}_k)(\sigma_{k-i}^{(r-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_{k-1})\tau_{j-k-q}^{(r)}(\mathbf{x}_{k+1},\dots,\mathbf{x}_{j-1}) \\ &+ \sigma_{k-i-1}^{(r-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_{k-1})x_k^{(r-1)}\tau_{j-k-q}^{(r)}(\mathbf{x}_{k+1},\dots,\mathbf{x}_{j-1}) \\ &+ \sigma_{k-i-1}^{(r-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_{k-1})\tau_{j-k-q+1}^{(r+1)}(\mathbf{x}_{k+1},\dots,\mathbf{x}_{j-1}) \\ &+ \sigma_{k-i}^{(r-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_{k-1})x_k^{(r)}\tau_{j-k-1-q}^{(r-1)}(\mathbf{x}_{k+1},\dots,\mathbf{x}_{j-1})) \\ =\; & \sigma_{k-i}^{(r-k+i+1)}(\mathbf{x}_i,\dots,\mathbf{x}_k)(\sigma_{k-i-1}^{(r-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_{k-1})(x_k^{(r-1)}\tau_{j-k-q}^{(r)}(\mathbf{x}_{k+1},\dots,\mathbf{x}_{j-1}) + \tau_{j-k-q+1}^{(r+1)}(\mathbf{x}_{k+1},\dots,\mathbf{x}_{j-1})) \\ &+ \sigma_{k-i}^{(r-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_{k-1})(\tau_{j-k-q}^{(r)}(\mathbf{x}_{k+1},\dots,\mathbf{x}_{j-1}) + x_k^{(r)}\tau_{j-k-1-q}^{(r-1)}(\mathbf{x}_{k+1},\dots,\mathbf{x}_{j-1}))) \\ =\; & \sigma_{k-i}^{(r-k+i+1)}(\mathbf{x}_i,\dots,\mathbf{x}_k)\sigma_{k-i-1}^{(r-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_{k-1})\tau_{j-k-q+1}^{(r-1)}(\mathbf{x}_k,\dots,\mathbf{x}_{j-1}) \\ &+ \sigma_{k-i}^{(r-k+i+1)}(\mathbf{x}_i,\dots,\mathbf{x}_k)\sigma_{k-i}^{(r-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_{k-1})\tau_{j-k-q}^{(r)}(\mathbf{x}_k,\dots,\mathbf{x}_{j-1}).\end{aligned}$$

This is precisely the terms of the left hand side that contains q x_j's after we divide by all the x_j's. $\square$

If we can prove Conjecture 2.8, then we have the following explicit formulas for the action of transpositions.

Theorem 2.10. Given $i<k<j$, let $s = s_k \dots s_{j-2}s_{j-1}s_{j-2}\dots s_i$. If Conjecture 2.8 holds, then

$$s(x_k^{(r)}) = x_j^{(r+j-k)}\frac{\sigma_{(n-1)(k-i)}^{(r-k+i-1)}(\mathbf{x}_i,\dots,\mathbf{x}_k)\ {}^{(k)}\Omega_{(n-1)(j-i)}^{(r-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_j)}{\sigma_{(n-1)(k-i)}^{(r-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_k)\ {}^{(k)}\Omega_{(n-1)(j-i)}^{(r-k+i-1)}(\mathbf{x}_i,\dots,\mathbf{x}_j)}$$

and

$$\kappa_r(s(x_{k-1}),s(x_k)) = \frac{\sigma_{(n-1)(k-i)}^{(r-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_k)\ {}^{(k-1)}\Omega_{(n-1)(j-i)}^{(r-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_j)}{\sigma_{(n-1)(k-i-1)}^{(r-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_{k-1})\ {}^{(k)}\Omega_{(n-1)(j-i)}^{(r-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_j)}.$$

Proof. We proceed by induction on k. When $k=j-1$, by Theorem 2.5, indeed

$$s(x_{j-1}^{(r)}) = s_{j-1}s_{j-2}\dots s_i(x_{j-1}^{(r)}) = x_j^{(r+1)}\frac{\sigma_{(n-1)(j-i-1)}^{(r-j+i)}(x_i,\dots,x_{j-1})\ \sigma_{(n-1)(j-i)}^{(r-j+i+1)}(\mathbf{x}_i,\dots,\mathbf{x}_j)}{\sigma_{(n-1)(j-i-1)}^{(r-j+i+1)}(x_i,\dots,x_{j-1})\ \sigma_{(n-1)(j-i)}^{(r-j+i)}(\mathbf{x}_i,\dots,\mathbf{x}_j)}.$$

Now let $s = s_k \dots s_{j-2}s_{j-1}s_{j-2}\dots s_i$ and suppose that

$$s(x_k^{(r)}) = x_j^{(r+j-k)} \frac{\sigma_{(n-1)(k-i)}^{(r-k+i-1)}(\mathbf{x}_i,\dots,\mathbf{x}_k)\ {}^{(k)}\Omega_{(n-1)(j-i)}^{(r-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_j)}{\sigma_{(n-1)(k-i)}^{(r-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_k)\ {}^{(k)}\Omega_{(n-1)(j-i)}^{(r-k+i-1)}(\mathbf{x}_i,\dots,\mathbf{x}_j)}.$$

By Theorem 2.5,

$$s(x_{k-1}^{(r)}) = s_{j-1}s_{j-2}\dots s_i(x_{k-1}^{(r)}) = \frac{x_{k+1}^{(r+1)}\sigma_{(n-1)(k+1-i)}^{(r-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_{k+1})\sigma_{(n-1)(k-i)}^{(r-k+i-1)}(\mathbf{x}_i,\dots,\mathbf{x}_k)}{\sigma_{(n-1)(k+1-i)}^{(r-k+i-1)}(\mathbf{x}_i,\dots,\mathbf{x}_{k+1})\sigma_{(n-1)(k-i)}^{(r-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_k)}.$$

By definition,

$$\begin{aligned}
&\kappa_r(s(x_{k-1}),s_k)\\
&= \sum_{s=0}^{n-1}\prod_{t=r+1}^{r+s} s(x_k^{(t)}) \prod_{t=r+s+1}^{r+n-1} s(x_{k-1}^{(t)})\\
&= \sum_{s=0}^{n-1}\prod_{t=r+s+1}^{r+n-1} x_j^{(r+j-k)} \frac{\sigma_{(n-1)(k-i)}^{(t-k+i-1)}(\mathbf{x}_i,\dots,\mathbf{x}_k)\ {}^{(k)}\Omega_{(n-1)(j-i)}^{(t-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_j)}{\sigma_{(n-1)(k-i)}^{(t-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_k)\ {}^{(k)}\Omega_{(n-1)(j-i)}^{(t-k+i-1)}(\mathbf{x}_i,\dots,\mathbf{x}_j)}\\
&\prod_{t=r+1}^{r+s} \frac{x_{k+1}^{(t+1)}\sigma_{(n-1)(k+1-i)}^{(t-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_{k+1})\sigma_{(n-1)(k-i)}^{(t-k+i-1)}(\mathbf{x}_i,\dots,\mathbf{x}_k)}{\sigma_{(n-1)(k+1-i)}^{(t-k+i-1)}(\mathbf{x}_i,\dots,\mathbf{x}_{k+1})\sigma_{(n-1)(k-i)}^{(t-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_k)}\\
&= \left(\prod_{t=r+1}^{r+n-1} x_k^{(t+1)}\right) \frac{\sigma_{(n-1)(k-i)}^{(r-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_k)\sigma_{(n-1)(k-i-1)}^{(r-k+i+1)}(\mathbf{x}_i,\dots,\mathbf{x}_{k-1})}{\sigma_{(n-1)(k-i)}^{(r-k+i+1)}(\mathbf{x}_i,\dots,\mathbf{x}_k)\sigma_{(n-1)(k-i-1)}^{(r-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_{k-1})}\\
&+\sum_{s=1}^{n-2}\left(\prod_{t=r+1}^{r+s} x_j^{(t+j-k)}\right)\left(\prod_{t=r+s+1}^{r+n-1} x_k^{(t+1)}\right) \frac{\sigma_{(n-1)(k-i)}^{(r-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_k)\ {}^{(k)}\Omega_{(n-1)(j-i)}^{(r-k+i+s)}(\mathbf{x}_i,\dots,\mathbf{x}_j)}{\sigma_{(n-1)(k-i)}^{(r-k+i+s)}(\mathbf{x}_i,\dots,\mathbf{x}_k)\ {}^{(k)}\Omega_{(n-1)(j-i)}^{(r-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_j)}\\
&\frac{\sigma_{(n-1)(k-i)}^{(r-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_k)\sigma_{(n-1)(k-i-1)}^{(r-k+i+s)}(\mathbf{x}_i,\dots,\mathbf{x}_{k-1})}{\sigma_{(n-1)(k-i)}^{(r-k+i+s+1)}(\mathbf{x}_i,\dots,\mathbf{x}_k)\sigma_{(n-1)(k-i-1)}^{(r-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_{k-1})}\\
&+\left(\prod_{t=r+1}^{r+n-1} x_j^{(t+j-k)}\right) \frac{\sigma_{(n-1)(k-i)}^{(r-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_k)\ {}^{(k)}\Omega_{(n-1)(j-i)}^{(r-k+i-1)}(\mathbf{x}_i,\dots,\mathbf{x}_j)}{\sigma_{(n-1)(k-i)}^{(r-k+i-1)}(\mathbf{x}_i,\dots,\mathbf{x}_k)\ {}^{(k)}\Omega_{(n-1)(j-i)}^{(r-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_j)}\\
&= \frac{\sigma_{(n-1)(k-i)}^{(r-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_k)}{{}^{(k)}\Omega_{(n-1)(j-i)}^{(r-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_j)\sigma_{(n-1)(k-i-1)}^{(r-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_{k-1})}\\
&\Bigg[\left(\prod_{t=r+1}^{r+n-1} x_k^{(t+1)}\right) \frac{\sigma_{(n-1)(k-i-1)}^{(r-k+i+1)}(\mathbf{x}_i,\dots,\mathbf{x}_{k-1})\,{}^{(k)}\Omega_{(n-1)(j-i)}^{(r-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_j)}{\sigma_{(n-1)(k-i)}^{(r-k+i+1)}(\mathbf{x}_i,\dots,\mathbf{x}_k)}\\
&+\sum_{s=1}^{n-2}\left(\prod_{t=r+1}^{r+s} x_j^{(t+j-k)}\right)\left(\prod_{t=r+s+1}^{r+n-1} x_k^{(t+1)}\right) \frac{{}^{(k)}\Omega_{(n-1)(j-i)}^{(r-k+i+s)}(\mathbf{x}_i,\dots,\mathbf{x}_j)}{\sigma_{(n-1)(k-i)}^{(r-k+i+s)}(\mathbf{x}_i,\dots,\mathbf{x}_k)} \frac{\sigma_{(n-1)(k-i)}^{(r-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_k)\sigma_{(n-1)(k-i-1)}^{(r-k+i+s)}(\mathbf{x}_i,\dots,\mathbf{x}_{k-1})}{\sigma_{(n-1)(k-i)}^{(r-k+i+s+1)}(\mathbf{x}_i,\dots,\mathbf{x}_k)}\\
&+\left(\prod_{t=r+1}^{r+n-1} x_j^{(t+j-k)}\right) \frac{{}^{(k)}\Omega_{(n-1)(j-i)}^{(r-k+i-1)}(\mathbf{x}_i,\dots,\mathbf{x}_j)\sigma_{(n-1)(k-i-1)}^{(r-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_{k-1})}{\sigma_{(n-1)(k-i)}^{(r-k+i-1)}(\mathbf{x}_i,\dots,\mathbf{x}_k)}\Bigg],
\end{aligned}$$

but Conjecture 2.8 is precisely the statement that the bracketed part on the last line is equal to ${}^{(k-1)}\Omega_{(n-1)(j-i)}^{(r-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_j)$. This proves the second claim in the theorem. We can now also prove the first claim to finish the inductive step:

$$\begin{aligned}
&s_{k-1}s_k\dots s_{j-2}s_{j-1}s_{j-2}\dots s_i(x_{k-1}^{(r)})\\
&= s_{k-1}(s(x_{k-1}^{(r)}))\\
&= s(x_k^{(r+1)})\frac{\kappa_{r+1}(s(\mathbf{x}_{k-1}),s(\mathbf{x}_k))}{\kappa_r(s(\mathbf{x}_{k-1}),s(\mathbf{x}_k))}\\
&= x_j^{(r+j-k+1)}\frac{\sigma^{(r-k+i)}_{(n-1)(k-i)}(\mathbf{x}_i,\dots,\mathbf{x}_k)\ {}^{(k)}\Omega^{(r-k+i+1)}_{(n-1)(j-i)}(\mathbf{x}_i,\dots,\mathbf{x}_j)}{\sigma^{(r-k+i+1)}_{(n-1)(k-i)}(\mathbf{x}_i,\dots,\mathbf{x}_k)\ {}^{(k)}\Omega^{(r-k+i)}_{(n-1)(j-i)}(\mathbf{x}_i,\dots,\mathbf{x}_j)}\\
&\frac{\sigma^{(r-k+i+1)}_{(n-1)(k-i)}(\mathbf{x}_i,\dots,\mathbf{x}_k)\ {}^{(k-1)}\Omega^{(r-k+i+1)}_{(n-1)(j-i)}(\mathbf{x}_i,\dots,\mathbf{x}_j)}{\sigma^{(r-k+i+1)}_{(n-1)(k-i-1)}(\mathbf{x}_i,\dots,\mathbf{x}_{k-1})\ {}^{(k)}\Omega^{(r-k+i+1)}_{(n-1)(j-i)}(\mathbf{x}_i,\dots,\mathbf{x}_j)}\ \frac{\sigma^{(r-k+i)}_{(n-1)(k-i-1)}(\mathbf{x}_i,\dots,\mathbf{x}_{k-1})\ {}^{(k)}\Omega^{(r-k+i)}_{(n-1)(j-i)}(\mathbf{x}_i,\dots,\mathbf{x}_j)}{\sigma^{(r-k+i)}_{(n-1)(k-i)}(\mathbf{x}_i,\dots,\mathbf{x}_k)\ {}^{(k-1)}\Omega^{(r-k+i)}_{(n-1)(j-i)}(\mathbf{x}_i,\dots,\mathbf{x}_j)}\\
&= x_j^{(r+j-k+1)}\frac{{}^{(k-1)}\Omega^{(r-k+i+1)}_{(n-1)(j-i)}(\mathbf{x}_i,\dots,\mathbf{x}_j)\sigma^{(r-k+i)}_{(n-1)(k-i-1)}(\mathbf{x}_i,\dots,\mathbf{x}_{k-1})}{\sigma^{(r-k+i+1)}_{(n-1)(k-i-1)}(\mathbf{x}_i,\dots,\mathbf{x}_{k-1}){}^{(k-1)}\Omega^{(r-k+i)}_{(n-1)(j-i)}(\mathbf{x}_i,\dots,\mathbf{x}_j)}
\end{aligned}$$

as desired. ☐

And now we are able to describe how a transposition acts on the intermediate variables.

Corollary 2.11. Given $1 \le i < j \le m$, and $i < k < j$, if Conjecture 2.8 holds,

$$s_i\dots s_{j-2}s_{j-1}s_{j-2}\dots s_i(x_k^{(r)}) = x_k^{(r)}\frac{{}^{(k)}\Omega^{(r-k+i)}_{(n-1)(j-i)}(\mathbf{x}_i,\dots,\mathbf{x}_j)\ {}^{(k-1)}\Omega^{(r-k+i-1)}_{(n-1)(j-i)}(\mathbf{x}_i,\dots,\mathbf{x}_j)}{{}^{(k-1)}\Omega^{(r-k+i)}_{(n-1)(j-i)}(\mathbf{x}_i,\dots,\mathbf{x}_j)\ {}^{(k)}\Omega^{(r-k+i-1)}_{(n-1)(j-i)}(\mathbf{x}_i,\dots,\mathbf{x}_j)}.$$

Proof. Let $s = s_k\dots s_{j-2}s_{j-1}s_{j-2}\dots s_i$.

$$\begin{aligned}
s_i\dots s_{j-2}s_{j-1}s_{j-2}\dots s_i(x_k^{(r)}) &= s_{k-1}s(x_k^{(r)})\\
&= \frac{s(x_{k-1}^{(r-1)})\kappa_{r-1}(s(\mathbf{x}_{k-1}),s(\mathbf{x}_k))}{\kappa_r(s(\mathbf{x}_{k-1}),s(\mathbf{x}_k))}\\
&= \frac{s_{j-1}s_{j-2}\dots s_i(x_{k-1}^{(r-1)})\kappa_{r-1}(s(\mathbf{x}_{k-1}),s(\mathbf{x}_k))}{\kappa_r(s(\mathbf{x}_{k-1}),s(\mathbf{x}_k))}.
\end{aligned}$$

Plugging in the formulas from Theorem 2.5 and Theorem 2.10 yields the desired result. ☐

As a special case of the previous corollary,

$$s_is_{i+1}s_i(x_{i+1}^{(r)}) = \frac{x_{i+1}^{(r)}\sigma^{(r-1)}_{2(n-1)}(\mathbf{x}_i,\mathbf{x}_{i+1},\mathbf{x}_{i+2})\bar\sigma^{(r-2)}_{2(n-1)}(\mathbf{x}_i,\mathbf{x}_{i+1},\mathbf{x}_{i+2})}{\bar\sigma^{(r-1)}_{2(n-1)}(\mathbf{x}_i,\mathbf{x}_{i+1},\mathbf{x}_{i+2})\sigma^{(r-2)}_{2(n-1)}(\mathbf{x}_i,\mathbf{x}_{i+1},\mathbf{x}_{i+2})}.$$

2.3. **Identity of the Ω functions.** In our attempt to prove Conjecture 2.8, we discovered an identity of σ functions that would follow from the conjecture, which we prove below.

Theorem 2.12.

$$\prod_{t=1}^{n-1}\sigma^{(r-j+i+t+1)}_{(n-1)(j-i-1)}(\mathbf{x}_i,\dots,\mathbf{x}_{j-1}) = \sum_{s=0}^{n-1}\prod_{t=r+s+1}^{r+n-1}x_{j-1}^{(t+1)}\left[\prod_{\substack{0\le t\le n-1\\ t\ne s,s+1}}\sigma^{(r-j+i+t+1)}_{(n-1)(j-i-1)}(\mathbf{x}_i,\dots,\mathbf{x}_{j-1})\right]$$
$$\sigma^{(r-j+i+s+2)}_{(n-1)(j-i-2)}(\mathbf{x}_i,\dots,\mathbf{x}_{j-2})x_i^{(r-j+i+s+1)}\dots x_i^{(r-j+i+3)}x_i^{(r-j+i+2)}.$$

Preliminarily, we may factor out

$$\left(\prod_{t=r+2}^{r+n}x_j^{(t)}\right)\sigma^{(r-j+i+1)}_{(n-1)(j-i-1)}(\mathbf{x}_i,\dots,\mathbf{x}_{j-1})$$

from both sides, which makes the identity equivalent to

$$\prod_{t=1}^{n-1}\sigma^{(r-j+i+t+1)}_{(n-1)(j-i-1)}(\mathbf{x}_i,\dots,\mathbf{x}_{j-1}) = \sum_{s=0}^{n-1}\prod_{t=r+s+1}^{r+n-1}x_{j-1}^{(t+1)}\left[\prod_{\substack{0\le t\le n-1\\ t\ne s,s+1}}\sigma^{(r-j+i+t+1)}_{(n-1)(j-i-1)}(\mathbf{x}_i,\dots,\mathbf{x}_{j-1})\right]$$
$$\sigma^{(r-j+i+s+2)}_{(n-1)(j-i-2)}(\mathbf{x}_i,\dots,\mathbf{x}_{j-2})x_i^{(r-j+i+s+1)}\dots x_i^{(r-j+i+3)}x_i^{(r-j+i+2)}.$$

For example, when $r = n = 4$, $j = 4$, $i = 1$, this identity says that

$$\begin{aligned}\sigma_6^{(3)}(\mathbf{a},\mathbf{b},\mathbf{c})\sigma_6^{(4)}(\mathbf{a},\mathbf{b},\mathbf{c})\sigma_6^{(1)}(\mathbf{a},\mathbf{b},\mathbf{c}) &= c_2c_3c_4\sigma_3^{(3)}(\mathbf{a},\mathbf{b})\sigma_6^{(4)}(\mathbf{a},\mathbf{b},\mathbf{c})\sigma_6^{(1)}(\mathbf{a},\mathbf{b},\mathbf{c})\\ &+ a_4c_3c_4\sigma_6^{(2)}(\mathbf{a},\mathbf{b},\mathbf{c})\sigma_3^{(4)}(\mathbf{a},\mathbf{b})\sigma_6^{(1)}(\mathbf{a},\mathbf{b},\mathbf{c})\\ &+ a_4a_3c_4\sigma_6^{(2)}(\mathbf{a},\mathbf{b},\mathbf{c})\sigma_6^{(3)}(\mathbf{a},\mathbf{b},\mathbf{c})\sigma_3^{(1)}(\mathbf{a},\mathbf{b})\\ &+ a_1a_4a_3\sigma_3^{(2)}(\mathbf{a},\mathbf{b})\sigma_6^{(3)}(\mathbf{a},\mathbf{b},\mathbf{c})\sigma_6^{(4)}(\mathbf{a},\mathbf{b},\mathbf{c}).\end{aligned}$$

But in fact the sum on the right hand side is also nice if we are summing from $s = 0$ up to $0 \le k \le n-1$. To show the theorem, we proceed by showing the following claim about what the sum evaluates to when we sum up to a certain k.

Lemma 2.13. For $0 \le k \le n-1$,

$$\begin{aligned}&\sum_{s=0}^{k}\prod_{t=r+s+1}^{r+n-1} x_{j-1}^{(t+1)} \prod_{\substack{0\le t\le n-1\\ t\ne s,s+1}} \sigma_{(n-1)(j-i-1)}^{(r-j+i+t+1)}(\mathbf{x}_i,\dots,\mathbf{x}_{j-1})\\ &\sigma_{(n-1)(j-i-2)}^{(r-j+i+s+2)}(\mathbf{x}_i,\dots,\mathbf{x}_{j-2})x_i^{(r-j+i+s+1)}\dots x_i^{(r-j+i+3)}x_i^{(r-j+i+2)}\\ =&\prod_{t=r+k+1}^{r+n-1} x_{j-1}^{(t+1)}\prod_{t=1}^{k}\sigma_{(n-1)(j-i-1)}^{(r-j+i+t+1)}(\mathbf{x}_i,\dots,\mathbf{x}_j)P_k^{(r-j+i+k+2)}(\mathbf{x}_i,\dots,\mathbf{x}_{j-1})\prod_{t=k+2}^{n-1}\sigma_{(n-1)(j-i-1)}^{(r-j+i+t+1)}(\mathbf{x}_i,\dots,\mathbf{x}_{j-1})\end{aligned}$$

where

$$P_k^{(r-j+i+k+2)}(\mathbf{x}_i,\dots,\mathbf{x}_{j-1}) := \sum_{t=0}^{k}\prod_{s=0}^{t-1} x_i^{(r-j+i+2+k-s)}\sigma_{(n-1)(j-i-2)}^{(r-j+i+2+k-t)}(\mathbf{x}_i,\dots,\mathbf{x}_{j-2})\prod_{s=0}^{k-t-1}x_{j-1}^{(r+k-t-s)}.$$

Since

$$P_{n-1}^{(r-j+i+1)}(\mathbf{x}_i,\dots,\mathbf{x}_{j-1}) = \sigma_{(n-1)(j-i-1)}^{(r-j+i+2)}(\mathbf{x}_i,\dots,\mathbf{x}_{j-1}),$$

this claim reduces to the desired identity when $k = n-1$.

Proof. When $k = 0$, the equality is by definition. For $0 \le k < n-1$, suppose that the claim is true for k. Then it suffices to prove that

$$\begin{aligned}&\prod_{t=r+k+1}^{r+n-1} x_{j-1}^{(t+1)}\prod_{t=1}^{k}\sigma_{(n-1)(j-i-1)}^{(r-j+i+t+1)}(x_i,\dots,x_j)P_k^{(r-j+i+k+2)}(x_i,\dots,x_{j-1})\prod_{t=k+2}^{n-1}\sigma_{(n-1)(j-i-1)}^{(r-j+i+t+1)}(x_i,\dots,x_{j-1})\\ +&\prod_{t=r+k+2}^{r+n-1} x_{j-1}^{(t+1)}\prod_{\substack{0\le t\le n-1\\ t\ne k+1,k+2}}\sigma_{(n-1)(j-i-1)}^{(r-j+i+t+1)}(x_i,\dots,x_{j-1})\sigma_{(n-1)(j-i-2)}^{(r-j+i+k+3)}(x_i,\dots,x_{j-2})\prod_{t=2}^{k+2}x_i^{(r-j+i+t)}\\ =&\prod_{t=r+k+2}^{r+n-1} x_{j-1}^{(t+1)}\prod_{t=1}^{k+1}\sigma_{(n-1)(j-i-1)}^{(r-j+i+t+1)}(x_i,\dots,x_j)P_{k+1}^{(r-j+i+k+3)}(x_i,\dots,x_{j-1})\prod_{t=k+3}^{n-1}\sigma_{(n-1)(j-i-1)}^{(r-j+i+t+1)}(x_i,\dots,x_{j-1}).\end{aligned}$$

We may factor out

$$\prod_{t=r+k+2}^{r+n-1} x_{j-1}^{(t+1)}\prod_{\substack{1\le t\le n-1\\ t\ne k+1,k+2}}\sigma_{(n-1)(j-i-1)}^{(r-j+i+t+1)}(x_i,\dots,x_{j-1})$$

so that it suffices to prove

$$\begin{aligned}&x_{j-1}^{(r+k+2)}P_k^{(r-j+i+k+2)}(x_i,\dots,x_{j-1})\sigma_{(n-1)(j-i-1)}^{(r-j+i+k+3)}(x_i,\dots,x_{j-1})\\ &+\sigma_{(n-1)(j-i-1)}^{(r-j+i+1)}(x_i,\dots,x_{j-1})\sigma_{(n-1)(j-i-2)}^{(r-j+i+k+3)}(x_i,\dots,x_{j-2})\prod_{t=2}^{k+2}x_i^{(r-j+i+t)}\\ &=\sigma_{(n-1)(j-i-1)}^{(r-j+i+k+2)}(x_i,\dots,x_j)P_{k+1}^{(r-j+i+k+3)}(x_i,\dots,x_{j-1}).\end{aligned}$$

□

For example, when $r = n = 4$, $j = 4$, $i = 1$, this identity says that

$$\begin{aligned}\sigma_6^{(3)}(a,b,c)\sigma_6^{(4)}(a,b,c)\sigma_6^{(1)}(a,b,c) &= c_2c_3c_4\sigma_3^{(3)}(a,b)\sigma_6^{(4)}(a,b,c)\sigma_6^{(1)}(a,b,c)\\ &+ a_4c_3c_4\sigma_6^{(2)}(a,b,c)\sigma_3^{(4)}(a,b)\sigma_6^{(1)}(a,b,c)\\ &+ a_4a_3c_4\sigma_6^{(2)}(a,b,c)\sigma_6^{(3)}(a,b,c)\sigma_3^{(1)}(a,b)\\ &+ a_1a_4a_3\sigma_3^{(2)}(a,b)\sigma_6^{(3)}(a,b,c)\sigma_6^{(4)}(a,b,c).\end{aligned}$$

We briefly discuss how Conjecture 2.8 implies Theorem 2.12. ((FEIYANG: fill in))

2.4. **Other Permutations.** We did not venture far into permutations other than cyclic shifts by 1 and transpositions, but we record our preliminary observations here.

Consider $s = s_3s_1s_2$ and denote $\mathbf{x}_1, \ldots, \mathbf{x}_4$ with $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$. Since $s_1s_3s_2 = s_3s_1s_2$,

$$\begin{aligned}s(a_r) &= s_1s_2(a_r),\\ s(b_r) &= s_1s_2(b_r),\\ s(c_r) &= s_3s_2(c_r),\\ s(d_r) &= s_3s_2(d_r).\end{aligned}$$

We know what the right hand side should be because we know how permutations such as s_1s_2 and s_3s_2 act. In cycle notation, $s_1s_3s_2 = (1243) = (12)(432)$. Also, when we act with $s = (13)(35)$, $s(\mathbf{c}_r)$ also has a nice expression in terms of known functions.

In contrast, when we act with the permutation $s_2s_3s_1s_2 = (13)(24)$, factors that we are very unfamiliar with arise. For instance, the following factor arises in the numerator of $s_2s_3s_1s_2(b_1)$:

$$\begin{aligned}&a_1a_2a_3^2b_1^2b_2b_3 + a_1a_2a_3^2b_1b_2b_3c_1 + a_1a_2a_3b_1b_2b_3^2c_1 + a_1a_2a_3^2b_1b_3c_1c_2 + a_1a_2a_3b_1b_3^2c_1c_2 + a_2a_3b_1^2b_3^2c_1c_2+\\ &a_1a_2a_3b_1b_3c_1c_2c_3 + a_2a_3b_1^2b_3c_1c_2c_3 + a_2a_3b_1b_3c_1^2c_2c_3 + a_1a_2a_3^2b_1b_2b_3d_1 + a_1a_2a_3b_1b_2b_3^2d_1 + a_1a_2a_3^2b_1b_3c_2d_1+\\ &a_1a_2a_3b_1b_3^2c_2d_1 + a_2a_3b_1^2b_3^2c_2d_1 + a_1a_2a_3b_1b_2b_3c_3d_1 + a_1a_2a_3^2b_1c_2c_3d_1 + 2a_1a_2a_3b_1b_3c_2c_3d_1 + 2a_2a_3b_1^2b_3c_2c_3d_1+\\ &a_3b_1^2b_2b_3c_2c_3d_1 + a_2a_3b_1b_3c_1c_2c_3d_1 + a_1a_2a_3b_1c_2c_3^2d_1 + a_2a_3b_1^2c_2c_3^2d_1 + a_3b_1^2b_2c_2c_3^2d_1 + a_2a_3b_1c_1c_2c_3^2d_1+\\ &a_3b_1b_2c_1c_2c_3^2d_1 + a_1a_2a_3^2b_1b_3d_1d_2 + a_1a_2a_3b_1b_3^2d_1d_2 + a_2a_3b_1^2b_3^2d_1d_2 + a_1a_2a_3^2b_1c_3d_1d_2 + 2a_1a_2a_3b_1b_3c_3d_1d_2+\\ &2a_2a_3b_1^2b_3c_3d_1d_2 + a_3b_1^2b_2b_3c_3d_1d_2 + a_1a_2a_3^2c_1c_3d_1d_2 + a_1a_2a_3b_3c_1c_3d_1d_2 + 2a_2a_3b_1b_3c_1c_3d_1d_2 + a_3b_1b_2b_3c_1c_3d_1d_2+\\ &b_1b_2b_3^2c_1c_3d_1d_2 + a_1a_2a_3b_1c_3^2d_1d_2 + a_2a_3b_1^2c_3^2d_1d_2 + a_3b_1^2b_2c_3^2d_1d_2 + a_1a_2a_3c_1c_3^2d_1d_2 + 2a_2a_3b_1c_1c_3^2d_1d_2+\\ &2a_3b_1b_2c_1c_3^2d_1d_2 + b_1b_2b_3c_1c_3^2d_1d_2 + a_2a_3c_1^2c_3^2d_1d_2 + a_3b_2c_1^2c_3^2d_1d_2 + b_2b_3c_1^2c_3^2d_1d_2 + a_1a_2a_3b_1b_3d_1d_2d_3+\\ &a_2a_3b_1^2b_3d_1d_2d_3 + a_2a_3b_1b_3c_1d_1d_2d_3 + a_1a_2a_3b_1c_3d_1d_2d_3 + a_2a_3b_1^2c_3d_1d_2d_3 + a_3b_1^2b_2c_3d_1d_2d_3 + a_1a_2a_3c_1c_3d_1d_2d_3+\\ &2a_2a_3b_1c_1c_3d_1d_2d_3 + 2a_3b_1b_2c_1c_3d_1d_2d_3 + b_1b_2b_3c_1c_3d_1d_2d_3 + a_2a_3c_1^2c_3d_1d_2d_3 + a_3b_2c_1^2c_3d_1d_2d_3 + b_2b_3c_1^2c_3d_1d_2d_3+\\ &a_3b_1c_1c_2c_3d_1d_2d_3 + a_3c_1^2c_2c_3d_1d_2d_3 + b_3c_1^2c_2c_3d_1d_2d_3 + a_2a_3b_1b_3d_1^2d_2d_3 + a_2a_3b_1c_3d_1^2d_2d_3 + a_3b_1b_2c_3d_1^2d_2d_3+\\ &a_2a_3c_1c_3d_1^2d_2d_3 + a_3b_2c_1c_3d_1^2d_2d_3 + b_2b_3c_1c_3d_1^2d_2d_3 + a_3c_1c_2c_3d_1^2d_2d_3 + b_3c_1c_2c_3d_1^2d_2d_3 + c_1c_2c_3^2d_1^2d_2d_3\end{aligned}$$

Such factors are unfamiliar because there are terms with a coefficient of 2 in front and because sometimes variables in $\mathbf{b}$ and $\mathbf{c}$ are squared. We were hopeful that all functions that arise have a combinatorial interpretion in terms of weights of highway paths (see the next section). Our current way of using highway paths to provide combinatorial interpretaions implies that there should be at most one of every term, which is why the coefficient of 2 came as a surprise. Expecting Ω functions to appear again, we did not expect variables that are not $\mathbf{a}$ or $\mathbf{d}$ to be squared. However, this is also not so surprising since we are performing the permutation $(13)(24)$, and we would get squared $\mathbf{b}$ variables from first performing (24). This suggests that there are families of functions out there to be understood that will better explain what is happening beyond the case of transpositions.

We call permutations such as $(13)(24)$ overlapping, and permutations such as $(13)(35)$ non-overlapping. We suspect that the key distinction is whether a permutation can be written as a product of 1-shifts and transpositions that don't "overlap". When this happens, we can write down nice expressions for the resultant variables in terms of Ω functions. When the permutation cannot be written as a non-overlapping product, however, there are factors that we need to explore and understand further.

3. Combinatorial Interpretation of τ and σ functions

Following Section 4.3 of [LP08], let $N(n, m)$ denote the following grid cylindrical network:

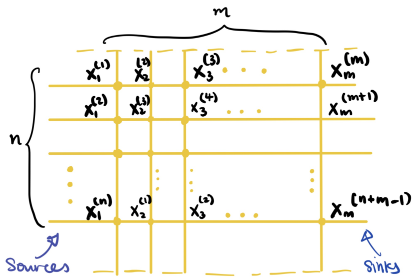

FIGURE 1. Illustration of the network $N(n,m)$

Notice that the dashed top and bottom boundaries are identified and that the r-th crossing of the k-th vertical loop, counting from the dashed boundary at the top, is given the weight $x_k^{(r)}$. The horizontal lines are extended indefinitely to the left and right, leading to the sources and sinks of $N(n,m)$. A *highway path* from a source to a sink is a path that only goes to the right and up, and never goes up twice in a row. See below for an example and a non-example.

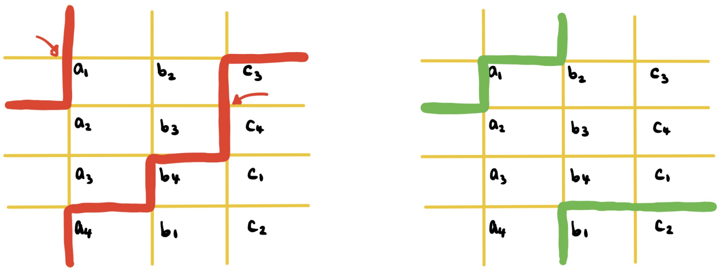

FIGURE 2. Illustration of a highway path and a non-highway path in $N(3,4)$

We can prescribe *weights* to highway paths by letting its weight be the product of the weights of the crosses that it passes whenever it goes to the right twice in a row. For example, the highway path on the right in Figure 2 has weight c_2. Given a family of non-crossing highway paths, we let its weight be the product of the weights of the paths in the family. Note that non-crossing highway paths are allowed to touch at corners.

The following theorem is proven partly in [LP08] through a lemma that shows that certain τ functions are cylindric loop Schur functions (Lemma 6.5) and a proposition that establishes a weight-preserving bijection between cylindric semistandard Young tableaux and noncrossing families of highway paths in $N(n,m)$ between specific sets of sources and sinks (Proposition 4.7). We prove this in the generality of all τ functions below, directly appealing to the properties of noncrossing paths.

Theorem 3.1. Let $k = \ell(n-1)+t$ where $0 \le t < n-1$. Define $s_i = r - i + 1$ and

$$r_i = \begin{cases} s_i + \ell & i \le t, \\ s_i + \ell - 1 & i > t, \end{cases}$$

where i ranges from 1 to $n-1$. Then

$$\tau_k^{(r)}(\mathbf{x}_1,\ldots,\mathbf{x}_m) = \sum_{\substack{\text{families of noncrossing} \\ \text{highway paths } P \\ \text{from } s_i \to r_i \text{ in } N(n,m)}} \mathrm{wt}(P).$$

Proof. Consider the term in $\tau_k^{(r)}$ where each index in the sum is as low as possible. Then we get the following term:

$$x_1^{(r)} x_1^{(r-1)} \ldots x_1^{(r-n+2)} x_2^{(r-n+1)} \ldots x_\ell^{(r-k+t+1)} x_{\ell+1}^{(r-k+t)} \ldots x_{\ell+1}^{(r-k+1)}.$$

We get this term as a family of paths in $N(n,m)$ by starting at every source except $r+1$. Every path goes straight through the first ℓ crossings. Then the path that started at r down through the path that

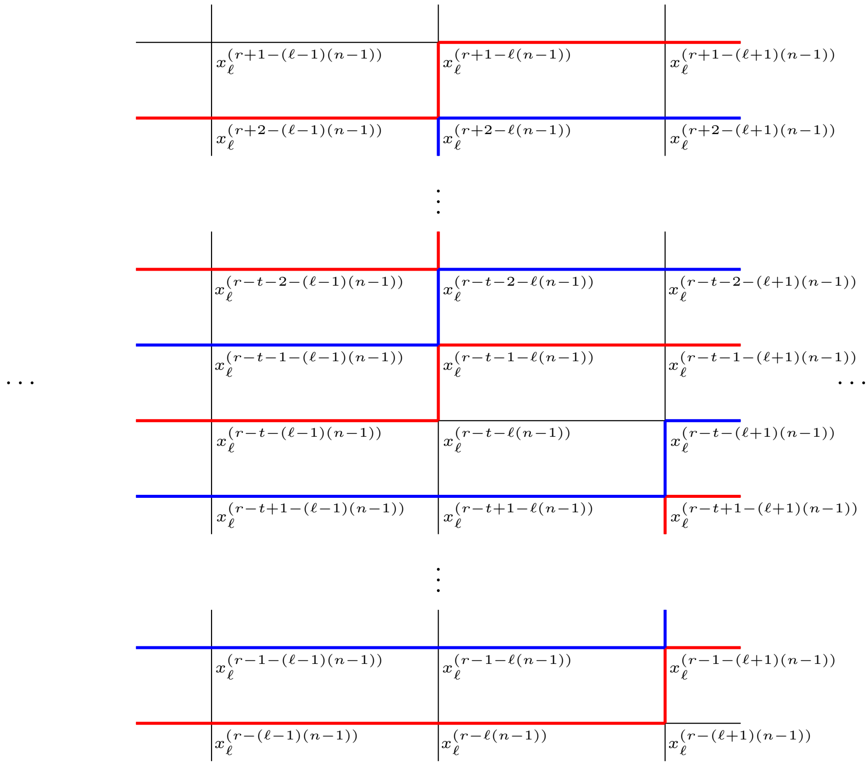

FIGURE 3. An illustration of the proof of Theorem 3.1.

started $r-t+1$ go straight once more while the other paths make a zigzag, which does not result in crossing (see Figure 3). After this all the paths zigzag until they reach a sink. We can compute which sink each path will end at by starting with the source it started at and then increasing by 1 for each crossing it went straight through (zigzagging doesn't change the index). We have to subtract one at the end because the indices for the sinks are not shifted from the indices of the vertices in the previous column. Thus, we know this gives us a family of paths between the correct sources and sinks.

We can get other terms in $\tau_k^{(r)}$ by increasing in index. Let's suppose that $i_j = a$ in some term of $\tau_k^{(r)}$ and shifting to $i_j = a+1$ gives another term. Note that if changing this index is allowed, this means $i_{j+1} > a$. We will first show that if changing the index is allowed then in family of paths corresponding to the first term, the path that goes through the vertex with weight $x_a^{(r-j+1)}$ does not go straight through the next crossing. If it did, it would pick up the weight $x_{a+1}^{(r-j+2)}$. So, the first term must have been $\dots x_a^{(r-j+1)} x_{a+1}^{(r-j)} x_{a+1}^{(r-j-1)} \dots x_{a+1}^{(r-j-n+2)} \dots$. Here $a+1$ appears as an index $n-1$ times, so we would not be allowed to change i_j from a to $a+1$. Thus the path in family of paths corresponding to the first term goes straight through $x_a^{(r-j+1)}$ and then zigzags at the next crossing. If we switch the order of these steps so that the path zigzags at $x_a^{(r-j+1)}$ and then goes straight through $x_{a+1}^{(r-j+1)}$, we get a family of paths that corresponds to the second term. All we need to do now is check that this family is noncrossing. Since $i_{j+1} > a$ and a appears as an index at most $n-1$ times, there is no path that goes through $x_a^{(r-j)}$. This means that moving the path's zigzag to $x_a^{(r-j+1)}$ does not result in adding a crossing. $\square$

Lemma 3.2. If $k \le m(n-1)$, let $k = an + b$ where $0 \le b < n$. Define $s_i = r - i + 1$,

$$r_i^{(a)} = \begin{cases} s_i & i \le b, \\ s_i - 1 & i > b, \end{cases}$$

where i ranges from 1 to $n-1$. We can recursively define $r_i^{(j)} := r_{i-1}^{(j+1)}$. Then

$$\sigma_k^{(r)} = \sum_{j=0}^{a} \left(\prod_{i=0}^{n-1} x_1^{(i)} \right)^j \sum_{\substack{\text{families of noncrossing} \\ \text{highway paths } P \\ \text{from } s_i \to r_i^{(j)} \text{ in } N(n,m)}} \mathrm{wt}(P).$$

Proof. When $k \leq m(n-1)$, we can rewrite $\sigma_k^{(r)}$ as

$$\sigma_k^{(r)}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m) = \sum_{j=0}^{a} \left(\prod_{i=0}^{n-1} x_1^{(i)} \right)^j \tau_{k-jn}^{(r)}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m).$$

We can check that the definitions for s_i and $r_i^{(j)}$ match those in Lemma 3. □

Theorem 3.3. If $k > m(n-1)$, define $s_i = r - k + mn - m - i + 1, r_i^{(m)} = s_i - 1$, where i ranges from 1 to $n-1$. We can recursively define $r_i^{(j)} := r_{i-1}^{(j+1)}$.

$$\sigma_k^{(r)}(\mathbf{x}_1, \ldots, \mathbf{x}_m) = \left(\prod_{i=r-k+mn-m+1}^{r} x_1^{(i)} \right) \sum_{j=0}^{m} \left(\prod_{i=0}^{n-1} x_1^{(i)} \right)^j \sum_{\substack{\text{families of noncrossing} \\ \text{highway paths } P \\ \text{from } s_i \to r_i^{(j)} \text{ in } N(n,m)}} \mathrm{wt}(P).$$

Proof. When $k > m(n-1)$,

$$\sigma_k^{(r)}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m) = x_1^{(r)} x_1^{(r-1)} \ldots x_1^{(r-k+mn-m+1)} \sigma_{m(n-1)}^{(r-k+mn-m)}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m).$$

Then the theorem follows from Lemma 3.2. □

We believe there should be similar interpretations of the Ω functions.

4. Acknowledgements

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