Filtering Grasmannian Cohomology via k-Schur Functions

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Abstract

This project concerns the cohomology rings of complex Grassmannians. In 2003, Reiner and Tudose conjectured the form of the Hilbert series for certain subalgebras of these cohomology rings. We build on their work in two ways. First, we conjecture two natural bases for these subalgebras that would imply their conjecture using notions from the theory of k-Schur functions. Second, we formulate an analogous conjecture for Lagrangian Grassmannians.

Preliminaries

- The cohomology ring of the complex Grassmannian $Gr(\ell, \mathbb{C}^{\ell+k})$ with coefficients in $\mathbb{Q}$ can be interpreted as the graded vector space:

$$R^{\ell,k} \cong \mathbb{Q}[h_1, h_2, \dots, h_k]/(e_{\ell+1}, \cdots, e_{\ell+k}) = \Lambda^{(k)}/(e_{\ell+1}, \cdots, e_{\ell+k})$$

where $deg(e_i) = deg(h_i) = i$, and the e_i's are the i'th **Jacobi-Trudi determinants**

$$\det \begin{pmatrix} h_1 & h_2 & \cdots & & \\ 1 & h_1 & \cdots & & \\ \vdots & \ddots & \ddots & & \\ 0 & \cdots & 1 & h_1 & h_2 \\ 0 & \cdots & 0 & 1 & h_1 \end{pmatrix}.$$

- $R^{l,k,m}$ is the subalgebra of $R^{l,k}$ generated by $h_1, \dots, h_m$. See that

$$\mathbb{Q} = R^{\ell,k,0} \subset R^{\ell,k,1} \subset R^{\ell,k,2} \subset \cdots \subset R^{\ell,k,m} \subset \cdots \subset R^{\ell,k}$$

- A **partition** is a weakly decreasing sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)$. It can be represented by its Ferrer's diagram:

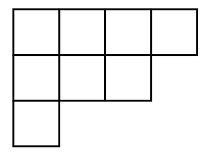


Figure 1:The Ferrer's diagram of the partition $\lambda = (4, 3, 1)$.

The q-**binomial coefficient** is $\begin{bmatrix} k+\ell \\ k \end{bmatrix}_q = \sum_{\lambda \subset (k^\ell)} q^{|\lambda|}$.

- Given any graded vector space $R = \bigoplus_{d=0}^{\infty} R_d$ over $\mathbb{Q}$, the **Hilbert series** of R is $\mathrm{Hilb}(R, q) = \sum_{d=0}^{\infty} \dim_{\mathbb{Q}}(R_d) q^d$.

The Problem (R-T Conjecture)[3]

For each $m = 0, 1, 2, \dots, \min(k, \ell)$, one has

$$\mathrm{Hilb}(R^{\ell,k,m}, q) = 1 + \sum_{i=1}^{m} q^i \begin{bmatrix} k \\ i \end{bmatrix}_q \sum_{j=0}^{\ell-i} q^{j(k-i+1)} \begin{bmatrix} i+j-1 \\ j \end{bmatrix}_q \quad (1)$$

Visualization and Boundary Cases

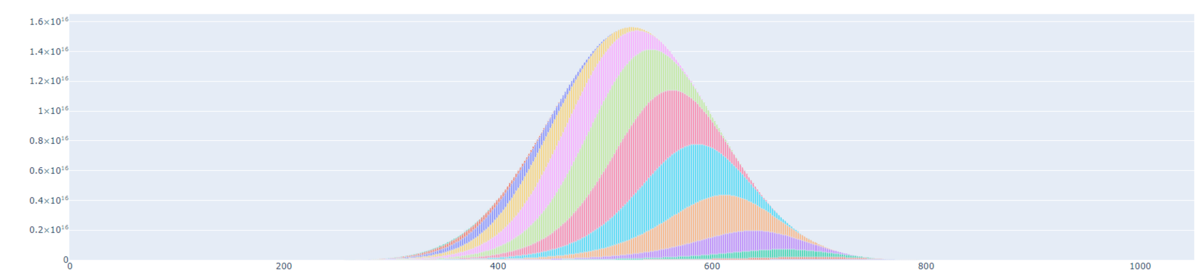


Figure 2:An illustration of the R-T Conjecture for $k = 30$, $l = 35$.

- One can check that for $m = 1$, this conjecture reduces to $\mathrm{Hilb}(R^{\ell,k,1}, q) = 1 + q + \ldots + q^{k\ell}$, which can be deduced from either Schubert calculus or the hard Lefschetz theorem.
- For $m = \min(k, \ell)$, this conjecture must be consistent with $\mathrm{Hilb}(R^{\ell,k}, q) = \begin{bmatrix} \ell+k \\ \ell \end{bmatrix}_q$. We can verify that the RHS of the R-T Conjecture reduces to this q-binomial coefficient via a combinatorial interpretation of the R-T conjecture involving the notion of i-vacant partitions.

i-vacant Partitions

A k-bounded partition λ is i-**vacant** if i is the largest integer for which the complementary *skew diagram* $(k^{\ell(\lambda)})/\lambda$ contains an $i \times (i-1)$ rectangle in its southeast corner. We will call $(k^{\ell(\lambda)})$ the **ambient** k-**rectangle** of λ.

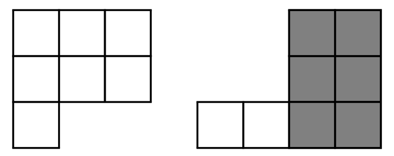
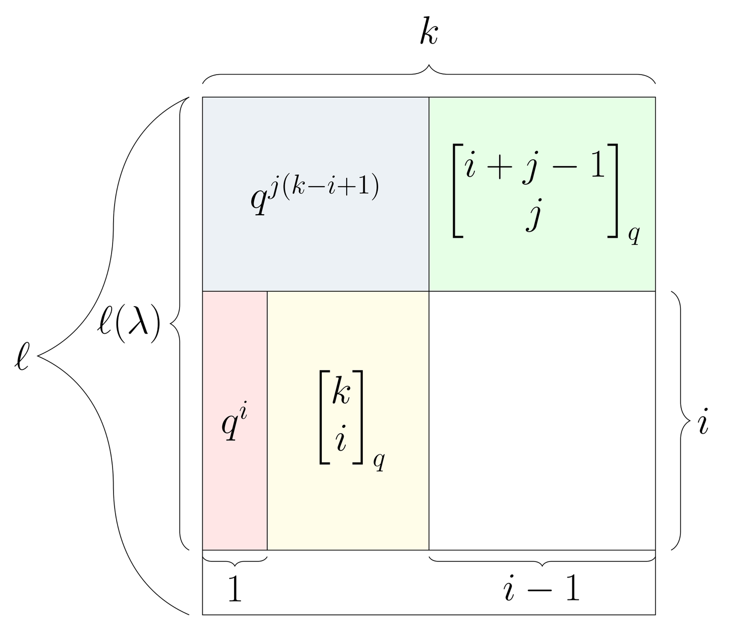


Figure 3:The 5-bounded partition $\lambda = (3, 3, 1)$ is 3-vacant.

Combinatorial Interpretation of the R-T Conjecture

For each $m = 0, 1, 2, \dots, \min(k, \ell)$,

$$\sum_{\substack{i\text{-vacant, } i \leq m \\ \lambda \subseteq (k^\ell)}} q^{|\lambda|} = 1 + \sum_{i=1}^{m} q^i \begin{bmatrix} k \\ i \end{bmatrix}_q \sum_{j=0}^{\ell-i} q^{j(k-i+1)} \begin{bmatrix} i+j-1 \\ j \end{bmatrix}_q$$



k-conjugation

A k-**bounded partition** is a partition $\lambda = (\lambda_1 \geq \ldots \geq \lambda_d)$ where $\lambda_1 \leq k$. We denote the set of all k-bounded partitions by $\mathcal{P}^k$.

A $(k+1)$-**core** is a partition $\lambda = (\lambda_1 \geq \ldots \geq \lambda_d)$ where no box has hook-length equal to $k+1$. We denote the set of all $(k+1)$-cores by $\mathcal{C}^{k+1}$.

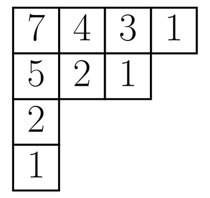
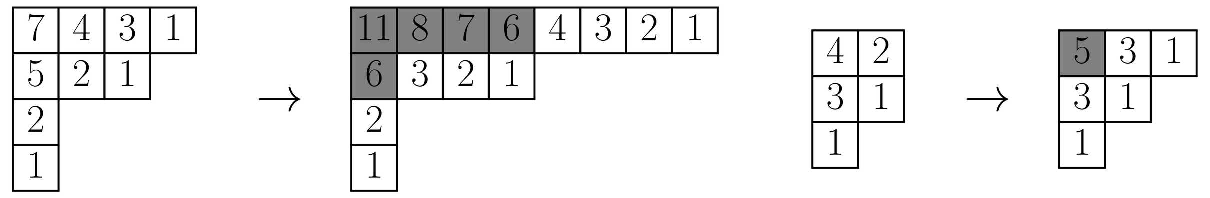
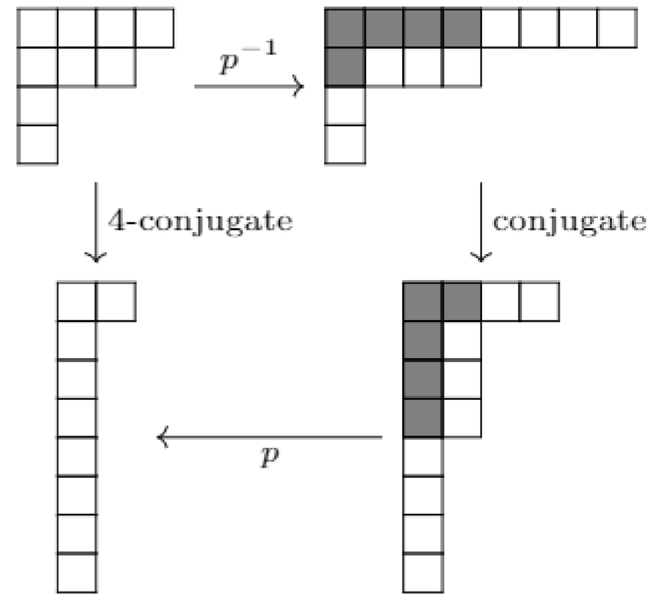


Figure 4:$(4, 3, 1, 1)$ is 4-bounded and a 6-core.

There is a bijection between $\mathcal{P}^k$ and $\mathcal{C}^{k+1}$:

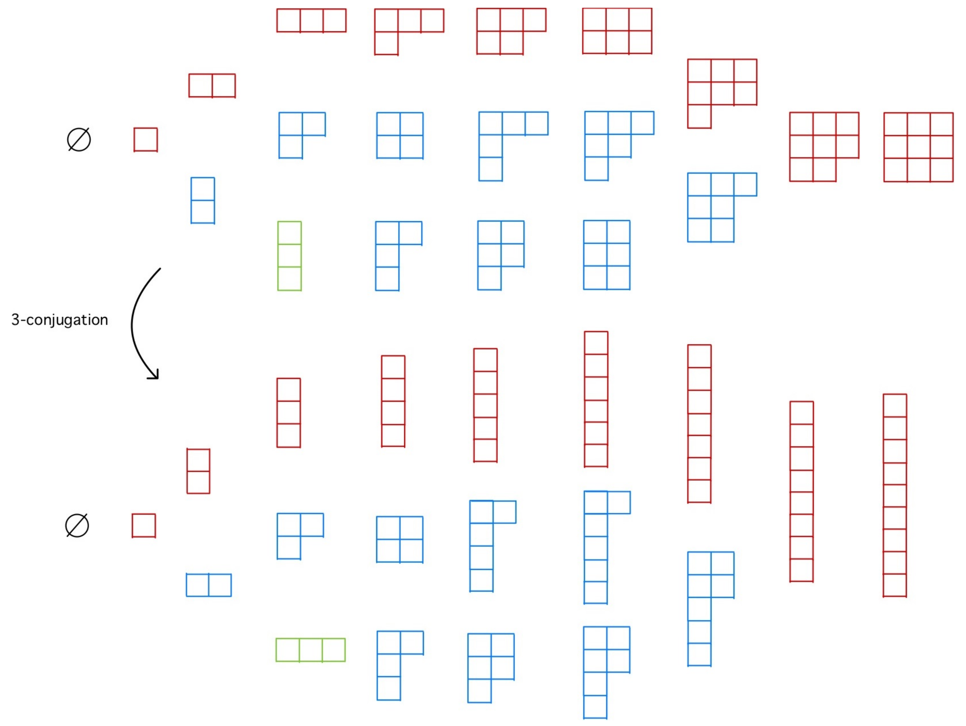


Theorem: (Lapointe, Lascoux and Morse, 2003) [2]
The map $\omega(k) : \mathcal{P}^k \to \mathcal{P}^k$ defined by sending a k-bounded partition λ to $\lambda^{\omega(k)} := \mathfrak{p}(\mathfrak{c}(\lambda)')$ is an involution, where $(-)'$ denotes usual conjugation. We call $\lambda^{\omega(k)}$ the k-**conjugate** of λ.



Proposition: For any $i = 1, 2, \dots, \min(\ell, k)$, one has

$$\sum_{\substack{i\text{-vacant} \\ \lambda \subseteq (k^\ell)}} q^{|\lambda|} = \sum_{\substack{\mu : \mu_1 = i, \\ \mu^{\omega(k)} \subseteq (k^\ell)}} q^{|\mu|}.$$



Conjectured Filtered Bases

Write $\{\mu : \mu_1 \leq m, \mu^{\omega(k)} \subseteq (k^\ell)\}$ as $P^{k,\ell,m}$, and $P^{k,\ell,\min\{k,\ell\}}$ as $P^{k,\ell}$.

Conjecture:
(a) The set $\{h_\lambda \mid \lambda \in P^{k,\ell}\}$ is a basis of $R^{\ell,k}$; moreover,
(b) the set $\{h_\lambda \mid \lambda \in P^{k,\ell,m}\}$ is a basis of $R^{\ell,k,m}$ for all m.

- (a) will show half of the R-T conjecture (LHS $\geq$ RHS)
- (b) implies the full R-T conjecture

Lagrangian Analogue

For a strictly decreasing partition $\lambda = (\lambda_1 > \cdots > \lambda_\ell)$, we define its **shifted Young diagram** to be a diagram with λ_i boxes in row i with each row shifted one unit right of the previous one. An **ambient triangle of size** n, denoted as Δ_n, is a shifted Young diagram $\lambda = (n > n-1 > \cdots > 1)$.

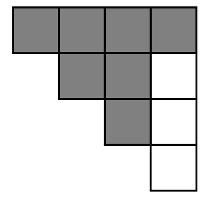


Figure 5:An ambient triangle Δ_4 and a shifted Young diagram $\lambda = (4, 2, 1)$.

In Lie type C, we replace $Gr(\ell, \mathbb{C}^{k+\ell})$ by the Lagrangian Grassmannian $\mathbb{LG}(n, \mathbb{C}^{2n})$ and define the ring $R^n_{\mathbb{LG}} := H^*(\mathbb{LG}(n, 2n); \mathbb{Q})$. Then,

$$R^n_{\mathbb{LG}} \cong [e_1, e_2, \dots, e_n] \ / \ \left(e_i^2 + 2\sum_{k=1}^{n-i}(-1)^k e_{i+k} e_{i-k}\right)_{i=1,2,\dots,n}$$

$$\mathrm{Hilb}(R^n_{\mathbb{LG}}, q) = \sum_{\lambda \subset \Delta^n} q^{|\lambda|} = (1+q)(1+q^2)(1+q^3)\cdots(1+q^n).$$

The R-T Conjecture (Type C Analogue)

For each $m = 0, 1, \cdots, n$, write the subalgebra of $R^n_{\mathbb{LG}}$ generated by $e_1, \dots, e_m$ as $R^{n,m}_{\mathbb{LG}}$, then

$$\mathrm{Hilb}(R^{n,m}_{\mathbb{LG}}, q) = 1 + \sum_{\substack{1 \leq i \leq m \\ i \text{ odd}}} q^i \sum_{j=0}^{n-i} q^{\binom{j+1}{2}} \begin{bmatrix} i+j \\ i \end{bmatrix}_q. \quad (2)$$

References

[1] Thomas Lam, Luc Lapointe, Jennifer Morse, Anne Schilling, Mark Shimozono, Mike Zabrocki, k-Schur Functions and Affine Schubert Calculus. Fields Institute Monographs 33, 2014.
[2] Luc Lapointe, Alain Lascoux and Jennifer Morse, Tableau atoms and a new Macdonald positivity conjecture. Duke Mathematical Journal, 116(1):103–146, January 2003
[3] V. Reiner and G. Tudose, Conjectures on the cohomology of the Grassmannian. preprint 2003. arXiv:math/0309281 [math.CO].
[4] H. Tamvakis, Quantum cohomology of Lagrangian and orthogonal Grassmannians. Talk at Mathematische Arbeitstagung 2001