FILTERING GRASSMANNIAN COHOMOLOGY VIA K-SCHUR FUNCTIONS

THE 2020 POLYMATH JR. "Q-B-AND-G" GROUP†

Abstract. This paper is concerned with studying the Hilbert Series of the subalgebras of the cohomology rings of the complex Grassmannians and Lagrangian Grassmannians. We build upon a previous conjecture by Reiner and Tudose for the complex Grassmannian and present an analogous conjecture for the Lagrangian Grassmannian. Additionally, we summarize several potential approaches involving Gröbner bases, homological algebra, and algebraic topology. Then we give a new interpretation to the conjectural expression for the complex Grassmannian by utilizing k-conjugation. This leads to two conjectural bases that would imply the original conjecture for the Grassmannian. Finally, we comment on how this new approach foreshadows the possible existence of analogous k-Schur functions in other Lie types.

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1. Introduction

Complex Grassmannians $Gr(k, \mathbb{C}^{k+\ell})$ and their cohomology rings, denoted as $R^{k,\ell} := H^*(Gr(k, \mathbb{C}^{k+\ell}); \mathbb{Q})$, are some of the most well-studied objects in algebra and topology. Their rich structure gives rise to the study of Schubert calculus, a subject that strongly overlaps with algebraic combinatorics, intersection theory and representation theory. Despite being a salient object in algebraic geometry and algebraic combinatorics, there still remains basic yet unsolved problems involving these cohomology rings.

In 1953, Borel published a detailed account of the generators and relations of $R^{k,\ell}$. In Borel's picture, $R^{k,\ell} = \mathbb{Q}[e_1, \dots, e_k]/(h_{\ell+1}, \dots, h_{\ell+k})$ is generated by the Chern classes e_i's of the canonical bundle of $Gr(k, \mathbb{C}^{k+\ell})$ subject to polynomial relations h_i's. These relations are known as the first Jacobi-Trudi relations as in this Wikipedia page. Naturally, $R^{k,\ell}$ can be equipped with a graded algebra structure by assigning to e_i the degree i. Subsequently, we are able to pose questions about the Hilbert series of $R^{k,\ell}$. The Hilbert Series is a power series that encodes information about the dimensions of each graded component of $R^{k,\ell}$. It turns out that $\mathrm{Hilb}(R^{k,\ell}, q)$ is the q-binomial coefficient $\begin{bmatrix} k+\ell \\ k \end{bmatrix}_q$. The q-binomial coefficient is the q-analogue of the binomial coefficients $\binom{k+\ell}{k}$.

Now let $0 \leq m \leq k \leq \ell$ and consider the graded subalgebra $R^{k,\ell,m}$ of $R^{k,\ell}$ generated by the first m Chern classes $e_1, \dots, e_m$. The filtration

$$\mathbb{Q} = R^{k,\ell,0} \subset R^{k,\ell,1} \subset \cdots \subset R^{k,\ell,k-1} \subset R^{k,\ell,k} = R^{k,\ell}$$

induces a sequence of degree-wise inequalities of their respective Hilbert series

$$1 = \mathrm{Hilb}(R^{k,\ell,0}, q) \leq \mathrm{Hilb}(R^{k,\ell,1}, q) \leq \cdots \leq \mathrm{Hilb}(R^{k,\ell,k-1}, q) \leq \mathrm{Hilb}(R^{k,\ell,k}, q) = \begin{bmatrix} k+\ell \\ k \end{bmatrix}_q.$$

At first glance, it seems that these objects are mysterious since there are no known geometric interpretation of the subalgebras. However, by observation of data, Reiner and Tudose conjectured in [5] the form of $\mathrm{Hilb}(R^{k,\ell,m}, q)$ in terms of a complicated yet intriguing expression.

$$\mathrm{Hilb}(R^{k,\ell,m}, q) = 1 + \sum_{i=1}^{m} q^i \begin{bmatrix} \ell \\ i \end{bmatrix}_q \left(\sum_{j=0}^{k-i} q^{j(\ell-i+1)} \begin{bmatrix} i+j-1 \\ j \end{bmatrix}_q \right). \tag{1.1}$$

From now on we refer to (1.1) as the Riener-Tudose Conjecture (or the R-T Conjecture). By applying the Hard Lefschetz Theorem from complex geometry, Reiner and Tudose were able to prove their conjecture for the edge case $m = 1$ [5].

This can also be proved by invoking results from Schubert calculus known as Pieri's rule. On the other end, the edge case $m = k$ was also proved by a combinatorial argument counting Ferrers' diagrams with certain properties. Beyond that, there was no clue which directions one should probe nor how to naturally interpret the conjectured expression. Thus, we started the REU program with two initial goals:

- Proving the Riener-Tudose Conjecture formulated in [5, Conj. 1], and
- Finding its analogue for the cohomology of Lagrangian Grassmannians.

In reaching the first goal, we found a surprising connection between the R-T conjecture and the theory of k-Schur functions; this led us to make conjectures about the existence of certain filtered bases, which will be explained in Section 5. On the other hand, we were very fortunate to achieve the second goal at a very early stage in our research. We conjectured that

$$\mathrm{Hilb}(R_{LG}^{n,m}, q) = 1 + \sum_{1\le k\le m,\ k \text{ odd}} q^k \left(\sum_{j=0}^{n-k} q^{\binom{j+1}{2}} \begin{bmatrix} k+j \\ j \end{bmatrix}_q \right)$$

and proved the edge cases $m = 1$ and $m = n$.

Sections 3 and 4 summarize our initial approaches to proving these conjectures. Although those approaches have not been very successful, they nonetheless afford us a deeper understanding of the nature of these conjectures.

In Section 5, we apply a powerful technique called k-Schur conjugation to give a simple interpretation of the complicated expression in the R-T Conjecture and make two conjectures on the existence of certain filtered bases. Finally, Section 6 discusses two important implications of our results and future research directions.

2. Conjectures

In this section, we introduce the main conjectures for the Grassmannian and Lagrangian Grassmannian. We also present some of their important properties that we will rely on in our subsequent exposition.

2.1. **The Grassmannian.** The conjecture is about the cohomology ring of the Grassmannian of all k-linear subspaces of $\mathbb{C}^{k+\ell}$. This is a commutative graded $\mathbb{Q}$-algebra, with several natural descriptions[1], among them this one:

$$R^{k,\ell} \cong \mathbb{Q}[e_1, e_2, \ldots, e_k, h_1, h_2, \ldots, h_\ell]/J^{k,\ell} \tag{2.1}$$

where $J^{k,\ell}$ is the ideal generated by these elements:

$$\left\{ \sum_{i+j=d} (-1)^i e_i h_j \right\}_{d=1,2,\ldots,k+\ell}$$

Here the grading is defined by letting $\deg(e_i) = \deg(h_i) = i$, and by convention, $e_0 = h_0 = 1$, while $e_i = 0 = h_j$ if $i \notin [0, k]$ or $j \notin [0, \ell]$. Its *Hilbert series* is

$$\mathrm{Hilb}(R^{k,\ell}, q) := \sum_{d\ge 0} \dim_{\mathbb{Q}} R_d^{k,\ell} \cdot q^d = \begin{bmatrix} k+\ell \\ k \end{bmatrix}_q = \frac{[k+\ell]!_q}{[k]!_q[\ell]!_q}$$
$$= \frac{(1-q)(1-q^2)\cdots(1-q^{k+\ell})}{(1-q)(1-q^2)\cdots(1-q^k)\cdot(1-q)(1-q^2)\cdots(1-q^\ell)}$$

[1] More background and descriptions can be found in Fulton [1, §9.4]

where $[n]!_q := [n]_q[n-1]_q \cdots [2]_q[1]_q$ and $[n]_q := \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}$.

Let $R^{k,\ell,m}$ denote the $\mathbb{Q}$-subalgebra of $R^{k,\ell}$ generated by $e_1, e_2, \ldots, e_m$.

Conjecture 2.1 (Reiner-Tudose Conjecture)**.** *For $m \geq 0$,*

$$\mathrm{Hilb}(R^{k,\ell,m}, q) = 1 + \sum_{i=1}^{m} q^i \begin{bmatrix} \ell \\ i \end{bmatrix}_q \left(\sum_{j=0}^{k-i} q^{j(\ell-i+1)} \begin{bmatrix} i+j-1 \\ j \end{bmatrix}_q \right).$$

Proposition 2.2. $\mathrm{Hilb}(R^{k,\ell,m}, q) - \mathrm{Hilb}(R^{k,\ell,m-1}, q) = \mathrm{Hilb}(R^{k,\ell,m}/R^{k,\ell,m-1}, q)$

Proof. Let $V = \bigoplus_{d=0}^{\infty} V_d$ be a graded vector space, and let U be a subspace $U \subset V$ which is *homogeneous* in the sense that $U = \bigoplus_{d=0}^{\infty} U_d$, where $U_d := V_d \cap U$. Then the quotient vector space V/U is also graded, with

$$V/U = \bigoplus_{d=0}^{\infty} V_d/U_d$$

and has the following Hilbert series

$$\begin{aligned} \mathrm{Hilb}(V/U, q) &:= \sum_{d=0}^{\infty} \dim_k V_d/U_d \cdot q^d \\ &= \sum_{d=0}^{\infty} (\dim_k V_d - \dim_k U_d) \cdot q^d \\ &= \mathrm{Hilb}(V, q) - \mathrm{Hilb}(U, q) \end{aligned}$$

Since both $R^{k,\ell,m}$ and $R^{k,\ell,m-1}$ are graded algebras, and $R^{k,\ell,m-1}$ is a subalgebra of $R^{k,\ell,m}$, we can apply the above argument with $V = R^{k,\ell,m}$ and $U = R^{k,\ell,m-1}$, giving us what we wanted. ☐

Thus, equivalently, by Proposition 2.2 one could phrase the Riener-Tudose Conjecture as conjecturing that, for $m \geq 1$, one has

$$\begin{aligned} &\mathrm{Hilb}(R^{k,\ell,m}, q) - \mathrm{Hilb}(R^{k,\ell,m-1}, q) \quad \left(= \mathrm{Hilb}(R^{k,\ell,m}/R^{k,\ell,m-1}, q)\right) \\ &= q^m \begin{bmatrix} \ell \\ m \end{bmatrix}_q \left(\sum_{j=0}^{k-m} q^{j(\ell-m+1)} \begin{bmatrix} m+j-1 \\ j \end{bmatrix}_q \right). \end{aligned} \tag{2.2}$$

2.2. **The Lagrangian Grassmannian.** The Lagrangian Grassmannian denoted $\mathrm{LG}(n, \mathbb{C}^{2n})$ is the space of all maximal isotropic (n-dimensional) subspaces of $\mathbb{C}^{2n}$ endowed with a symplectic bilinear form. Its cohomology ring has the following description as a $\mathbb{Q}$-algebra as in [7, p. 2]:

$$R^n_{LG} \cong \mathbb{Q}[e_1, e_2, \ldots, e_n] \Big/ \left(e_i^2 + 2\sum_{k=1}^{n-i} (-1)^k e_{i+k} e_{i-k} : i = 1, 2, \ldots, n \right)$$

with $\deg(e_i) = i$, setting $e_0 = 1$, and $e_i = 0$ if $i \notin [0, n]$. Its Hilbert series is

$$\mathrm{Hilb}(R^n_{LG}, q) = [2]_q[2]_{q^2}[2]_{q^3} \cdots [2]_{q^n} = (1+q)(1+q^2)(1+q^3)\cdots(1+q^n).$$

Let $R^{n,m}_{LG}$ denote the $\mathbb{Q}$-subalgebra of R^n_{LG} generated by $e_1, e_2, \ldots, e_m$.

Conjecture 2.3. *For the Lagrangian Grassmannian, we have*

$$\mathrm{Hilb}(R_{LG}^{n,m}/R_{LG}^{n,m-1},q) = \begin{cases} q^m \left(\sum_{j=0}^{n-m} q^{\binom{j+1}{2}} \begin{bmatrix} m+j \\ j \end{bmatrix}_q \right), & \text{if } m \text{ is odd} \\ 0 \ , & \text{if } m \text{ is even} \end{cases}$$

The assertion in the even case is not hard.

Proposition 2.4. $R_{LG}^{n,m} = R_{LG}^{n,m+1}$ *whenever* $1 \leq m \leq n-1$ *and* m *is odd.*

Proof. Let $m \in \mathbb{N}$ be an odd number $1 \leq m \leq n-1$ and let

$$I = \left(e_d^2 + 2\sum_{k=1}^{n-d} (-1)^k e_{d-k} e_{d+k} : d \in \{1, \dots, n\} \right)$$

be an ideal in $\mathbb{Q}[e_1, \dots, e_n]$, where $e_0 = 1$ and $e_i = 0$ for all $i > n$ or $i < 0$. Since m is odd, there exists an $\ell < n \in \mathbb{N} \cup \{0\}$ such that $m = 2\ell + 1$. Consider the polynomial generator of I given by $d = \ell + 1$. We know that it is of the form

$$e_{\ell+1}^2 + 2 \cdot \left(\sum_{k=1}^{n-(\ell+1)} (-1)^k e_{\ell+1-k} e_{\ell+1+k} \right)$$

which we can rewrite as

$$e_{\ell+1}^2 + 2 \cdot \left(\sum_{\substack{k=1 \\ k \neq \ell+1}}^{n-(\ell+1)} (-1)^k e_{\ell+1-k} e_{\ell+1+k} \right) + 2(-1)^{\ell+1} e_{m+1}. \tag{2.3}$$

However, since $e_{\ell+1-k} = 0$ for all $k > \ell + 1$ we can actually write (2.3) as

$$2(-1)^{\ell+1} e_{m+1} + e_{\ell+1}^2 + 2 \cdot \left(\sum_{k=1}^{\ell} (-1)^k e_{\ell+1-k} e_{\ell+1+k} \right) \in I. \tag{2.4}$$

Equation (2.4) shows that, within the quotient ring R_{LG}^n, the image of the element e_{m+1} lies in the subalgebra $R_{LG}^{n,m}$ generated by the images of $e_1, e_2, \dots, e_m$. Hence $R_{LG}^{n,m+1}$ also lies inside $R_{LG}^{n,m}$, so these two subalgebras are equal. □

Since one has

$$\mathrm{Hilb}(R_{LG}^n, q) = 1 + \sum_{m=1}^{n} \mathrm{Hilb}(R_{LG}^{n,m}/R_{LG}^{n,m-1}, q),$$

Conjecture 2.3 implies the following combinatorial q-identity.

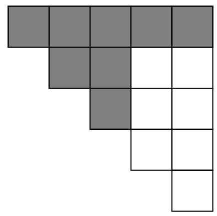
Theorem 2.5 (Lagrangian filtration q-identity)**.**

$$(1+q)(1+q^2)(1+q^3)\cdots(1+q^n) = 1 + \sum_{\substack{m \text{ odd} \\ 1 \leq m \leq n}} q^m \left(\sum_{j=0}^{n-m} q^{\binom{j+1}{2}} \begin{bmatrix} m+j \\ j \end{bmatrix}_q \right)$$

But before proving this theorem, we need a way to interpret the left hand side of this identity.

Definition 2.6. For a strictly decreasing partition $\lambda = (\lambda_1 > \cdots > \lambda_\ell)$, we define its **shifted Young diagram**, also denoted as λ, to be a diagram with λ_i boxes in row i with each row shifted one unit right of the previous one. An **ambient triangle of size** n, denoted as Δ_n, is a shifted Young diagram $\lambda = (n > n-1 > \cdots > 1)$.[2]

Example 2.7. An ambient triangle Δ_5 and a shifted Young diagram $\lambda = (5, 2, 1)$ colored gray are illustrated below.



Lemma 2.8. *We can interpret* $[2]_q[2]_{q^2}[2]_{q^3}\cdots[2]_{q^n}$ *as a generating function counting shifted Young diagrams in* Δ_n*, namely*

$$(1+q)(1+q^2)(1+q^3)\cdots(1+q^n) = \sum_{\lambda\subseteq\Delta_n} q^{|\lambda|}$$

Proof. Any partition $\lambda \subseteq \Delta_n$ is completely determined by its row sizes, which are an arbitrary subset $\{\lambda_1, \ldots, \lambda_\ell\}$ of $[n]$. Hence we have

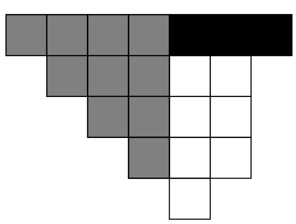
$$\sum_{\lambda\subset\Delta_n} q^{|\lambda|} = \sum_{\{\lambda_1,\ldots,\lambda_\ell\}\subset[n]} q^{\lambda_1+\cdots+\lambda_\ell} = (1+q)(1+q^2)\cdots(1+q^n). \quad \square$$

Proof of Theorem 2.5. The lemma above allows us to interpret the left hand side as the generating function counting shifted Young diagrams that fit inside an ambient n-triangle according to their weights. To prove the q-identity, whenever λ is non-empty, we will uniquely define a triple (m, j, μ) where m is odd, j is an integer, and μ is a Ferrers diagram fitting inside a (m^j) ambient rectangle. In terms of diagrams, this decomposes the shifted Young diagram λ into three parts:

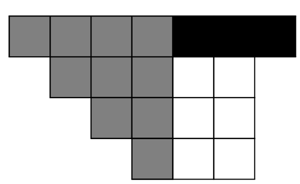
(1) a triangle Δ_j.
(2) a row of odd length m.
(3) a Ferrers diagram μ fitting inside a (m^j) ambient rectangle.

A few examples of this decomposition are illustrated below, where the colors correspond to the three portions described previously: **part 1 is colored gray, part 2 black, and part 3 white**. For example, in figure (1) and (2), the diagrams both have $m = 3$ and $j = 4$, but they have different part 3 (white) fitting inside a (4^3) ambient rectangle; In figure (3), $m = 1$ and $j = 5$, and the part 3 (white) is fitting inside a (5^1) ambient rectangle.

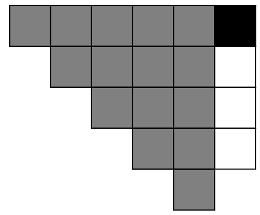
(1)



(2)



[2]See section 6 of Maria Gillespie's "Variations on a Theme of Schubert Calculus"

(3) 

It is easy to see that for any such triple (m, j, μ), we can construct a shifted Young diagram that fits into the ambient n-triangle. Conversely, given a shifted Young diagram $\lambda = (\lambda_1 > \cdots > \lambda_\ell) \subseteq \Delta_n$, we shall show that there is a unique decomposition. Let j be the maximum value such that $\Delta_j \subseteq \lambda$ and $m := \lambda_1 - j$ is odd. So

$$j = \begin{cases} \ell & \text{, if } \lambda_1 - \ell \text{ is odd} \\ \ell - 1 & \text{, if } \lambda_1 - \ell \text{ is even} \end{cases}$$

where $\ell = \ell(\lambda)$ is the number of parts of λ.[3] Finally, it is clear that removing Δ_j and the top row of length m would result in a Ferrers diagram fitting inside a (m^j) ambient rectangle.

Therefore, the set of such triples (m, j, μ) are in bijection with the set of shifted Young diagrams λ fitting in the ambient triangle of size n. In particular, for any fixed m, j, the triples (m, j, μ) ranging over μ are in 1-to-1 correspondence with the terms in the expansion of $q^m q^{\binom{j+1}{2}} \begin{bmatrix} m+j \\ j \end{bmatrix}_q$. The q-identity then easily follows from this bijection and the fact that $|\lambda| = m + |\mu| + \binom{j+1}{2}$. $\square$

The following is the analoque of [5, Thm. 6, Rem. 7] for the Grassmannian.

Proposition 2.9. *The $m = 1$ case of Conjecture 2.3 is correct.*

Proof. We claim the $m = 1$ case of Conjecture 2.3 can be rephrased as asserting

$$\mathrm{Hilb}(R_{LG}^{n,1}, q) = [N]_q, \tag{2.5}$$

where $N := \binom{n+1}{2}$. To see this, note that it asserts

$$\begin{aligned} \mathrm{Hilb}(R_{LG}^{n,1}/R_{LG}^{n,0}, q) &= q^1 \sum_{j=0}^{n-1} q^{\binom{j+1}{2}} \begin{bmatrix} 1+j \\ j \end{bmatrix}_q \\ &= q \sum_{j=0}^{n-1} q^{\binom{j+1}{2}} [j+1]_q \\ &= q(1 + q^1(1+q) + \cdots + q^{\binom{n}{2}}[n]_q) \\ &= q[N-1]_q. \end{aligned}$$

As $R_{LG}^{n,0} = \mathbb{Q}$, this is equivalent to

$$\mathrm{Hilb}(R_{LG}^{n,1}, q) = 1 + q[N-1]_q = [N]_q$$

as in (2.5).

[3] This condition exactly separates whether the bottom row of part 3 (white) is the last row in the (j^m) ambient rectangle or not. As compared in figure (1) and (2), although $\lambda_1 = 7$ in both cases, the values of ℓ are different. However, we set $m = 3$ and $j = 4$ in both cases, and part 3 (white) should both be viewed as Ferrers diagrams in a (3^4) ambient rectangle.

Note that (2.5) is equivalent to saying that inside the ring R^n_{LG}, the subalgebra $R^{n,1}_{LG} \cong \mathbb{Q}[e_1]/(e_1^{N+1})$. Since R^n_{LG} is the cohomology of the Lagrangian Grassmannian, a smooth complex projective variety of dimension N, the fact that $e_1^N \neq 0$ (but $e_1^{N+1} = 0$) follows by appealing to the Hard Lefschetz Theorem[4] [2, p. 122]. □

Remark 2.10. Closely related to the Lagrangian Grassmannian is the Orthogonal Grassmannian, discussed in [7] and [3]. It has a cohomology ring R^n_{OG} with a presentation (see, e.g., [3, Thm. 1] with $q = 0$) extremely similar to that of R^n_{LG}, and in fact it has the same Hilbert series. For example,

$$R^4_{OG} = \mathbb{Q}[e_1, e_2, e_3, e_4]/(e_1^2 - e_2, e_2^2 - 2e_1e_3 + e_4, e_3^2 - 2e_2e_4, e_4^2),$$
$$R^4_{LG} = \mathbb{Q}[e_1, e_2, e_3, e_4]/(e_1^2 - 2e_2, e_2^2 - 2e_1e_3 + \mathbf{2}e_4, e_3^2 - 2e_2e_4, e_4^2).$$

Proposition 2.11. *The subalgebras $R^{n,m}_{OG}, R^{n,m}_{LG}$ have the same Hilbert series.*

Proof. We claim that there should be an isomorphism of graded rings

$$R^n_{OG} \cong R^n_{LG}, \tag{2.6}$$

meaning a ring isomorphism that preserves degrees. If we knew this, then this isomorphism would restrict to an isomorphism of their subalgebras

$$R^{n,m}_{OG} \cong R^{n,m}_{LG},$$

since each of $R^{n,m}_{OG}, R^{n,m}_{LG}$, is defined as the subalgebra of R^n_{OG}, R^n_{LG}, respectively, generated by its elements of degree at most m.

The isomorphism (2.6) comes from Borel's picture[5] of the cohomology of the Lagrangian Grasmmannian X and the Orthogonal Grassmannian X'. One can think of both X, X' as *homogeneous spaces*, that is, quotient spaces G/P_J where G is a complex semisimple algebra group, and P_J is a parabolic subgroup generated by the Borel subgroup of G together with a certain subset J of generators for the Weyl group W of G. Borel showed that the cohomology of G/P has this description:

$$H^*(G/P, \mathbb{Q}) \cong H^*(G/B, \mathbb{Q})^{W_J},$$

that is, the subring of W_J-invariant elements inside $H^*(G/B, \mathbb{Q})$, where W_J is the subgoup of W generated by J. He also showed that the cohomology $H^*(G/B, \mathbb{Q})$ is isomorphic to the quotient

$$\mathbb{Q}[x_1, \ldots, x_n]/(\mathbb{Q}[x_1, \ldots, x_n]^W_+)$$

of the polynomial ring $\mathbb{Q}[x_1, \ldots, x_n]$ by the ideal generated by the W-invariant polynomials $\mathbb{Q}[x_1, \ldots, x_n]^W_+$ of positive degree.

One has $X = Sp_{2n}/P$ and $X' = SO(2n+1)/P'$, and the groups $Sp_{2n}, SO(2n+1)$ share the same Weyl group W, isomorphic to the *hyperoctahedral group* B_n of all $n \times n$ signed permutation matrices. One also has that P, P' both correspond to the same subset of generators for W (generating the symmetric group $W_J = S_n$ inside the hyperoctahedral group $W = B_n$). Thus we should have both rings R^n_{OG}, R^n_{LG} isomorphic to this graded ring:

$$\Big(\mathbb{Q}[x_1, \ldots, x_n]/(\mathbb{Q}[x_1, \ldots, x_n]^{B_n}_+)\Big)^{S_n}.$$

[4]Alternatively, one can use a Pieri formula [4, eqn. (52)] for R^n_{LG} to compute that e_1^N is a certain nonzero scalar multiple of the orientation class of the Lagrangian Grassmannian, given by an explicit power of 2 times the number of standard tableaux of shifted staircase shape.

[5]See [6] for some references, and its Section 3 for some of the assertions.

□

Remark 2.12. Let us explain here how this Borel picture fits with our presentation of the cohomology of the Grassmannian $Gr(k, k+\ell)$ as the ring $R^{k,\ell}$. Let us freely use the notation $n := k+\ell$ here, so that $\ell = n-k$.

In this case, the complex semisimple[6] algebraic group $G = GL_n(\mathbb{C})$, the group of all invertible $n \times n$ complex matrices. One starts by thinking about how G acts on the *(complete) flag manifold*

$$Fl_n := \{(V_1, V_2, \dots, V_{n-1}) : \{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n \text{ and } \dim_{\mathbb{C}}(V_i) = i\}$$

where each V_i is a $\mathbb{C}$-linear subspace of $\mathbb{C}^n$. Here G acts transitively on Fl_n (that is, with only one orbit), and if we choose the particular base flag having $V_i = \mathrm{span}_{\mathbb{C}}\{e_1, e_2, \dots, e_i\}$, it is stabilized by the Borel subgroup B consisting of all upper triangular matrices. Hence one can identity $Fl_n = G/B$, a homogeneous space for G. In this case, the Weyl group W is the symmetric group S_n of permutation matrices, and the first part of Borel's picture tells us that

$$\begin{aligned} H^*(FL_n, \mathbb{Q}) = H^*(G/B, \mathbb{Q}) &\cong \mathbb{Q}[x_1, \dots, x_n]/(\mathbb{Q}[x_1, \dots, x_n]_+^{S_n}) \\ &= \mathbb{Q}[x_1, \dots, x_n]/(e_1, e_2, \dots, e_n) \end{aligned}$$

where we have used (Newton's) fundamental theorem of symmetric functions that says the subalgebra $\mathbb{Q}[x_1, \dots, x_n]^{S_n}$ of symmetric polynomials within $\mathbb{Q}[x_1, \dots, x_n]$ is itself again a polynomial algebra generated by the elementary symmetric functions $e_1, \dots, e_n$.

So how do we think of the Grassmannian $Gr(k, \mathbb{C}^{k+\ell}) = Gr(k, \mathbb{C}^n)$? It is the homogeneous space G/P where P is the subgroup of block upper triangular invertible matrices $\left[\begin{smallmatrix} A & B \\ 0 & C \end{smallmatrix}\right]$ where A lies in $GL_k(\mathbb{C})$, C lies in $GL_{n-k}(\mathbb{C})$, and B lies in $\mathbb{C}^{k \times (n-k)}$; this is exactly the subgroup that stabilizes the particular k-dimensional subspace $\mathrm{span}_{\mathbb{C}}\{e_1, e_2, \dots, e_k\}$ from the base flag. To use the rest of Borel's picture to understand its cohomology, we note that $P = P_J$ is the parabolic subgroup G generated by the Borel subgroup B together with the subset J inside $W = S_n$ that generates $W_J = S_k \times S_\ell$, where S_k permutes only $\{1, 2, \dots, k\}$ among themselves and S_ℓ permutes only $\{k+1, k+2, \dots, k+\ell (= n)\}$ among themselves. Consequently, Borel tells us the following, to be explained line-by-line below, using the abbreviation

$$e_i(\mathbf{x}_{[a,b]}) := e_i(x_a, x_{a+1}, \dots, x_b)$$

for elementary symmetric functions in a variable set $x_a, x_{a+1}, \dots, x_b$:

Proposition 2.13. $H^*(Gr(k, n)) =: R^{k,\ell}$

[6]Actually, to be really semisimple, we should take $G = SL_n(\mathbb{C})$, but this will have no material effect on the discussion.

Proof.

$$
\begin{aligned}
H^*(Gr(k,n)) &\overset{(1)}{=} H^*(G/P_J)\\
&\overset{(2)}{\cong} \left(\mathbb{Q}[x_1,\ldots,x_n]/(\mathbb{Q}[x_1,\ldots,x_n]_+^W)\right)^{W_J}\\
&\overset{(3)}{\cong} \mathbb{Q}[x_1,\ldots,x_n]^{W_J}/(\mathbb{Q}[x_1,\ldots,x_n]_+^W)\\
&\overset{(4)}{=} \mathbb{Q}[x_1,\ldots,x_n]^{S_k\times S_\ell}/(\mathbb{Q}[x_1,\ldots,x_n]_+^{S_n})\\
&\overset{(5)}{=} \mathbb{Q}[e_1(\mathbf{x}_{[1,k]}), e_2(\mathbf{x}_{[1,k]}),\ldots,e_k(\mathbf{x}_{[1,k]}),\\
&\qquad e_1(\mathbf{x}_{[k+1,k+\ell]}), e_2(\mathbf{x}_{[k+1,k+\ell]}),\ldots,e_k(\mathbf{x}_{[k+1,k+\ell]})]\\
&\qquad\quad /(e_1(\mathbf{x}_{[1,n]}), e_2(\mathbf{x}_{[1,n]}),\ldots,e_n)(\mathbf{x}_{[1,n]})\\
&\overset{(6)}{\cong} \mathbb{Q}[e_1,\ldots,e_k,h_1,\ldots,h_\ell]/\left(\sum_{i=0}^{d}(-1)^i e_i h_{d-i}\right)_{d=0,1,\ldots,k+\ell}\\
&\overset{(7)}{=}: R^{k,\ell}.
\end{aligned}
$$

∎

The equality (1) is simply because $Gr(k,n)$ is the homogeneous space G/P_J, as explained above.

The isomorphism (2) is Borel's picture for the cohomology of G/P_J in general.

The isomorphism (3) is actually trickier than it looks. It uses the fact that we are working with coefficients in $\mathbb{Q}$ that have characteristic zero. There is a general averageing argument that one can use to show that, in characteristic zero, if W is a finite group acting on a graded ring S, and W' a subgroup of W, then the composite map

$$S^{W'} \hookrightarrow S \twoheadrightarrow S/(S_+^W).$$

surjects onto the W'-invariants in the quotient $\left(S/(S_+^W)\right)^{W'}$, and has kernel equal to the ideal (S_+^W) inside S^W, so it induces an isomorphism

$$S^{W'}/(S_+^W) \cong \left(S/(S_+^W)\right)^{W'}.$$

This isomorphism is what gets used in (3), with $S=\mathbb{Q}[x_1,\ldots,x_n]$ and $W'=W_J$.

The equality (4) is just reminding us who W and W_J are in this instance.

The equality (5) uses a variant on Newton's fundamental theorem of symmetric functions, saying that the $S_k\times S_\ell$-invariant polynomials inside $\mathbb{Q}[x_1,\ldots,x_n]$ are generated by elementary symmetric polynomials in the first k-variables and in the last ℓ variables.

The isomorphism (6) is induced by sending

$$
\begin{aligned}
e_i &\mapsto (-1)^i e_i(\mathbf{x}_{[1,k]}) \text{ for } i=1,2,\ldots,k\\
h_j &\mapsto e_j(\mathbf{x}_{[k+1,k+\ell]}) \text{ for } j=1,2,\ldots,\ell
\end{aligned}
$$

and noting the easy identity

$$e_d(\mathbf{x}_{[1,n]}) = e_d(x_1, \dots, x_n) = \sum_{i=0}^{d} e_i(x_1, x_2, \dots, x_k) e_{d-i}(x_{k+1}, x_{k+2}, \dots, x_n)$$
$$= \sum_{i=0}^{d} e_i(\mathbf{x}_{[1,k]}) e_{d-i}(\mathbf{x}_{[k+1,k+\ell]}).$$

The equality (7) was our definition of $R^{k,\ell}$.

Lemma 2.14. *$Gr(k, \mathbb{C}^{k+\ell})$ is homeomorphic to $Gr(\ell, \mathbb{C}^{k+\ell})$.*

Proof. Points in $Gr(k, \mathbb{C}^{k+\ell})$ are the k-dimensional $\mathbb{C}$-linear subspaces of $\mathbb{C}^{k+\ell}$. Let V be a $(k+\ell)$-dimensional complex vector space, and let W be a k-plane of V (i.e. a k-dimensional linear subspace of V), then $(V/W)^*$ is an ℓ-plane of V^*. This allows us to say that every k-dimensional subspace W of V determines an ℓ-dimensional subspace $(V/W)^*$ of V^*. More precisely, from the short exact sequence (or SES)[7]:

$$0 \longrightarrow W \xrightarrow{f} V \xrightarrow{g} V/W \longrightarrow 0 \tag{2.7}$$

we can take the dual to get another SES

$$0 \longrightarrow (V/W)^* \xrightarrow{g^*} V^* \xrightarrow{f^*} W^* \longrightarrow 0. \tag{2.8}$$

Since the double dual V^{**} is naturally isomorphic to V. Taking the dual of the SES in (2) again yields the SES in (1). The above procedure describes a 1-to-1 correspondence between k-dimensional subspaces of V, and ℓ-dimensional subspaces of V^*. In terms of the Grassmannian this means:

$$Gr(k, V) \cong Gr(\ell, V^*). \tag{2.9}$$

Because V is a finite dimensional vector space, V and V^* are isomorphic. Choosing an isomorphism form V to V^* determines an isomorphism from $Gr(\ell, V)$ to $Gr(\ell, V^*)$. This in turn yields an isomorphism between $Gr(\ell, V)$ and $Gr(k, V)$, i.e. $Gr(k, \mathbb{C}^{k+\ell}) \cong Gr(\ell, \mathbb{C}^{k+\ell})$.[8] □

Proposition 2.15. *The map of polynomial rings*

$$\omega : \mathbb{Q}[e_1, \dots, e_k, h_1, \dots, h_\ell] \longrightarrow \mathbb{Q}[e_1, \dots, e_\ell, h_1, \dots, h_k]$$

that sends $e_i \mapsto h_i$ for $i = 1, 2, \dots, k$ and $h_j \mapsto e_j$ for $j = 1, 2, \dots, \ell$ induces a graded ring isomorphism $R^{k,\ell} \cong R^{\ell,k}$.

Proof. This follows from examination of the ring presentation (2.1). □

Remark 2.16. There are at least two other ways to think of this graded ring isomorphism $\omega : R^{k,\ell} \to R^{\ell,k}$.

On one hand, one can view $R^{k,\ell}$ as the quotient of the ring of symmmetric functions $\Lambda_{\mathbb{Q}}$ in infinitely many variables, in which one quotients by the $\mathbb{Q}$-linear span of the Schur functions s_λ for which λ does not lie inside a $k \times \ell$ rectangle. Then

---

[7] f is injective because it is the inclusion map; g is clearly surjective; moreover, $\mathrm{Im}(f) = W = \mathrm{Ker}(g)$. Satisfying these properties makes the sequence in (1) to be a so-called **short exact sequence**.

[8] Remark: The term "isomorphism" is being utilised in the context of the category of topological spaces, the Grassmannian being an object in this category. And, in this category, isomorphisms are homeomorphisms.

it is not hard to see that $\omega : R^{k,\ell} \to R^{k,\ell}$ above is induced from the *fundamental involution* ω on $\Lambda_{\mathbb{Q}}$, which swaps $e_i \leftrightarrow h_i$ for $i = 1, 2, \ldots$, and $s_\lambda \leftrightarrow s_{\lambda'}$, where λ' is the transpose or conjugate partition of λ.

On the other hand, it can also be shown that $\omega : R^{k,\ell} \to R^{k,\ell}$ is the ring isomorphism induced by the homeomorphism in Proposition 2.13.

Proposition 2.17. *The set* $\{h_\lambda : \lambda \subset (\ell^k)\}$ *is a basis for* $R^{k,\ell}$.

Proof. By [11, Thm. 7.4.4], $\{e_\lambda : \lambda \subset (\ell^k)\}$ is a basis of $R^{k,\ell}$. Now apply ω from Proposition 2.15, which sends $e_\lambda \mapsto h_\lambda$. □

3. A Gröbner Basis Approach

Perhaps the most brute force approach to proving the Reiner-Tudose Conjecture is by appealing to Gröbner bases. While this approach ultimately failed to prove the conjecture it is still worthwhile to examine what went wrong. Recall that

$$R^{k,\ell} \cong \mathbb{Q}[e_1, e_2, \ldots, e_k, h_1, h_2, \ldots, h_\ell]/I^{k,\ell}$$

where we have

$$I^{k,\ell} = \left(\sum_{i+j=d} (-1)^i e_i h_j : d = 1, 2, \ldots, k+\ell \right).$$

Throughout section 3, we will fix the monomial ordering to be lexicographic with $e_1 < e_2 < \cdots < e_m$. There are two main ways to go about using Gröbner Bases, each of which focus on one side of the Reiner-Tudose Conjecture. The first, as done in section 3.1, is to find a $\mathbb{Q}$-basis for $R^{k,\ell}$ via Gröbner Basis calculations of $I^{k,\ell}$ and trying to understand in general what form the standard monomials take. The second, as done in section 3.2, is to examine the conjectured Hilbert series and from it, try and derive what the standard monomials must be. The latter relies on an algorithmic way to go from the conjectured Hilbert series to a set of monomials.

3.1. Patterns, Patterns and more Patterns.

For this approach we would need the general structure of the leading terms of a Gröbner Basis of $I^{k,\ell}$. Using a computer algebra system, e.g., Macaulay2 we were able to conjecture the structure of the leading terms of the Gröbner Basis for $I^{k,\ell}$ for $k = 2, 3$.

Conjecture 3.1. *For an ideal I we will let $\mathcal{G}(I)$ denote a Gröbner basis of I. We have that*

$$LT(\mathcal{G}(I^{2,\ell})) = \left\{ \begin{matrix} e_1^{2\ell+1-4i} e_2^i \quad \text{for } 0 \le i \le \lfloor \frac{\ell}{2} \rfloor, \\ \\ e_2^{\lfloor \frac{\ell}{2} \rfloor + 1} \end{matrix} \right\}$$

$$LT(\mathcal{G}(I^{3,\ell})) = \left\{ \begin{array}{cc} e_1^{3\ell+1}, & \\ e_1^{3(\ell-i)}e_2^{i} & \textit{for } 1 \leq i \leq \ell-1, \\ e_1^{\ell-3i}e_3^{i} & \textit{for } 1 \leq i \leq \left\lfloor \frac{\ell-1}{3} \right\rfloor, \\ e_1^{3i}e_2^{\ell-(4+3j)-(i-1)}e_3^{j+1} & \textit{for } 1 \leq i \leq \ell-(4+3j),\ \textit{for } j \in \mathbb{N} \textit{ and if } \ell > 4+3j, \\ e_2^{\ell-3i}e_3^{i} & \textit{for } 1 \leq i \leq \left\lfloor \frac{\ell}{3} \right\rfloor \textit{ and } \ell \neq 3, \\ e_2e_3^{\left\lfloor \frac{\ell}{3} \right\rfloor} & \textit{if } \ell \equiv 0 \textit{ (mod 3)}, \\ e_2^{\ell}, & \\ e_3^{\left\lfloor \frac{\ell}{3} \right\rfloor+1} & \end{array} \right\}$$

We will first focus on $LT(\mathcal{G}(I^{2,\ell}))$. Recall,

$$R^{2,\ell} = \mathbb{Q}[e_1, e_2]/(h_{\ell+1}, h_{\ell+2})$$

where h_r is defined as a polynomial in e_1, e_2 via an $r \times r$ Jacobi-Trudi determinant.

Lemma 3.2. *Let J_d be the d-th Jacobi-Trudi Matrix. Then we have that*

$$\sum_{i=1}^{d}(-1)^{i-1}e_i \det(J_{d-i}) = \det(J_d)$$

Proof. Let $J_{d,i}$ be the $(d-1)\times(d-1)$ matrix obtained from J_d by omitting the first row and i-th column. By definition of determinant, we have

$$\det(J_d) = \sum_{i=1}^{d}(-1)^{i+1}e_i \det(J_{d,i}).$$

To complete the proof, it suffices to show

$$\det(J_{d,i}) = \det(J_{d-i}),$$

for arbitrary d and $1 \leq i \leq d$. Indeed, when $d = i = 1$, both $J_{d,i}$ and J_{d-i} are the empty 0×0 matrix, which has determinant 1. When d is arbitrary and $i = 1$, we see that $J_{d,1} = J_{d-1}$.

Now suppose $\det(J_{n,i}) = \det(J_{n-i})$ for all $n < d$ and $1 \leq i \leq n$. We can notice that

$$\det(J_{d,i}) = \det(J_{d-1,i-1}),$$

if $1 < i \leq d$. This is seen by expanding along the first column of $J_{d,i}$ which contains a single entry with the value 1 and the rest 0's. Finally,

$$\det(J_{d-1,i-1}) = \det(J_{d-i})$$

by the induction hypothesis. Therefore,

$$\det(J_{d,i}) = \det(J_{d-i}),$$

as required. □

Proposition 3.3. *For $R^{k,\ell}$ we have that $e_k^{\lfloor \frac{\ell}{k} \rfloor + 1}$ is the lead monomial of one of $h_{\ell+1}, \dots, h_{\ell+k}$ with respect to lexicographical ordering.*

Proof. We will first choose $\alpha \in \{\ell+1, \dots, \ell+k\}$ such that $\alpha = c \cdot k$ for some $c \in \mathbb{N}$. Let J_α denote the α-th Jacobi-Trudi Matrix. Then from Lemma 3.2 we have that

$$\begin{aligned} h_\alpha &= \det(J_\alpha) \\ &= \sum_{i=1}^{\alpha} (-1)^{i-1} e_i \det(J_{\alpha-i}) \\ &= (-1)^{k-1} e_k \det(J_{\alpha-k}) + \sum_{\substack{i=1 \\ i \neq k}}^{\alpha} (-1)^{i-1} e_i \det(J_{\alpha-i}) \end{aligned}$$

where

$$(-1)^{k-1} e_k \det(J_{\alpha-k}) >_{lex} \sum_{\substack{i=1 \\ i \neq k}}^{\alpha} (-1)^{i-1} e_i \det(J_{\alpha-i})$$

since $e_{k+1}, \dots, e_\alpha = 0$. Thus, we can see that

$$LM(h_\alpha) = e_k \cdot LM(h_{\alpha-k}). \tag{3.1}$$

It directly follows from Lemma 3.2 that $LM(h_k) = e_k$. Thus, using the recurrence given in equation 3.1, we see that

$$\begin{aligned} LM(h_\alpha) &= e_k^{\frac{\alpha-k}{k}} \cdot LM(h_{\alpha-(\alpha-k)}) \\ &= e_k^{\frac{\alpha}{k}-1} \cdot e_k \\ &= e_k^{\frac{\alpha}{k}}. \end{aligned}$$

However, since $\ell < \alpha$ and $\alpha = c \cdot k$ we have that

$$\frac{\alpha}{k} = \left\lfloor \frac{\ell}{k} \right\rfloor + 1$$

and we can conclude that

$$LM(h_\alpha) = e_k^{\frac{\alpha}{k}} = e_k^{\lfloor \frac{\ell}{k} \rfloor + 1}$$

as desired. □

We can see that Proposition 3.3 implies that $e_2^{\lfloor \frac{\ell}{2} \rfloor + 1} \in LT(I^{2,\ell})$. We now want an inductive scheme that produces the rest of the elements

$$e_1^{2\ell+1-4i} e_2^i \quad \text{for } 1 \leq i \leq \lfloor \frac{\ell}{2} \rfloor$$

as leading terms $LT(f)$ for some $f \in I^{2,\ell} = (h_{\ell+1}, h_{\ell+2})$.

The following is an attempt to find such an inductive scheme using a two step recursive formula defined in Proposition 3.4. The proof of Proposition 3.4 relies heavily on Lemmas 3.5, 3.6 and 3.7. While two of these lemmas have been proven we have so far failed to prove Lemma 3.6. As described below in some sense what one would need to prove this lemma illuminates the extremely hands on nature of a Gröbner basis approach as even for the case $k = 2$ it is extremely hands on. While, we are optimistic someone could modify Proposition 3.4 to allow one to prove Lemma 3.6, it seems that any hope of doing this for higher k would be fruitless. It is for this reason that we don't attempt to prove the conjectured structure of $LT(\mathcal{G}(I^{3,\ell})$. However, we will carry out the following attempt to illustrate it's difficulty and hands on nature.

Proposition 3.4. *For a polynomial q we will let $LT_i(q) = LC_i(q) \cdot LM_i(q)$ denote the i-th leading term of q. For ℓ even we have*

$$h_{\ell+1} \qquad \textit{where} \quad LM_1(h_{\ell+1}) = e_1 e_2^{\frac{\ell}{2}}$$

$$f_1 = \tfrac{e_2 h_{\ell+1}}{LC_1(h_{\ell+1})} - \tfrac{e_1 h_{\ell+2}}{LC_1(h_{\ell+2})}$$

$$f_2 = \tfrac{e_1^2 h_{\ell+1}}{LC_1(h_{\ell+1})} - \tfrac{f_1}{LC_1(f_1)} \qquad \textit{where} \quad LM_1(f_2) = e_1^5 e_2^{\frac{\ell}{2}-1}$$

and for ℓ odd we have that

$$f_0 = h_{\ell+2}$$

$$f_1 = \tfrac{e_1 h_{\ell+1}}{LC_1(h_{\ell+1})} - \tfrac{f_0}{LC_1(f_0)} \qquad \textit{where} \quad LM_1(f_1) = e_1^3 e_2^{\frac{\ell-1}{2}}.$$

Then for $j \in \{3, 5, \ldots, \ell - 1\}$ and ℓ even or $j \in \{2, 4, \ldots, \ell - 1\}$ and ℓ odd we have that

$$f_j = \tfrac{e_1^2 f_{j-2}}{LC_1(f_{j-2})} - \tfrac{e_2 f_{j-1}}{LC_1(f_{j-1})}$$

$$f_{j+1} = \tfrac{e_1^2 f_{j-1}}{LC_1(f_{j-1})} - \tfrac{f_j}{LC_1(f_j)} \qquad \textit{where} \quad LM_1(f_{j+1}) = e_1^{2(j+1)+1} e_2^{\frac{\ell}{2}-\frac{j}{2}-\frac{1}{2}}$$

Lemma 3.5. *For any $\ell \in \mathbb{N}$ and $k = 2$, we have*

$$h_\ell = \sum_{i=0}^{\lfloor \frac{\ell}{2} \rfloor} (-1)^i \binom{\ell - i}{i} e_1^{\ell-2i} e_2^i.$$

Proof. We will proceed using induction. For the case of $\ell = 1$, we have that

$$h_1 = e_1 = \sum_{i=0}^{\lfloor \frac{1}{2} \rfloor} (-1)^i \binom{1 - i}{i} e_1^{1-2i} e_2^i.$$

Now assume that this proposition holds for all j such that $1 \le j \le \ell$. We now want to prove that it holds for $\ell + 1$. We know that $h_{\ell+1} = e_1 \cdot h_\ell - e_2 \cdot h_{\ell-1}$. By our induction hypothesis, we then have that

$$\begin{aligned}
h_{\ell+1} &= e_1 \cdot h_\ell - e_2 \cdot h_{\ell-1}\\
&= e_1 \cdot \left(\sum_{i=0}^{\lfloor \frac{\ell}{2} \rfloor} (-1)^i \binom{\ell-i}{i} e_1^{\ell-2i} e_2^i\right) - e_2 \cdot \left(\sum_{j=0}^{\lfloor \frac{\ell-1}{2} \rfloor} (-1)^j \binom{\ell-1-j}{j} e_1^{\ell-1-2j} e_2^j\right)\\
&= \left(\sum_{i=0}^{\lfloor \frac{\ell}{2} \rfloor} (-1)^i \binom{\ell-i}{i} e_1^{\ell-2i+1} e_2^i\right) - \left(\sum_{j=0}^{\lfloor \frac{\ell-1}{2} \rfloor} (-1)^j \binom{\ell-1-j}{j} e_1^{\ell-1-2j} e_2^{j+1}\right)\\
&= \left[\binom{\ell}{0} e_1^{\ell+1} - \binom{\ell-1}{1} e_1^{\ell-1} e_2 + \cdots + (-1)^{\lfloor \frac{\ell}{2} \rfloor} \binom{\ell - \lfloor \frac{\ell}{2} \rfloor}{\lfloor \frac{\ell}{2} \rfloor} e_1^{\ell - 2\lfloor \frac{\ell}{2} \rfloor + 1} e_2^{\lfloor \frac{\ell}{2} \rfloor}\right]\\
&- \left[\binom{\ell-1}{0} e_1^{\ell-1} e_2 - \binom{\ell-2}{1} e_1^{\ell-3} e_2^2 + \cdots + (-1)^{\lfloor \frac{\ell-1}{2} \rfloor} \binom{\ell-1-\lfloor \frac{\ell-1}{2} \rfloor}{\lfloor \frac{\ell-1}{2} \rfloor} e_1^{\ell-2\lfloor \frac{\ell-1}{2} \rfloor - 1} e_2^{\lfloor \frac{\ell-1}{2} \rfloor + 1}\right].
\end{aligned}$$

We have two cases depending upon the parity of ℓ. First, assume $\ell = 2p+1$ where $p = \lfloor \frac{\ell}{2} \rfloor$. Then we have that

$$\begin{aligned}
h_{\ell+1} =& \binom{\ell}{0} e_1^{\ell+1} - \left[\binom{\ell-1}{0} + \binom{\ell-1}{1}\right] e_1^{\ell-1} e_2 + \cdots + (-1)^p \left[\binom{\ell-p}{p-1} + \binom{\ell-p}{p}\right] e_1^{\ell-2p+1} e_2^p\\
&+ (-1)^{p+1} \binom{\ell-1-p}{p} e_1^{\ell-2p-1} e_2^{p+1}.
\end{aligned}$$

Within each coefficient, we have by Pascal's binomial identity that

$$\begin{aligned}
h_{\ell+1} &= \binom{\ell+1}{0} e_1^{\ell+1} - \binom{\ell}{1} e_1^{\ell-1} e_2 + \cdots + (-1)^{p+1} \binom{\ell-1-p}{p} e_1^{\ell-2p-1} e_2^{p+1}\\
&= \binom{\ell+1}{0} e_1^{\ell+1} - \binom{\ell}{1} e_1^{\ell-1} e_2 + \cdots + (-1)^{p+1} \binom{\ell+1-(p+1)}{p+1} e^{\ell-2p-1} e_2^{p+1}\\
&= \sum_{i=0}^{\lfloor \frac{\ell+1}{2} \rfloor} (-1)^i \binom{\ell+1-i}{i} e_1^{\ell+1-2i} e_2^i
\end{aligned}$$

proving the case for ℓ odd.

Now, suppose that $\ell = 2p$ and $p = \lfloor \frac{\ell}{2} \rfloor$. Thus, we have that

$$\begin{aligned}
h_{\ell+1} &= \left[\binom{\ell}{0} e_1^{\ell+1} - \binom{\ell-1}{1} e_1^{\ell-1} e_2 + \cdots + (-1)^p \binom{\ell-p}{p} e^{\ell-2p+1} e_2^p\right]\\
&- \left[\binom{\ell-1}{0} e_1^{\ell-2} e_2 - \binom{\ell-2}{1} e_1^{\ell-3} e_2^2 + \cdots + (-1)^{p-1} \binom{\ell-1-(p-1)}{p-1} e_1^{\ell-2(p-1)-1} e_2^{(p-1)+1}\right]\\
&= \binom{\ell}{0} e_1^{\ell+1} - \left[\binom{\ell-1}{0} + \binom{\ell-1}{1}\right] e_1^{\ell-1} e_2 + \cdots + (-1)^p \left[\binom{\ell-p}{p} + \binom{\ell-p}{p-1}\right] e_1^{\ell-2p+1} e_2^p.
\end{aligned}$$

Similarly by Pascal's binomial identity, it follows that

$$\begin{aligned}
h_{\ell+1} &= \binom{\ell+1}{0} e_1^{\ell+1} - \binom{\ell}{1} e_1^{\ell-1} e_2 + \binom{\ell-1}{2} e_1^{\ell-3} e_2^2 - \cdots + (-1)^p \binom{\ell-p+1}{p} e_1^{\ell-2p+1} e_2^p\\
&= \sum_{i=0}^{\lfloor \frac{\ell}{2} \rfloor} \binom{\ell+1-i}{i} (-1)^i e_1^{\ell+1-2i} e_2^i.
\end{aligned}$$

Since ℓ was assumed to be even we have that

$$\left\lfloor \frac{\ell}{2} \right\rfloor = \left\lfloor \frac{\ell+1}{2} \right\rfloor$$

and so in both cases we get that

$$h_{\ell+1} = \sum_{i=0}^{\lfloor \frac{\ell+1}{2} \rfloor} \binom{\ell+1-i}{i} (-1)^i e_1^{\ell+1-2i} e_2^i.$$

Thus, we can conclude that

$$h_\ell = \sum_{i=0}^{\lfloor \frac{\ell}{2} \rfloor} \binom{\ell-i}{i} (-1)^i e_1^{\ell-2i} e_2^i \qquad , \forall\ \ell \geq 1.$$

□

Lemma 3.6. (Still Conjectural) *Let f_i, for $i \in [\ell]$, be as defined in Proposition 3.4 and let $|f_i|$ be the number of monomials in f_i. Then we have that*

$$|f_i| = \left\lfloor \frac{\ell}{2} \right\rfloor - \left\lfloor \frac{i}{2} \right\rfloor + 1, \qquad \textit{for } \ell \textit{ even}$$
$$|f_i| = \left\lfloor \frac{\ell}{2} \right\rfloor - \left\lfloor \frac{i-1}{2} \right\rfloor + 1, \qquad \textit{for } \ell \textit{ odd}$$

As mentioned above we have so far failed to prove Lemma 3.6. To overcome this obstacle, one would need to reformulate Proposition 3.4 to either write explicitly what the coefficients of the f_i's are, or write f_i in terms of $h_{\ell+1}$ and $h_{\ell+2}$ instead of f_{i-1} and f_{i-2}. While the latter seems easier, finding such expressions have proven to be very difficult.

Lemma 3.7. *Let $f_j = \sum_i \gamma_i \cdot e_2^{\rho_i} e_1^{\mu_i}$, for $j \in [\ell]$, be as defined in Proposition 3.4. If one ordered the monomials $e_2^{\rho_i} e_1^{\mu_i}$ in $>_{lex}$ order from greatest to least, then the powers on e_2 would decrease by 1 as you went down the list and the powers of e_1 would increase by 2.*

Proof. For a polynomial q we will let $LT_i(q) = LC_i(q) \cdot LM_i(q)$ denote the i-th leading term of q. No matter the parity of ℓ it follows directly from Lemma 3.5 that $h_{\ell+1}$ and $h_{\ell+2}$ have the desired property.

For ℓ even Lemma 3.6 tells use that f_1 has $\frac{\ell}{2} + 1$ monomials where

$$LM_i(f_1) = LM_{i+1}(e_1 \cdot h_{\ell+2})$$

and f_2 and f_3 have $\frac{\ell}{2}$ monomials where

$$LM_i(f_2) = LM_{i+1}(f_1) = LM_{i+2}(e_1 \cdot h_{\ell+2})$$
$$LM_i(f_3) = e_1^2 \cdot LM_i(f_1) = LM_{i+1}(e_1^3 \cdot h_{\ell+2}).$$

Similary for ℓ odd Lemma 3.6 tells use that f_1 has $\frac{\ell-1}{2} + 1$ monomials where

$$LM_i(f_1) = LM_{i+1}(h_{\ell+2})$$

and f_2 has $\frac{\ell-1}{2}+1$ monomials where

$$LM_i(f_2) = LM_{i+1}(e_1^2 \cdot f_0) = LM_{i+1}(e_1^2 \cdot h_{\ell+2}).$$

Thus, we can see that for f_1, f_2, f_3 for ℓ even and f_1, f_2 for ℓ odd, that if one ordered the monomials in $>_{lex}$ order from greatest to least, the powers on e_2 would decrease by 1 as you went down the list and the powers of e_1 would increase by 2 since this property is also held by $h_{\ell+2}$.

Now, for ℓ even assume that for all $m < j$, for j odd, that f_m has the desired property. We now want to show that f_j and f_{j+1} have this desired property as well. Lemma 3.6 tells use that

$$\begin{aligned} LM_i(f_j) &= LM_i(e_1^2 \cdot f_{j-2}) \\ LM_{i'}(f_{j+1}) &= LM_{i'+1}(e_1^2 \cdot f_{j-1}) \end{aligned}$$

for $i \in [\frac{\ell}{2} - \frac{j-1}{2} + 1]$ and $i' \in [\frac{\ell}{2} - \frac{j+1}{2} + 1]$ and since by assumption f_{j-1} and f_{j-2} has the desired property we can conclude that f_j and f_{j+1} do as well. Lastly, for ℓ odd if we replace the assumption above with, $\forall m < j$, for j even, we have that f_m has the desired property, then it follows in a similar manner that f_j and f_{j+1} do as well. $\square$

Proof of Proposition 3.4. Using Lemmas 3.5 and 3.6 it is not hard to check that for ℓ even we have that $LM_1(h_{\ell+1}) = e_1 e_2^{\frac{\ell}{2}}$ and that $LM_1(f_2) = e_1^5 e_2^{\frac{\ell}{2}-1}$. In the same manner for ℓ odd, using Lemmas 3.5 and 3.6 it is not hard to see that $LM_1(f_1) = e_1^3 e_2^{\frac{\ell-1}{2}}$. We will now proceed using induction on j.

For ℓ even, the base case is when $j = 3$ and we want to show that $LM_1(f_4) = e_1^9 e_2^{\frac{\ell}{2}-2}$. We have that

$$f_3 = \frac{e_1^2 f_1}{LC_1(f_1)} - \frac{e_2 f_2}{LC_1(f_2)} \qquad f_4 = \frac{e_1^2 f_2}{LC_1(f_2)} - \frac{f_3}{LC_1(f_3)}$$

We know from above that $LM_1(f_2) = e_1^5 e_2^{\frac{\ell}{2}-1}$ and it is not hard to see that $LM_1(f_1) = e_1^3 e_2^{\frac{\ell}{2}}$. Because $LM_1(e_1^2 \cdot f_1) = LM_1(e_2 \cdot f_2)$, Lemma 3.6 tells us that we must have $LM_1(f_3) = e_2 \cdot LM_2(f_2)$. Using this fact and Lemma 3.7 we have that

$$\begin{aligned} LM_1(f_2) &= e_1^5 e_2^{\frac{\ell}{2}-1} \\ LM_2(f_2) &= e_1^7 e_2^{\frac{\ell}{2}-2} \\ LM_1(f_3) &= e_1^7 e_2^{\frac{\ell}{2}-1} \\ LM_2(f_3) &= e_1^9 e_2^{\frac{\ell}{2}-2}. \end{aligned}$$

Thus,

$$\begin{aligned} f_4 &= e_1^2 \cdot LM_1(f_2) + \frac{e_1^2 \cdot LT_2(f_2)}{LC_1(f_2)} - LM_1(f_3) - \frac{LT_2(f_3)}{LC_1(f_3)} + \text{ lex smaller terms} \\ &= c \cdot e_1^9 e_2^{\frac{\ell}{2}-2} + \text{lex smaller terms.} \end{aligned}$$

where

$$c = \left(\frac{LC_2(f_2)}{LC_1(f_2)} - \frac{LC_2(f_3)}{LC_1(f_3)}\right) \neq 0$$

by Lemma 3.6 allowing us to conclude that $LM_1(f_4) = e_1^9 e_2^{\frac{\ell}{2}-2}$.

For ℓ odd, the base case is when $j = 2$ and we want to show that $LM_1(f_3) = e_1^7 e_2^{\frac{\ell-1}{2}-1}$. Using Lemmas 3.6 and 3.7 the same line of reasoning used above gives us

$$\begin{aligned}
LM_1(f_1) &= e_1^3 e_2^{\frac{\ell-1}{2}} \\
LM_2(f_1) &= e_1^5 e_2^{\frac{\ell-1}{2}-1} \\
LM_1(f_2) &= e_1^5 e_2^{\frac{\ell-1}{2}} \\
LM_2(f_2) &= e_1^7 e_2^{\frac{\ell-1}{2}-1}.
\end{aligned}$$

Thus,

$$\begin{aligned}
f_3 &= e_1^2 \cdot LM_1(f_1) + \frac{e_1^2 \cdot LT_2(f_1)}{LC_1(f_1)} - LM_1(f_2) - \frac{LT_2(f_2)}{LC_1(f_2)} + \text{ lex smaller terms} \\
&= c \cdot e_1^7 e_2^{\frac{\ell-1}{2}-1} + \text{lex smaller terms.}
\end{aligned}$$

where

$$c = \left(\frac{LC_2(f_1)}{LC_1(f_1)} - \frac{LC_2(f_2)}{LC_1(f_2)}\right) \neq 0$$

by Lemma 3.6 allowing us to conclude that $LM_1(f_3) = e_1^7 e_2^{\frac{\ell-1}{2}-1}$.

For the inductive step there are two cases depending on the parity of ℓ. We will first attack the case for ℓ even. Pick an arbitrary $j \in \{5, \ldots, \ell-1\}$ and assume that for all even $m < j$ we have that $LM_1(f_m) = e_1^{2m+1} e_2^{\frac{\ell}{2}-\frac{m}{2}}$ and we want to show that $LM_1(f_{j+1}) = e_1^{2(j+1)+1} e_2^{\frac{\ell}{2}-\frac{j}{2}-\frac{1}{2}}$. We have that

$$\begin{aligned}
f_{j-1} &= \frac{e_1^2 f_{j-3}}{LC_1(f_{j-3})} - \frac{f_{j-2}}{LC_1(f_{j-2})} \\
f_j &= \frac{e_1^2 f_{j-2}}{LC_1(f_{j-2})} - \frac{e_2 f_{j-1}}{LC_1(f_{j-1})} \\
f_{j+1} &= \frac{e_1^2 f_{j-1}}{LC_1(f_{j-1})} - \frac{f_j}{LC_1(f_j)}.
\end{aligned}$$

We know by assumption that $LM_1(f_{j-1}) = e_1^{2(j-1)+1} e_2^{\frac{\ell}{2}-\frac{j-1}{2}}$. Additionally, Lemma 3.6 is really saying that if we look at f_j then e_2 multiplied by the second largest lex order term in f_{j-1} survived the difference and so we know that $LM_1(f_j) =$

$e_2 \cdot LM_2(f_{j-1})$. Thus, Lemma 3.7 tells us

$$\begin{aligned} LM_1(f_{j-1}) &= e_1^{2(j-1)+1} e_2^{\frac{\ell}{2}-\frac{j-1}{2}} \\ LM_2(f_{j-1}) &= e_1^{2(j-1)+3} e_2^{\frac{\ell}{2}-\frac{j+1}{2}} \\ LM_1(f_j) &= e_1^{2(j-1)+3} e_2^{\frac{\ell}{2}-\frac{j-1}{2}} \\ LM_2(f_j) &= e_1^{2(j-1)+5} e_2^{\frac{\ell}{2}-\frac{j+1}{2}} \end{aligned}$$

and so we have that

$$\begin{aligned} f_{j+1} &= e_1^2 \cdot LM_1(f_{j-1}) + \frac{e_1^2 \cdot LT_2(f_{j-1})}{LC_1(f_{j-1})} - LM_1(f_j) - \frac{LT_2(f_j)}{LC_1(f_j)} + \text{ lex smaller terms} \\ &= \left(\frac{LC_2(f_{j-1})}{LC_1(f_{j-1})} - \frac{LC_2(f_j)}{LC_1(f_j)}\right) \cdot e_1^{2(j-1)+5} e_2^{\frac{\ell}{2}-\frac{j+1}{2}} + \text{lex smaller terms.} \\ &= \left(\frac{LC_2(f_{j-1})}{LC_1(f_{j-1})} - \frac{LC_2(f_j)}{LC_1(f_j)}\right) \cdot e_1^{2(j+1)+1} e_2^{\frac{\ell}{2}-\frac{j+1}{2}} + \text{lex smaller terms.} \end{aligned}$$

By Lemma 3.6 we only lose one term when going from f_j to f_{j+1} and since we lost $LM_1(f_j) = e_1^2 LM_1(f_{j-1})$ we have

$$\left(\frac{LC_2(f_{j-1})}{LC_1(f_{j-1})} - \frac{LC_2(f_j)}{LC_1(f_j)}\right) \neq 0$$

and so we can conclude that $LM(f_{j+1}) = e_1^{2(j+1)+1} e_2^{\frac{\ell}{2}-\frac{j+1}{2}}$ as desired.

We will now go about the case for ℓ odd. Pick an arbitrary $j \in \{2, \ldots, \ell - 1\}$ and assume that for all odd $m < j$ we have that $LM_1(f_m) = e_1^{2m+1} e_2^{\frac{\ell}{2}-\frac{m}{2}}$. Using the same line of reasoning used in the case for ℓ even it follows that $LM_1(f_{j+1}) = e_1^{2(j+1)+1} e_2^{\frac{\ell}{2}-\frac{j}{2}-\frac{1}{2}}$. □

The above failed attempt demonstrate that the first approach using Gröbner Bases is too hands-on, and suggests that we should instead look elsewhere.

3.2. **A Lex-greedy Approach.** Some of the observations about the data, collected using Macaulay2, on lex standard monomial bases for $R^{k,\ell}$ and R^n_{LG} prompted the following definitions and conjectures.

Definition 3.8. Given any graded algebra $R = \bigoplus_{d=0}^{\infty} R_d$ over a field k, with $a_d := \dim_k R_d$, so $\mathrm{Hilb}(R, q) = \sum_{d=0}^{\infty} a_d q^d$, say that a sequence of homogeneous elements $(x_t; x_1, x_2, \ldots)$ of R give a *lex-greedy generating sequence for* R if for each degree $d = 0, 1, 2, \ldots$, one has a k-basis for the d^{th} homogeneous component R_d consisting of the first a_d monomials $x_1^{\alpha_1} x_2^{\alpha_2} \cdots$ with a factor of x_t and with $\sum_j \alpha_j = d$ in the *lexicographic ordering* with $x_1 < x_2 < \cdots$ on the set of all monomials of degree d. One also writes $(x_1, x_2, \cdots)$ to represent $(1; x_1, x_2, \cdots)$.

In particular, this condition implies that $\{x_1, x_2, \ldots\}$ generate R as an algebra over k, since a subset of the monomials in these elements give a k-basis for R.

Example 3.9. Start with polynomials $\mathbb{Q}[e_1, e_2]$ having $\deg(e_1) := 1, \deg(e_2) := 2$. Then these two quotient rings

$$\begin{aligned} R := R^{2,2} \cong& \mathbb{Q}[e_1, e_2]/(e_1^3 - e_1 e_2, e_1^4 - 2e_1^2 e_2 + e_2^2), \\ R' :=& \mathbb{Q}[e_1, e_2]/(e_1^3, e_2^2) \end{aligned}$$

share the same Hilbert series

$$\mathrm{Hilb}(R, q) = \mathrm{Hilb}(R', q) = 1 + q + 2q^2 + q^3 + q^4 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q .$$

Then $R = R^{2,2}$ has (e_1, e_2) as a lex-greedy generating sequence, as it has a $\mathbb{Q}$-basis

degree	$\mathbb{Q}$-basis elements
0	$\{1\}$
1	$\{e_1\}$
2	$\{e_1^2, e_2\}$
3	$\{e_1^3\}$
4	$\{e_1^4\}$

But R' does *not* have (e_1, e_2) as lex-greedy generating sequence, since $a_3 = a_4 = 1$, but $e_1^3 = 0$ in $(R')_3$ and $e_1^4 = 0$ in $(R')_4$.

Conjecture 3.10. *Both $R^{k,\ell,m}/R^{k,\ell,m-1}$ and $R_{LG}^{n,m}/R_{LG}^{n,m-1}$for odd m, have lex-greedy generating sequences, specifically*

(a) *the sequence $(e_m; e_1, e_2, e_3, \ldots, e_m)$ for $R^{k,\ell,m}/R^{k,\ell,m-1}$, and*
(b) *the sequence $(e_m; e_1, e_3, e_5, \ldots, e_m)$ for $R_{LG}^{n,m}/R_{LG}^{n,m-1}$.*

Remark 3.11. Conjecture 3.10 is not equivalent to the statement that $R^{k,l}$ and R_{LG}^n have the following lex-greedy generating sequences:

(a) the sequence $(e_1, e_2, e_3, \ldots, e_k)$ for $R^{k,\ell}$, and
(b) the sequence $(e_1, e_3, e_5, \ldots, e_k)$ for R_{LG}^n, where k is the largest odd number less than n.

An example that illustrates this is when we consider the degree 12 part of the graded ring $R^{3,6}$, which has dimension 7. We would expect the degree 12 elements in the $\mathbb{Q}$-basis of $R^{3,6}$ to be

$$\{1, e_1^{12}, e_2 e_1^{10}, e_2^2 e_1^8, e_2^3 e_1^6, e_2^4 e_1^4, e_2^5 e_1^2, e_2^6\}$$

according to the definition of lex-greedy. However, the degree 12 elements in the $\mathbb{Q}$-basis of $R^{3,6}$ are

$$\{1, e_1^{12}, e_2 e_1^{10}, e_2^2 e_1^8, e_2^3 e_1^6, e_2^4 e_1^4, e_2^5 e_1^2, e_3 e_1^9\}.$$

Despite looking promising at first, after further examination, we developed a counterexample to Conjecture 3.10.

Proposition 3.12. *Conjecture 3.10 is false.*

Proof. The counterexample for $R^{k,\ell,m}/R^{k,\ell,m-1}$ is when $k = 6$, $l = 7$, and $m = 3$. In this case when we look at the degree 31 part of the $\mathbb{Q}$-basis for $R^{6,7,3}/R^{6,7,2}$, we have $e_3^2 e_1^{25}$, but if it had a lex-greedy generating sequence then we would expect to have $e_3 e_2^{14}$ instead.

The counterexample for $R_{LG}^{n,m}/R_{LG}^{n,m-1}$ is when $n = 11$ and $m = 5$. In this case, when we look at the degree 47 part of the $\mathbb{Q}$-basis for $R_{LG}^{11,5}/R_{LG}^{11,4}$, we have $e_5^2 e_1^{37}$, but if it had a lex-greedy generating sequence then we would expect to have $e_5^2 e_1^{37}$ instead. $\square$

Before we knew that Conjecture 3.10 was false, the reason it seemed helpful was due to the following purely combinatorial conjecture, which felt very approachable.

Conjecture 3.13. *Conjectures 3.10(a),(b) imply the Reiner-Tudose Conjecture, and Conjecture 2.3, respectively.*

Despite Conjecture 3.10 being false, Lex-greedy generating sequences starting with a degree one element interact well with the ideas in the Hard Lefschetz Theorem [2, p. 122], as we now explain.

Definition 3.14. Recall that one says the graded ring R has *symmetric* Hilbert series

$$\mathrm{Hilb}(R,q) = a_0 + a_1 q + a_2 q^2 + \cdots + a_{N-1} q^{N-1} + a_N q^N$$

if $a_d = a_{N-d}$ for $d = 0, 1, 2, \ldots, \lfloor \frac{N}{2} \rfloor$.

One furthermore says that an element x in R is a *Lefschetz element* if for every $d = 0, 1, \ldots, \lfloor \frac{N}{2} \rfloor$, the map $R_d \longrightarrow R_{N-d}$ which multiplies by x^{N-2d} gives a k-vector space isomorphism $R_d \cong R_{N-d}$.

Note that having a Lefschetz element x in R will imply that its Hilbert series is *symmetric unimodal*, meaning not only that $a_d = a_{N-d}$, but also

$$a_0 \leq a_1 \leq \ldots a_{\lfloor \frac{N}{2} \rfloor} = a_{\lceil \frac{N}{2} \rceil} \geq \cdots \geq a_{N-1} \geq a_N.$$

This is because the map $R_d \longrightarrow R_{d+1}$ which multiplies by x has to be injective for $d < \frac{N}{2}$, since multiplying by $x^{N-2d} = x \cdot x^{N-2d-1}$ is bijective.

Proposition 3.15. *Assuming that the graded ring R has symmetric Hilbert series, if it has a lex-greedy generating sequence $(x_1, x_2, \ldots)$ with $\deg(x_1) = 1$, then x_1 is a Lefschetz element for R.*

In particular, this implies that the Hilbert series is symmetric unimodal.

Proof. Fix a degree d in the range $0, 1, 2, \ldots, \lfloor \frac{N}{2} \rfloor$. By our assumption that $(x_1, x_2, \ldots)$ is a lex-greedy generating sequence, the first a_d monomials

$$\{m_1, m_2, \ldots, m_{a_d}\} \tag{3.2}$$

of degree d in lex order give a k-basis for R_d. Since x_1 is the smallest variable in our lex order with $x_1 < x_2 < \cdots$, it is also true that

$$\{m_1 \cdot x_1^{N-2d}, m_2 \cdot x_1^{N-2d}, \ldots, m_{a_d} \cdot x_1^{N-2d}\} \tag{3.3}$$

are the first a_d monomials of degree $N - d$ in lex order. On the other hand, by symmetry, $a_{N-d} = a_d$, and hence by our lex-greedy assumption again, the monomials in (3.3) give a k-basis for R_{N-d}. But this then shows that the map $R_d \longrightarrow R_{N-d}$ which multiplies by x^{N-2d} gives a k-vector space isomorphism, since it carries the basis (3.2) to the basis (3.3). $\square$

Thus, while lex-greediness looked promising, it is clear that finding an algorithm to go from the right hand side of the Reiner-Tudose Conjecture to the standard monomials might be asking too much. Thus, we see that it would be advantageous to try another approach to prove this conjecture that doesn't heavily involve Gröbner Bases.

4. An Approach Using Natural Maps Between Rings

There are several natural maps between Grassmannians, giving rise to maps between $R^{k,\ell}$. The goal of this section is to utilise such maps, to construct commutative diagrams, to which the Hilbert additivity property affords us equivalent formulations of the Reiner-Tudose Conjecture and Conjecture 2.3, respectively. One such family of natural maps is as follows.

4.1. **Maps Between Grassmannians.** One has an inclusion of Grassmannians

$$Gr(k, k+\ell-1) \hookrightarrow Gr(k, k+\ell) \tag{4.1}$$

that comes from including $\mathbb{C}^{k+\ell-1} \hookrightarrow \mathbb{C}^{k+\ell}$, and then just sending a k-dimensional subspace of $\mathbb{C}^{k+\ell-1}$ to the same subspace inside $\mathbb{C}^{k+\ell}$. Additionally, 4.1 induces a map in the opposite direction on their respective cohomology rings.

$$H^*(Gr(k, k+\ell)) \twoheadrightarrow H^*(Gr(k, k+\ell-1)) \tag{4.2}$$

Lemma 4.1. $I^{k,\ell} \subseteq I^{k,\ell-1}$

Proof. It suffices to show that $h_{\ell+k}$ can be expressed as a linear combination of the $h_\ell, h_{\ell+1}, \ldots, h_{\ell+k-1}$. We notice that

$$\mathbb{Q}[e_1, e_2, \ldots, e_k] = \mathbb{Q}[e_1, e_2, \ldots]/(e_{k+1}, e_{k+2}, \ldots)\ ,$$

and also

$$h_d = \det \begin{pmatrix} e_1 & e_2 & \cdots & \cdots & e_d \\ 1 & e_1 & e_2 & \ddots & \vdots \\ 0 & 1 & e_1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & e_2 \\ 0 & \cdots & 0 & 1 & e_1 \end{pmatrix}.$$

Doing cofactor expansion with respect to the first row, we get

$$h_d = e_1 h_{d-1} - e_2 h_{d-2} + \cdots + (-1)^d e_d.$$

Letting $d = \ell + k$ and throwing away the last ℓ terms which are all zero, we are done. □

Map (4.2) in terms of $R^{k,\ell}$ is the following map:

$$R^{k,\ell} \twoheadrightarrow R^{k,\ell-1} \tag{4.3}$$

doing exactly what you would expect: it sends the image of each e_i in $R^{k,\ell}$ to the same image of e_i in $R^{k,\ell-1}$. This map is also well-defined when viewing $R^{k,\ell}$ as the quotient ring

$$R^{k,\ell} = \mathbb{Q}[e_1, \ldots, e_k]/I^{k,\ell} \text{ where } I^{k,\ell} := (h_{\ell+1}, h_{\ell+2}, \ldots, h_{\ell+k}).$$

since $I^{k,\ell} \subseteq I^{k,\ell-1}$ by Lemma 4.1.

As the surjection 4.3 sends $e_i \mapsto e_i$, it also induces surjections

$$R^{k,\ell,m} \twoheadrightarrow R^{k,\ell-1,m} \tag{4.4}$$

as one would have hoped.

On the other hand, by Proposition 2.15, we know that there exists a graded ring isomorphism: $R^{k,\ell} \longrightarrow R^{\ell,k}$ that maps (the images of) $e_i \mapsto h_i$ for $i = 1, 2, \ldots, k$

and $h_j \mapsto e_j$ for $j = 1, 2, \ldots, \ell$, this means that the above surjections $R^{k,\ell,m} \twoheadrightarrow R^{k,\ell-1,m}$ also give rise to the following surjections:

$$\varphi_{k,\ell,m} : R^{k,\ell} \twoheadrightarrow R^{k-1,\ell,m}.$$

4.2. **Maps Between Lagrangian Grassmannians.** Similarly, we obtain the following surjections:

$$R^n_{LG} \twoheadrightarrow R^{n-1}_{LG} \text{ and } R^{n,m}_{LG} \twoheadrightarrow R^{n-1,m}_{LG}$$

coming from restricting a nondegenerate symplectic bilinear form $(-,-)$ on $\mathbb{C}^{2n}$ to such a bilinear form on $\mathbb{C}^{2n-2}$, so that one can send a (maximal) isotropic $(n-1)$-dimensional subspace W of $\mathbb{C}^{2n-2}$ to $\hat{W} := W + \mathbb{C} \cdot v_0$ where v_0 is some fixed vector of $\mathbb{C}^{2n}$ that lies in $(\mathbb{C}^{2n-2})^{\perp}$. This would give an inclusion of the Lagrangian Grassmannians, that gives rise to the above surjections.

One can also check directly that the surjective map

$$\begin{array}{rcl} \mathbb{Q}[e_1, e_2, \ldots, e_{n-1}, e_n] & \longrightarrow & \mathbb{Q}[e_1, e_2, \ldots, e_{n-1}] \\ e_i & \longmapsto & e_i \text{ for } i = 1, 2, \ldots, n-1 \\ e_n & \longmapsto & 0 \end{array}$$

induces a surjective ring map $R^n_{LG} \twoheadrightarrow R^{n-1}_{LG}$ because it sends the ideal I^n_{LG} defining R^n_{LG} into the ideal I^{n-1}_{LG}.

4.3. **Diagrammatic Reformulation for the Two Main Conjectures.** In this section, we describe equivalent forms of the Reiner-Tudose Conjecture using methods from algebraic topology. We first construct the equivalent conjectures for the Lagrangian Grassmannian, because there are only two parameters, giving rise to a 2-D commutative diagram. Then, we naturally extend the methods to the Grassmanian, giving rise to a 3-D commutative diagram.

4.3.1. *Reformulation for R^n_{LG}.* By the surjections in section 4.2, we obtain the following commutative diagram:

$$\begin{tikzcd}
R^{1,1}_{LG} & & & & \\
R^{2,1}_{LG} \arrow[u, two heads] \arrow[r, hook] & R^{2,2}_{LG} \arrow[ul, two heads] & & & \\
R^{3,1}_{LG} \arrow[u, two heads] \arrow[r, hook] & R^{3,2}_{LG} \arrow[u, two heads] \arrow[r, hook] & R^{3,3}_{LG} \arrow[ul, two heads] & & \\
R^{4,1}_{LG} \arrow[u, two heads] \arrow[r, hook] & R^{4,2}_{LG} \arrow[u, two heads] \arrow[r, hook] & R^{4,3}_{LG} \arrow[u, two heads] \arrow[r, hook] & R^{4,4}_{LG} \arrow[ul, two heads] & \\
\vdots \arrow[u, two heads] & \vdots \arrow[u, two heads] & \vdots \arrow[u, two heads] & \vdots \arrow[u, two heads] & \ddots \arrow[ul, two heads]
\end{tikzcd}$$

Note that the rows and columns are not exact sequences, so we would like to investigate the following commutative square

$$\begin{array}{ccc} R_{LG}^{n-1,m-1} & \hookrightarrow & R_{LG}^{n-1,m} \\ \uparrow\!\!\uparrow & & \uparrow\!\!\uparrow \\ R_{LG}^{n,m-1} & \hookrightarrow & R_{LG}^{n,m}. \end{array}$$

First, we form the cokernels of the two horizontal injections and get an induced surjective map between them. So the rows are short exact sequences of graded $\mathbb{Q}$-algebras

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R_{LG}^{n-1,m-1} & \longrightarrow & R_{LG}^{n-1,m} & \longrightarrow & R_{LG}^{n-1,m}/R_{LG}^{n-1,m-1} & \longrightarrow & 0 \\ & & \uparrow\!\!\uparrow & & \uparrow\!\!\uparrow & & \uparrow\!\!\uparrow & & \\ 0 & \longrightarrow & R_{LG}^{n,m-1} & \longrightarrow & R_{LG}^{n,m} & \longrightarrow & R_{LG}^{n,m}/R_{LG}^{n,m-1} & \longrightarrow & 0 \end{array}$$

Then we apply the snake lemma to complete the commutative diagram where all rows and columns are short exact sequences

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & R_{LG}^{n-1,m-1} & \longrightarrow & R_{LG}^{n-1,m} & \longrightarrow & R_{LG}^{n-1,m}/R_{LG}^{n-1,m-1} & \longrightarrow & 0 \\ & & {\scriptstyle \phi_{n,m-1}}\uparrow & & {\scriptstyle \phi_{n,m}}\uparrow & & \uparrow & & \\ 0 & \longrightarrow & R_{LG}^{n,m-1} & \longrightarrow & R_{LG}^{n,m} & \longrightarrow & R_{LG}^{n,m}/R_{LG}^{n,m-1} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \ker(\phi_{n,m-1}) & \longrightarrow & \ker(\phi_{n,m}) & \longrightarrow & \ker(\phi_{n,m})/\ker(\phi_{n,m-1}) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

Property 4.2 (Hilbert Additivity Property for SES)**.** *Let*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be a short exact sequence of graded vector spaces. The Hilbert Series is Additive relatively to the below SES

$$\mathrm{Hilb}(B,q) = \mathrm{Hilb}(A,q) + \mathrm{Hilb}(C,q). \tag{4.5}$$

So Property 4.2 gives the following two equivalent expressions of Conjecture 2.3:

Conjecture 4.3. *Assume m is odd, then the Hilbert series of* $\ker(\phi_{n,m})$ *is*

$$\mathrm{Hilb}(\ker(\phi_{n,m}),q) = \sum_{\substack{1\le k\le m \\ k\ odd}} q^k \cdot q^{\binom{n-k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q$$

Conjecture 4.4. *Assume m is odd, then the Hilbert series of* $\ker(\phi_{n,m})/\ker(\phi_{n,m-1})$ *is*

$$\mathrm{Hilb}(\ker(\phi_{n,m})/\ker(\phi_{n,m-1}),q) = q^m \cdot q^{\binom{n-m+1}{2}} \begin{bmatrix} n \\ m \end{bmatrix}_q$$

Hence, proving any of the above conjectures would automatically prove Conjecture 2.3, and vice versa.

4.3.2. *Reformulation for $R^{k,\ell}$.* Similarly, by the surjections obtained in section 4.1, and the same construction in 4.3.1 we have the following

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & R^{k-1,\ell,m-1} \arrow[r] \arrow[u] & R^{k-1,\ell,m} \arrow[r] \arrow[u] & R^{k-1,\ell,m}/R^{k-1,\ell,m-1} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & R^{k,\ell,m-1} \arrow[r] \arrow[u, "\varphi_{k,\ell,m-1}"] & R^{k,\ell,m} \arrow[r] \arrow[u, "\varphi_{k,\ell,m}"] & R^{k,\ell,m}/R^{k,\ell,m-1} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \ker(\varphi_{k,\ell,m-1}) \arrow[r] \arrow[u] & \ker(\varphi_{k,\ell,m}) \arrow[r] \arrow[u] & \ker(\varphi_{k,\ell,m})/\ker(\varphi_{k,\ell,m-1}) \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Since there are three variables for the subalgebras for the Grassmannian case, m, ℓ and k, the diagrammatic reformulation is actually a 3D prism, each direction representing the three variables. Above, is a slice of the diagram and below are the first three layers of the $3D$-diagram:

$$\begin{tikzcd}
 & & & R^{1,1,1} \\
 & & R^{1,2,1} \arrow[ur, dashed, two heads] & \\
 & & R^{2,2,1} \arrow[u, two heads] \arrow[r, hook] & R^{2,2,2} \arrow[ul, two heads] \\
R^{1,3,1} \arrow[uurr, dashed, two heads] & & & \\
R^{2,3,1} \arrow[u, two heads] \arrow[r, hook] \arrow[uurr, dashed, two heads] & R^{2,3,2} \arrow[ul, two heads] \arrow[uurr, dashed, two heads] & & \\
R^{3,3,1} \arrow[u, two heads] \arrow[r, hook] & R^{3,3,2} \arrow[u, two heads] \arrow[r, hook] & R^{3,3,3} \arrow[ul, two heads] &
\end{tikzcd}$$

Likewise, applying Property 4.2, we obtain equivalent formulation of the Reiner-Tudose Conjecture.

Conjecture 4.5.

$$\operatorname{Hilb}(\ker(\varphi_{k,\ell,m})\ , q) = \sum_{i=1}^{m} q^{i+(k-i)(\ell-i+1)} \begin{bmatrix} \ell \\ i \end{bmatrix}_q \begin{bmatrix} k-1 \\ k-i \end{bmatrix}_q$$

Conjecture 4.6.

$$\operatorname{Hilb}(\ker(\varphi_{k,\ell,m})/\ker(\varphi_{k,\ell,m-1})\ , q) = q^m \cdot q^{(k-m)(\ell-(m-1))} \begin{bmatrix} \ell \\ m \end{bmatrix}_q \begin{bmatrix} k-1 \\ m-1 \end{bmatrix}_q$$

Analogously to the Lagrangian case, proving Conjecture 4.4 would automatically prove the Reiner-Tudose Conjecture by Property 4.2.

4.4. **Approach via Coefficient-Wise Inequalities.** One might approach the main conjectures by proving only a coefficientwise inequality, perhaps inductively. While we were not successful in proving the conjecture using this approach, we highlight why we considered it below. Proving an inequality forces a coefficientwise *equality*, for the following reason: by Proposition 7, in [5]

$$\operatorname{Hilb}(R^{k,\ell},q) = \begin{bmatrix} k+\ell \\ k \end{bmatrix}_q = 1 + \sum_{i=1}^{k} \sum_{j=0}^{k-i} q^i \binom{\ell}{i} \cdot q^{j(\ell-i+1)} \binom{i+j-1}{j}. \tag{4.6}$$

On the other hand, we know, by using a telescoping series argument, that:

$$\operatorname{Hilb}(R^{k,\ell},q) = \sum_{m=0}^{k} \operatorname{Hilb}(R^{k,\ell,m}/R^{k,\ell,m-1},q). \tag{4.7}$$

Hence if we show that the RHS of equation 4.6 is less (greater) than or equal to the RHS of equation 4.7 as *polynomials in q with nonnegative coefficients* at $q=1$, then these two sides are the same integer $\binom{k+\ell}{k}$. Therefore, the coefficientwise inequality must be an *equality*.

One might use recurrences to prove a coefficientwise inequality by induction. While we have come up with recursive formulas, it was not clear for us how to utilise them to demonstrate coefficientwise inequalities.

4.5. **Recursive Formula for $R^{k,\ell}$.** For the Grassmannian set-up, let's define our frequently occurring graded quotient $\mathbb{Q}$-vector space and Hilbert series

$$\begin{aligned} Q^{k,\ell,m} &:= R^{k,\ell,m}/R^{k,\ell,m-1}, \\ H^{k,\ell,m} &:= \operatorname{Hilb}(Q^{k,\ell,m},q) \end{aligned}$$

and let's also recall from from [5, Remark 9] that the inner sum in the Reiner-Tudose Conjecture is some kind of q-analogue of $\binom{k}{m}$ that satisfies a q-Pascal-like recurrence

$$f_m^{k,\ell} = f_{m-1}^{k-1,\ell} + q^{\ell-m+1} f_m^{k-1,\ell}.$$

Thus the conjecture can be rephrased as

$$\begin{aligned} H^{k,\ell,m} &= q^m \begin{bmatrix} \ell \\ m \end{bmatrix}_q f_m^{k,\ell} \\ &= q^m \begin{bmatrix} \ell \\ m \end{bmatrix}_q \left(f_{m-1}^{k-1,\ell} + q^{\ell-m+1} f_m^{k-1,\ell} \right) \\ &= q^m \cdot \frac{[\ell-m+1]_q}{[m]_q} \begin{bmatrix} \ell \\ m-1 \end{bmatrix}_q f_{m-1}^{k-1,\ell} + q^{\ell-m+1} \cdot q^m \begin{bmatrix} \ell \\ m \end{bmatrix}_q f_m^{k-1,\ell} \end{aligned} \tag{4.8}$$

hence

$$H^{k,\ell,m} = \frac{q-q^{\ell-m+2}}{1-q^m} H^{k-1,\ell,m-1} + q^{\ell-m+1} H^{k-1,\ell,m}.$$

We haven't yet been able to use this recursive formula to prove any inequalities.

4.6. **A "q-Pascal Short Exact Sequence".** Proposition 4.7 below uses some of the natural maps appearing in Diagram 4.3.2 to say something interesting about $R^{k,\ell}$, modeling this q-Pascal recurrence:

$$\begin{bmatrix} k+\ell \\ k \end{bmatrix}_q = \begin{bmatrix} k+\ell-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} k+\ell-1 \\ k \end{bmatrix}_q . \tag{4.9}$$

Proposition 4.7. *Consider this short exact sequence*

$$\begin{array}{ccccc} 0 \to \mathbb{Q}[e_1,\ldots,e_{k-1},e_k] & \xrightarrow{e_k \cdot} & \mathbb{Q}[e_1,\ldots,e_{k-1},e_k] & \xrightarrow{e_k=0} & \mathbb{Q}[e_1,\ldots,e_{k-1}] \to 0 \\ f(e_1,\ldots,e_k) & \longmapsto & e_k \cdot f(e_1,\ldots,e_k) & & \\ & & g(e_1,\ldots,e_{k-1},e_k) & \longmapsto & g(e_1,\ldots,e_{k-1},0) \end{array}$$

of maps of $\mathbb{Q}$-vectors spaces, whose first map raises degree by k while the second map preserves degree. Then it induces a short exact sequence on the quotients

$$0 \to R^{k,\ell-1} \xrightarrow{\cdot e_k} R^{k,\ell} \xrightarrow{e_k=0} R^{k-1,\ell} \to 0$$

which after taking Hilbert series, gives

$$\mathrm{Hilb}(R^{k,\ell},q) = \mathrm{Hilb}(R^{k-1,\ell},q) + q^k \mathrm{Hilb}(R^{k,\ell-1},q)$$

modeling the q-Pascal recurrence (4.9).

Proof. Recall that we can think of $R^{k,\ell}$ as being the quotient of the ring of symmetric functions $\Lambda_{\mathbb{Q}}$ in which one mods out by the $\mathbb{Q}$-span of Schur functions $\{s_\lambda\}$ for which λ does not fit in a (ℓ^k) rectangle. The second "$e_k = 0$" map can be identified with the map that sends s_λ to s_λ if λ has no columns of length k, and maps it to 0 if it has at least one column of length k. Thus the kernel of the second map is the span of $\{s_\lambda\}$ having λ inside the (ℓ^k) rectangle and having first column of length k. However, it is not hard to see using the Pieri formula for multiplying s_λ by e_k that this span is exactly the image of the first "$e_k \cdot$" map. $\square$

However, it was not clear how to utilise Proposition 4.7, in conjunction with the recursive formula 4.8, to analogously model recurrences involving the $R^{k,\ell,m}$ for the purpose of demonstrating coefficientwise inequalities.

Since the Reiner-Tudose Conjecture is equivalent to Conjecture 4.6; we considered it to be useful to formulate a recursive formula for

$$\mathrm{Hilb}(\ker \varphi_{k,\ell,m} / \ker \varphi_{k,\ell,m-1}, q)$$

which we will denote here by:

$$\mathrm{Hilb}(\ker \varphi_{k,\ell,m} / \ker \varphi_{k,\ell,m-1}, q) = \psi_{k,\ell,m} := q^m q^{(k-m)(\ell-m+1)} \begin{bmatrix} \ell \\ m \end{bmatrix}_q \begin{bmatrix} k-1 \\ m-1 \end{bmatrix}_q$$

to give a recursive formula, we first present the following:

Lemma 4.8 (q-analogue of Hockey Stick)**.** *Take arbitrary $n, m \in \mathbb{Z}$. The following holds:*

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \sum_{i=0}^{n-m} q^{m\cdot i} \begin{bmatrix} n-\ell-i \\ m-1 \end{bmatrix}_q$$

Proof. Fix $m \in \mathbb{Z}$. This is clear when $n \leq m$. We induct on n. We note:

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_q + q^m \begin{bmatrix} n-1 \\ m \end{bmatrix}_q$$

By induction, we have

$$q^m \begin{bmatrix} n-1 \\ m \end{bmatrix}_q = q^m \sum_{i=0}^{n-m-1} q^{m\cdot i} \begin{bmatrix} n-2-i \\ m-1 \end{bmatrix}_q = \sum_{i=1}^{n-m} q^{m\cdot i} \begin{bmatrix} n-\ell-i \\ m-1 \end{bmatrix}_q$$

Plugging back into the previous expression gives the desired form of $\begin{bmatrix} n \\ m \end{bmatrix}_q$. □

The following corollaries simplify the proof of Proposition 4.11.

Corollary 4.9. The following holds:

$$q^m q^{(k-m)(\ell-m+1)} q^m \begin{bmatrix} \ell-1 \\ m \end{bmatrix} \begin{bmatrix} k-2 \\ m-2 \end{bmatrix} = q \sum_{i=1}^{k-m} q^{k\cdot i} \psi_{k-1,\ell-1-i,m-1}$$

Proof. It is clear from the formula that the following holds for each i:

$$q^m q^{(k-m)(\ell-m+1)} \begin{bmatrix} \ell-1-i \\ m-1 \end{bmatrix} \begin{bmatrix} k-2 \\ m-2 \end{bmatrix} = q \cdot q^{i(k-m)} \psi_{k-1,\ell-1-i,m-1}.$$

Expanding the $\begin{bmatrix} \ell-1 \\ m \end{bmatrix}$ term in the corollary yields the following:

$$\begin{aligned} q^m q^{(k-m)(\ell-m+1)} q^m \begin{bmatrix} \ell-1 \\ m \end{bmatrix} \begin{bmatrix} k-2 \\ m-2 \end{bmatrix} &= q^m \sum_{i=0}^{\ell-m-1} q^{m\cdot i} q^{(k-m)(\ell-m+1)} q^m \begin{bmatrix} \ell-2-i \\ m-1 \end{bmatrix} \begin{bmatrix} k-2 \\ m-2 \end{bmatrix} \\ &= \sum_{i=1}^{\ell-m} q^{m\cdot i} q^{(k-m)(\ell-m+1)} q^m \begin{bmatrix} \ell-1-i \\ m-1 \end{bmatrix} \begin{bmatrix} k-2 \\ m-2 \end{bmatrix} \\ &= \sum_{i=1}^{\ell-m} q^{m\cdot i} q^{(k-m)i} q \psi_{k-1,\ell-1-i,m-1} \\ &= q \sum_{i=1}^{\ell-m} q^{ki} \psi_{k-1,\ell-1-i,m-1}. \end{aligned}$$

□

Corollary 4.10. Analogously, the following holds true:

$$q^m q^{(k-m)(\ell-m+1)} q^{m-1} \begin{bmatrix} \ell-1 \\ m-1 \end{bmatrix}_q \begin{bmatrix} k-2 \\ m-1 \end{bmatrix}_q = q \sum_{i=1}^{\ell-m} q^{\ell\cdot i} \psi_{k-1-i,\ell-1,m-1}$$

Proof. Similar to before, note that:

$$q^m q^{(k-m)(l-m+1)} \begin{bmatrix} \ell-1 \\ m-1 \end{bmatrix} \begin{bmatrix} k-2-i \\ m-1 \end{bmatrix} = q \cdot q^{i(l-m+1)} \psi_{k-1-i,\ell-1,m-1}$$

Expanding the $\begin{bmatrix} k-2 \\ m-1 \end{bmatrix}$ term in the corollary we have:

$$\begin{aligned}
q^m q^{(k-m)(\ell-m+1)} q^{m-1} \begin{bmatrix} \ell-1 \\ m-1 \end{bmatrix} \begin{bmatrix} k-2 \\ m-1 \end{bmatrix} &= q^{m-1} \sum_{i=0}^{k-m-1} q^{(m-1)i} q^{(k-m)(\ell-m+1)} q^m \begin{bmatrix} \ell-1 \\ m-1 \end{bmatrix} \begin{bmatrix} k-2-i \\ m-2 \end{bmatrix} \\
&= \sum_{i=1}^{k-m} q^{(m-1)i} q^{(k-m)(\ell-m+1)} q^m \begin{bmatrix} \ell-1 \\ m-1 \end{bmatrix} \begin{bmatrix} k-2-i \\ m-2 \end{bmatrix} \\
&= \sum_{i=1}^{k-m} q^{(m-1)i} q^{(\ell-m+1)i} q \psi_{k-1,\ell-1-i,m-1} \\
&= q \sum_{i=1}^{k-m} q^{\ell \cdot i} \psi_{k-\ell-i,\ell-1,m-1}
\end{aligned}$$

□

The above lead to the following:

Proposition 4.11. *ψ follows the following recurrence:*

$$\psi_{k,\ell,m} = q\psi_{k-1,\ell-1,m-1} + q^{k+\ell-1}\psi_{k-1,\ell-1,m} + q \sum_{i=1}^{\ell-m} q^{ki}\psi_{k-1,\ell-1-i,m-1} + q \sum_{i=1}^{k-m} q^{l\cdot i}\psi_{k-1-i,\ell-1,m-1}$$

Proof. We start with Pascal's identity to obtain the following:

$$\begin{aligned}
\psi_{k,\ell,m} &= q^m q^{(k-m)(\ell-m+1)} \left(\begin{bmatrix} \ell-1 \\ m-1 \end{bmatrix}_q + q^m \begin{bmatrix} \ell-1 \\ m \end{bmatrix}_q \right) \left(\begin{bmatrix} k-2 \\ m-2 \end{bmatrix}_q + q^{m-1} \begin{bmatrix} k-2 \\ m-1 \end{bmatrix} \right) \\
&= q^m q^{(k-m)(\ell-m+1)} \left(\begin{bmatrix} \ell-1 \\ m-1 \end{bmatrix}_q \begin{bmatrix} k-2 \\ m-2 \end{bmatrix}_q + q^{2m-1} \begin{bmatrix} \ell-1 \\ m \end{bmatrix}_q \begin{bmatrix} k-2 \\ m-1 \end{bmatrix}_q \right. \\
&\qquad \left. + q^m \begin{bmatrix} \ell-1 \\ m \end{bmatrix}_q \begin{bmatrix} k-2 \\ m-2 \end{bmatrix}_q + q^{m-1} \begin{bmatrix} \ell-1 \\ m-1 \end{bmatrix}_q \begin{bmatrix} k-2 \\ m-1 \end{bmatrix}_q \right) \\
&= q\psi_{k-1,\ell-1,m-1} + q^{(k+\ell-1)}\psi_{k-1,\ell-1,m} + q \sum_{i=1}^{\ell-m} q^{k\cdot i}\psi_{k-1,\ell-1-i,m-1} + q \sum_{i=1}^{k-m} q^{\ell\cdot i}\psi_{k-\ell-i,\ell-1,m-1}
\end{aligned}$$

□

4.7. **Recursive Formula for $R_{LG}^{n,m}$.** The following recursive formula avoids the problem posed by $R_{LG}^{n,m}$ vanishing at m even:

$$\begin{aligned}
\psi^n_m(q) &= q^m q^{\binom{n-m+1}{2}} \begin{bmatrix} n \\ m \end{bmatrix}_q \\
&= q^m q^{\binom{n-m+1}{2}} \left(\begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_q + q^m \begin{bmatrix} n-1 \\ m \end{bmatrix}_q \right) \\
&= q^m q^{\binom{n-m+1}{2}} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_q + q^{2m} q^{\binom{n-m+1}{2}} \begin{bmatrix} n-1 \\ m \end{bmatrix}_q \\
&= q^m q^{\binom{n-m+1}{2}} \left(\frac{1-q^{n-1}}{1-q^{m-1}} \begin{bmatrix} n-2 \\ m-2 \end{bmatrix}_q \right) + q^{2m} q^{\binom{(n-1)-m+1}{2}} q^{n-m} \begin{bmatrix} n-1 \\ m \end{bmatrix}_q \\
&= q^2 q^{m-2} q^{\binom{(n-2)-(m-2)+1}{2}} \left(\frac{1-q^{n-1}}{1-q^{m-1}} \right) \begin{bmatrix} n-2 \\ m-2 \end{bmatrix}_q + q^n q^m q^{\binom{(n-1)-m+1}{2}} \begin{bmatrix} n-1 \\ m \end{bmatrix}_q \\
&= q^2 \left(\frac{1-q^{n-1}}{1-q^{m-1}} \right) \psi^{n-2}_{m-2}(q) + q^n \psi^{n-1}_m(q)
\end{aligned}$$

Where we used the result that follows from the following identities:

$$\begin{aligned}
&(a) && \begin{bmatrix} n \\ m \end{bmatrix}_q = \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_q + q^m \begin{bmatrix} n-1 \\ m \end{bmatrix}_q \\
&(b) && \begin{bmatrix} n \\ m \end{bmatrix}_q = \begin{bmatrix} n-1 \\ m \end{bmatrix}_q + q^{n-m} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_q \\
& && \Rightarrow \begin{bmatrix} n-1 \\ m \end{bmatrix}_q = \begin{bmatrix} n \\ m \end{bmatrix}_q - q^{n-m} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_q \\
& && \Rightarrow \begin{bmatrix} n \\ m \end{bmatrix}_q = \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_q + q^m \left(\begin{bmatrix} n \\ m \end{bmatrix}_q - q^{n-m} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_q \right) \\
& && \Rightarrow \begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{1-q^n}{1-q^m} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_q .
\end{aligned}$$

4.8. **Doubly Filtered Basis.** In this section we outline an unsuccessful attempt to exploit the simplicity of the RHS of Conjecture 4.6. Utilising a combinatorial interpretation of the RHS, we attempted to construct a 1-1 correspondence between the doubly filtered basis elements and some Ferrers diagrams. We were motivated by the fact that, accomplishing this task, would have allowed us to take the union, over m, of the doubly filtered basis, yielding the kind of basis we are looking for, for the whole algebra.

Consider the right most Short Exact Column in Section 4.3.2 which is given by

$$0 \longrightarrow A \longrightarrow R^{k,\ell,m}/R^{k,\ell,m-1} \longrightarrow R^{k-1,\ell,m}/R^{k-1,\ell,m-1} \tag{4.10}$$

where

$$A := \ker(\varphi_{k,\ell,m})/\ker(\varphi_{k,\ell,m-1}).$$

By applying property 4.2 to the SES (4.10) we obtain:

$$\mathrm{Hilb}(R^{k,\ell,m}/R^{k,\ell,m-1},\ q) = \mathrm{Hilb}(A\ ,q) + \mathrm{Hilb}(R^{k-1,\ell,m}/R^{k-1,\ell,m-1}\ q). \tag{4.11}$$

where Hilb(A , q) and Hilb($R^{k-1,\ell,m})/R^{k-1,\ell,m-1}$ q) are considered as subspaces of Hilb($R^{k,\ell,m}/R^{k,\ell,m-1}$, q). By rearranging equation 4.11 we obtain:

$$\mathrm{Hilb}(A\ ,q) = \mathrm{Hilb}(R^{k,\ell,m}/R^{k,\ell,m-1},\ q) - \mathrm{Hilb}(R^{k-1,\ell,m}/R^{k-1,\ell,m-1},\ q). \tag{4.12}$$

Now let $\mathcal{B}^{k,l}$ denote the $\mathbb{Q}$-basis for $R^{k,\ell}$ given by the Gröbner basis with respect to a lexicographic monomial order in which $e_k > \cdots > e_2 > e_1$.

Lemma 4.12. $\mathcal{B}^{k,\ell,m} := \{x \in \mathcal{B}^{k,\ell} : e_j \nmid x$ *for* $j \geq m+1\}$ *is a* $\mathbb{Q}$*-basis for* $R^{k,\ell,m}$.

Proof. As in [15], this follows from the elimination of variables properties of the Gröbner bases with respect to lexographical ordering, and the fact that standard monomials give rise to bases for the quotients. □

Also, let

$$Q^{k-1,\ell,m} = R^{k-1,\ell,m}/R^{k-1,\ell,m-1},$$

Then it is clear that $B^{k-1,\ell,m} =: B^{k-1,\ell,m} := \mathcal{B}^{k-1,\ell,m} \setminus \mathcal{B}^{k-1,\ell,m-1}$ is a $\mathbb{Q}$-basis for $Q^{k-1,\ell,m}$ and we refer to it as the (once) filtered basis. Similarly, let

$$Q^{k,\ell,m} = R^{k,\ell,m}/R^{k,\ell,m-1}$$

and hence likewise $B^{k,\ell,m} =: B^{k,\ell,m} := \mathcal{B}^{k,\ell,m} \setminus \mathcal{B}^{k,l,m-1}$ is a $\mathbb{Q}$-basis for $Q^{k,\ell,m}$

Proposition 4.13. $B^{k-1,\ell,m} \subset B^{k,\ell,m}$.

Proof. We can write

$$R^{k,\ell,m} = \mathbb{Q}[e_1, \cdots, e_m]/I^{k,\ell,m}$$
$$R^{k-1,\ell,m} = \mathbb{Q}[e_1, \cdots, e_m]/I^{k-1,\ell,m}.$$

Furthermore, by definition of an elimination ideal, we can write the defining ideals as

$$I^{k,\ell,m} = I^{k,\ell} \cap \mathbb{Q}[e_1, \cdots, e_m]$$
$$I^{k-1,\ell,m} = I^{k-1,\ell} \cap \mathbb{Q}[e_1, \cdots, e_m].$$

Since $I^{k,\ell} \subset I^{k-1,\ell}$, it implies that $I^{k,\ell,m} \subset I^{k-1,\ell,m}$. Let $LT(G(I))$ denote the leading terms of a Gröbner basis for the ideal I with respect to lexicographical ordering. Thus, we have that $LT(G(I^{k,\ell,m})) \subset LT(G(I^{k,\ell,m}))$. Let

$$SM(R^{k,\ell,m}) = \{x \in R^{k,\ell,m} \quad : \quad y \not| x, \forall y \in LT(G(I^{k,\ell,m}))\}$$

denote the set of standard monomials with respect to the ideal $I^{k,\ell,m}$, and likewise, let

$$SM(R^{k-1,\ell,m}) = \{x \in R^{k-1,\ell,m} \quad : \quad y \not| x, \forall y \in LT(G(I^{k-1,\ell,m}))\}$$

denote the set of standard monomials with respect to the ideal $I^{k-1,\ell,m}$. Then, it follows that

$$SM(R^{k-1,\ell,m}) \subset SM(R^{k,\ell,m}).$$

It also immediately follows that

$$SM(R^{k-1,\ell,m-1}) \subset SM(R^{k,\ell,m-1}),$$

hence

$$B^{k-1,\ell,m} \subset B^{k,\ell,m}.$$

□

Now let

$$D^{k,\ell,m} = B^{k,\ell,m} \setminus B^{k-1,\ell,m}$$

denote the doubly filtered basis. Thus, when we compute the $\mathbb{Q}$-basis of the complement of $B^{k-1,\ell,m}$ inside $B^{k,\ell,m}$ (i.e. $D^{k,\ell,m}$), we can consecutively compute the RHS of equation 4.12 which is equal to Hilb(A , q), which is what we want.

Remark 4.14. Notice that the doubly-filtered basis $D^{k,\ell,m}$, is not the $\mathbb{Q}$-basis of the A. To see this let f be the map:

$$f : R^{k,l,m}/R^{k,l,m-1} \longrightarrow R^{k-1,l,m}/R^{k-1,l,m-1} \tag{4.13}$$

appearing in the commutative diagram in in Section 4.3.2.

From the exactness of the last column in the commutative diagram in subsection 4.3.2, namely:

$$0 \longrightarrow ker(\phi_{k,l,m})/ker(\phi_{k,l,m-1}) \xrightarrow{g} R^{k,\ell,m}/R^{k,\ell,m-1} \xrightarrow{f} R^{k-1,\ell,m}/R^{k-1,\ell,m-1} \longrightarrow 0 \tag{4.14}$$

whose general structure is:

$$0 \longrightarrow ker(f) \longrightarrow R^{k,\ell,m}/R^{k,\ell,m-1} \longrightarrow cokernel(g) \longrightarrow 0.$$

we deduce that:

$$ker(f) = ker(\phi_{k,l,m})/ker(\phi_{k,l,m-1}). \tag{4.15}$$

Subsequently, for elements $\gamma \in B^{k-1,\ell,m}$ such that $\gamma \in B^{k,\ell,m}$, the surjection f maps γ to itself.

For $\gamma \in D^{k,\ell,m} = B^{k,\ell,m} \setminus B^{k-1,\ell,m}$, $f(\gamma)$ is not necessarily zero, and could be some linear combination of the γ's. Hence, the $ker(f)$ is not necessarily $D^{k,\ell,m}$.

By definition of the Hilbert Series we get that:

$$\sum_{x \in D^{k,\ell,m}} q^{deg(x)} = \mathrm{Hilb}(R^{k,\ell,m}/R^{k,\ell,m-1},\ q) - \mathrm{Hilb}(R^{k-1,\ell,m}/R^{k-1,\ell,m-1}\ , q). \tag{4.16}$$

Hence using equation 4.16 and equation 4.12, it follows that

$$\mathrm{Hilb}(A,\, q) = \sum_{x \in D^{k,\ell,m}} q^{deg(x)}$$

By Conjecture 4.6, we obtain the equivalent conjecture in terms of the doubly filtered basis:

Conjecture 4.15.

$$\mathrm{Hilb}(A,\, q) = \sum_{x \in D^{k,\ell,m}} q^{deg(x)} = q^m \cdot q^{(k-m)(\ell-(m-1))} \begin{bmatrix} \ell \\ m \end{bmatrix}_q \begin{bmatrix} k-1 \\ m-1 \end{bmatrix}_q$$

Definition 4.16. We say a partition λ is **ℓ-bounded** if $\lambda_1 \leq \ell$.

Definition 4.17. A *ℓ-bounded* partition $\lambda = (\lambda_1, \ldots, \lambda_{\ell(\lambda)})$ is called *m-vacant* if m is the largest integer so that a $((m-1)^m)$ rectangle can fit inside the complement of λ in the $(\ell^{\ell(\lambda)})$ rectangle.

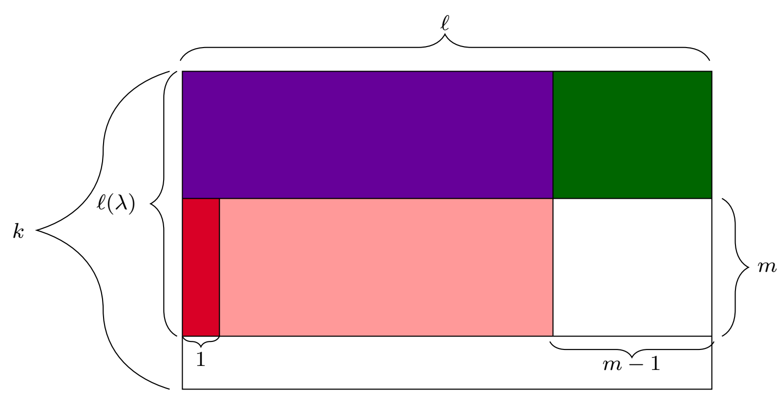
Definition 4.18. A partition $\lambda = (\lambda_1, \ldots, \lambda_{\ell(\lambda)}) \subset (\ell^k)$ is called *strong m-vacant* if it's m-vacant and $\ell(\lambda) = k$.

Proposition 4.19. *(Combinatorial Interpretation of the Reiner-Tudose Conjecture)*

$$\mathrm{Hilb}(R^{k,\ell,m}/R^{k,\ell,m-1}, q) = \sum_{\substack{\lambda\subset(\ell^k)\\ m-vacant}} q^{|\lambda|}$$

Proof. Let $\lambda \subset (\ell^k)$ be an m-vacant partition. Then we can decompose λ into four sub-partitions:

1. (1^m) (red)
2. $((\ell-m+1)^{l(\lambda)-m})$ (purple)
3. a partition $P \subset ((\ell-m)^m)$ (pink)
4. a partition $Q \subset ((m-1)^{l(\lambda)-m})$ (green)



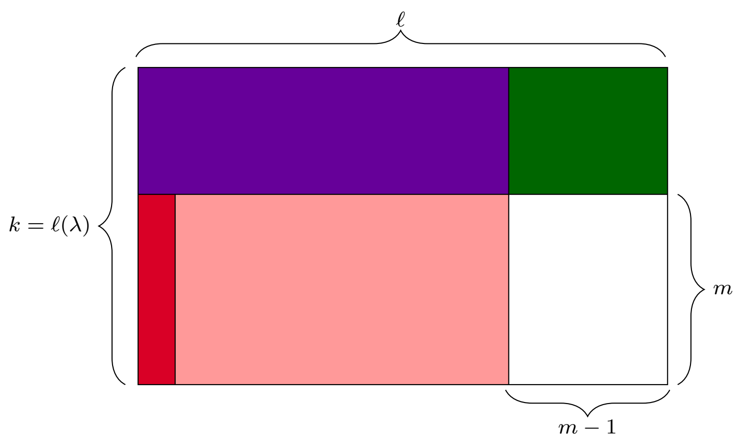
The pair of sub-partitions (P, Q) together with the number of parts $\ell(\lambda)$ specify the m-vacant partition λ uniquely. That is to say, by specifying a number j in the range $m \le j \le k$, and an arbitrary pair partitions (P, Q) with $P \subset ((\ell-m)^m)$ and $Q \subset ((m-1)^{j-m})$, there exists a unique m-vacant partition $\lambda \subset (\ell^k)$, such that (P, Q) is the pair of sub-partitions for λ and $j = l(\lambda)$.

Therefore,

$$\begin{aligned}\sum_{\substack{\lambda\subset(\ell^k)\\ m-\text{vacant}}} q^{|\lambda|} &= \sum_{j=m}^{k} \sum_{\substack{P\subset((\ell-m)^m)\\ Q\subset((m-1)^{j-m})}} q^m \cdot q^{(\ell-m+1)(j-m)} \cdot q^{|P|} \cdot q^{|Q|} \\ &= q^m \left(\sum_{P\subset((\ell-m)^m)} q^{|P|}\right) \left(\sum_{j=m}^{k} q^{(j-m)(\ell-m+1)} \sum_{Q\subset((m-1)^{j-m})} q^{|Q|}\right) \\ &= q^m \begin{bmatrix}\ell\\ m\end{bmatrix}_q \left(\sum_{j=0}^{k-m} q^{j(\ell-m+1)} \begin{bmatrix}m+j-1\\ j\end{bmatrix}_q\right)\end{aligned}$$

as required. □

Similarly, we now provide a combinatorial interpretation of the RHS in Conjecture 4.15 which requires strong m-vacant Ferrers diagrams. A general strong m-vacant Ferrers diagram is depicted below:



Recall that the RHS of Conjecture 4.15 is

$$q^m \cdot q^{(k-m)(\ell-(m-1))} \begin{bmatrix} \ell \\ m \end{bmatrix}_q \begin{bmatrix} k-1 \\ m-1 \end{bmatrix}_q .$$

The red strip whose height is m, corresponds to the multiplicity of the first q, the green box whose dimension is $((m-1)^{k-m})$ corresponds to the degree of $\begin{bmatrix} k-1 \\ m-1 \end{bmatrix}_q$, since $\begin{bmatrix} k-1 \\ m-1 \end{bmatrix}_q = \begin{bmatrix} (k-m)+m-1 \\ k-m \end{bmatrix}_q$. The pink box whose dimension is $((\ell-m)^m)$ corresponds to the degree of $\begin{bmatrix} \ell \\ m \end{bmatrix}_q = \begin{bmatrix} \ell-m+m \\ m \end{bmatrix}_q$. Lastly, the dimension of the purple box, $((\ell-m+1)^{k-m})$ corresponds to the multiplicity of the second q. Hence, combinatorially, we interpret the RHS as the number of *Strong m-vacant Ferrers diagrams* fitting inside (ℓ^k) rectangle. Therefore we get:

$$q^m \begin{bmatrix} \ell \\ m \end{bmatrix}_q q^{(k-m)(\ell-(m-1))} \begin{bmatrix} k-1 \\ m-1 \end{bmatrix}_q = \sum_{\substack{\lambda\in(k)^\ell \\ \lambda \text{ is } \textit{strong m-vacant}}} q^{|\lambda|}. \tag{4.17}$$

The correspondence in Proposition 4.19 between m-vacant partitions and the combinatorial formula in the Reiner-Tudose conjecture was first observed and explained by Reiner and Tudose in [5, Prop. 8].

This observation made us speculate that there exists a correspondence:

$$\{\lambda \subset (k)^\ell : \lambda \text{ is } \textit{strong m-vacant}\} \longleftrightarrow \{D^{k,\ell,m}\}$$

where a *m-vacant* partition λ corresponds to an element of the Gröbner basis of the same degree, but we were unsuccessful in finding one.

5. Schur and k-Schur Functions

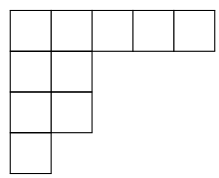
The first half of this section consists of a brief review of the classic picture of Schubert calculus (Schur functions, Pieri's rule, etc.) and its variation in the Lagrangian Grassmannians. It serves as a motivation for the second half on k-Schur functions (a fundamental tool in our approach to the Reiner-Tudose Conjecture).

We begin by fixing some notations. In this section, we denote a non-increasing sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{\ell(\lambda)})$ to be a *partition*. Recall that the English notation of the *Ferrers diagram* of λ consists of left-aligned rows of boxes placed from top to bottom, where the ith row consists of λ_i boxes. We shall use λ and its Ferrers diagram interchangeably.

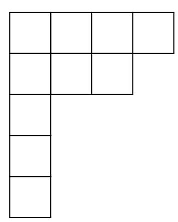
5.1. **Schur and Pieri for the Grassmannian.** The cohomology ring $R^{k,\ell}$ of the Grassmannian has a basis $\{s_\lambda\}$ consisting of *Schur functions* indexed by partitions λ in a (ℓ^k) rectangle. In terms of the generators $e_1, \dots, e_k$ for $R^{k,\ell}$, they can be expressed by the second Jacobi-Trudi determinant as coined in this Wikipedia page.

$$s_\lambda = \det(e_{\lambda'_i + j - i})_{i,j=1,2,\dots,\lambda_1}$$

where λ'_i is the number of squares in the i^{th} column of λ, with the usual conventions that $e_0 = 1$ and $e_i = 0$ if i is not in the range $[0, k]$. For example, the partition $\lambda = (5, 2, 2, 1)$ shown here



has $\lambda' = (\lambda'_1, \lambda'_2, \lambda'_3, \lambda'_4, \lambda'_5) = (4, 3, 1, 1, 1)$ as shown here



and

$$s_{(5,2,2,1)} = \det \begin{bmatrix} e_4 & e_5 & e_6 & e_7 & e_8 \\ e_2 & e_3 & e_4 & e_5 & e_6 \\ 0 & 1 & e_1 & e_2 & e_3 \\ 0 & 0 & 1 & e_1 & e_2 \\ 0 & 0 & 0 & 1 & e_1 \end{bmatrix}$$

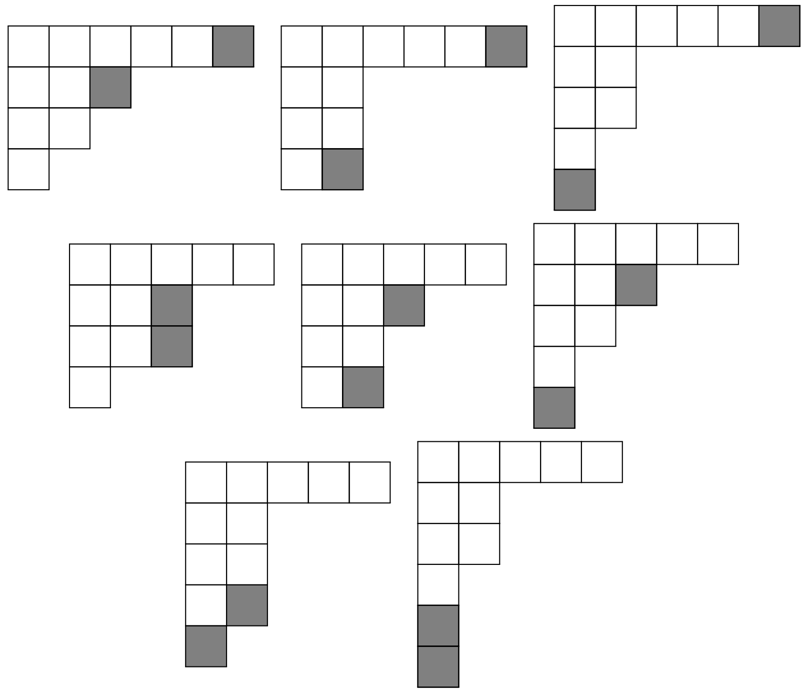
The Pieri's rule allows one to multiply one of the basis elements s_λ by any of the variables $e_k = (s_{(1,1,\dots,1)})$, and expand the product back into the basis $\{s_\mu\}$:

$$e_k \cdot s_\lambda = \sum_\mu s_\mu$$

where μ runs over all partitions that lie inside the (ℓ^k) that are obtained from λ by adding a *vertical k-strip*: a collection of k new boxes that have at most one box in each row. As an example, assuming that $k, \ell \geq 6$, we have

$$\begin{aligned} e_2 \cdot s_{(5,2,2,1)} = s_{(6,3,2,1)} + s_{(6,2,2,2)} + s_{(6,2,2,1,1)} + s_{(5,3,3,1)} \\ + s_{(5,3,2,2)} + s_{(5,3,2,1,1)} + s_{(5,2,2,2,1)} + s_{(5,2,2,1,1,1)} \end{aligned}$$

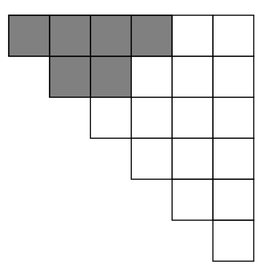
corresponding to these μ obtained from λ by adding a vertical 2-strip:



Note that if we had not assumed $k, \ell \geq 6$, then some of these terms would have been omitted. For example, if $\ell \geq 6$ but $k = 5$, then the last term $s_{(5,2,2,1,1,1)}$ would have been omitted. If both $k = \ell = 5$, then we would have also omitted the first terms $s_{(6,3,2,1)}, s_{(6,2,2,2)}, s_{(6,2,2,1,1)}$.

5.2. **Schur and Pieri for the Lagrangian Grassmannian.** We are using the talk by Tamvakis [7] and Kresch-Tamvakis [4] as references here.

The cohomology ring R^n_{LG} of the Lagrangian Grassmannian has a basis $\{\sigma_\lambda\}$ of *Schur Q-functions* indexed by strict partitions λ, whose *shifted* Young diagram fits inside an ambient triangle $\Delta_n = (n, n-1, \ldots, 2, 1)$. Equivalently, this says that $\lambda = (\lambda_1 > \cdots > \lambda_\ell)$ with $\lambda_1 \leq n$. For example, for $n = 6$, here is the ambient triangle and a strict partition $\lambda = (4, 2)$ contained within it:



In terms of the generators $e_1, \ldots, e_n$, for R^n_{LG}, with the usual conventions that $e_0 = 1$ and $e_i = 0$ if i is not in $[0, n]$, the σ_λ can be expressed in two steps:

- First, define for $i > j \geq 1$,

$$\sigma_{i,j} := e_i e_j + 2 \sum_{k=1}^{n-i} (-1)^k e_{i+k} e_{j-k}.$$

- Second, define

$$\sigma_\lambda := \mathrm{Pf}(\sigma_{\lambda_i,\lambda_j})_{1\le i<j\le r}$$

where $\mathrm{Pf}((\sigma_{\lambda_i,\lambda_j})_{1\le i<j\le r}$ denotes the *Pfaffian*[9] of the skew-symmetric matrix that has $\sigma_{\lambda_i,\lambda_j}$ as its above-diagonal (i,j)-entry if $i<j$; here r is the smallest even integer that is greater than or equal to $\ell(\lambda)$, the number of nonzero parts of λ.

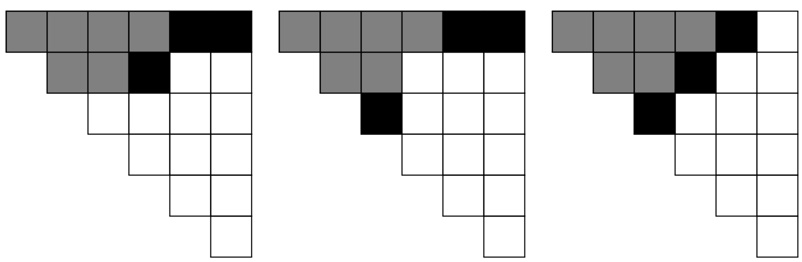
Then the *(Lagrangian) Pieri formula* [4, eqn. (51)] expresses the product of an e_k with a σ_λ back in the basis $\{\sigma_\mu\}$:

$$\sigma_\lambda \cdot e_k = \sum_\mu 2^{N(\lambda,\mu)}\sigma_\mu$$

where μ runs over all strict partitions lying in the ambient triangle Δ_n which are obtained from λ by adding a *horizontal k-strip*: a collection of new boxes that have at most one box in each column. Here $N(\lambda,\mu)$ is the number of rows within the horizontal k-strip $\mu\backslash\lambda$ that do not contain a "diagonal" box of the ambient triangle; the diagonal boxes are the leftmost box in each row, so in row 1 and column 1, in row 2 and column 2, etc. As an example, assuming that $n=6$, we have

$$\sigma_{(4,2)}\cdot e_3 = 4\sigma_{(6,3)} + 2\sigma_{(6,2,1)} + 4\sigma_{(5,3,1)}$$

corresponding to these μ obtained from λ by adding a horizontal 3-strip:



Note that if we had used a larger n, that is $n\ge 7$, an extra term $2\sigma_{(7,2)}$ would have appeared in the formula.

5.3. **k-Bounded Partitions and k-Conjugation.** The k-Schur functions $s_\lambda^{(k)}$ were first introduced in [8] to solve the Macdonald positivity conjecture. The set $\{s_\lambda^{(k)} \mid \lambda_1 \le k\}$ is a basis for the ring $\Lambda_k = \mathbb{Q}[h_1,\ldots,h_k]$, and they play the analogous role as Schur functions $\{s_\lambda\}$ in the symmetric function ring $\Lambda = \mathbb{Q}[h_1,h_2,h_3,\ldots]$.

Remark 5.1. Instead of working in $R^{k,\ell}$, we can equivalently work in $R^{\ell,k}$ by Proposition 2.15. We shall do that in the remaining sections of this paper for the following convenience. In the literature, the terms k-conjugate and k-Schur functions are standard. If we were instead working in $R^{k,\ell}$, then we would be using the non-standard terms ℓ-conjugate and ℓ-Schur functions.

Definition 5.2. The *hook-length* of a box c whose coordinate is (i,j) (where (1,1) is the upper left corner) in the diagram of λ is defined to be $\lambda_i+|\{m|\lambda_m\ge j\}|-i-j+1$. Intuitively, when we consider the diagram of λ in the English notation, the hook-length of $c=(i,j)$ is just the number of boxes directly below and to the right of c plus 1 (which counts c itself).

[9]For the definition of a Pfaffian, see this Wikipedia page.

As an example, we label the hook-length of every box in the partition $\lambda = (4, 3, 1, 1)$.

$$\begin{array}{|c|c|c|c|}\hline 7 & 4 & 3 & 1 \\ \hline 5 & 2 & 1 \\ \cline{1-3} 2 \\ \cline{1-1} 1 \\ \cline{1-1}\end{array}$$

Now let fix a positive integer k.

Definition 5.3. We say a partition λ is k**-bounded** if $\lambda_1 \leq k$.

Definition 5.4. We say a partition λ is a $(k+1)$**-core** if no cell in λ has hook-length $k+1$. Note that this definition permits that some boxes can have hooklength strictly greater than $k+1$.

Continuing with the above example, $\lambda = (4, 3, 1, 1)$ is a 4-bounded partition. It is also a 6-core.

Proposition 5.5. *The set of k-bounded partitions and the set of $(k+1)$-cores are in bijection with each other. More precisely, there is a bijective map*

$$p : \{(k+1)\text{-}cores\} \to \{k\text{-}bounded\ partitions\}$$

defined by removing all boxes in a $(k+1)$-core with hook-length greater than $k+1$ and then left-aligning the remaining boxes.

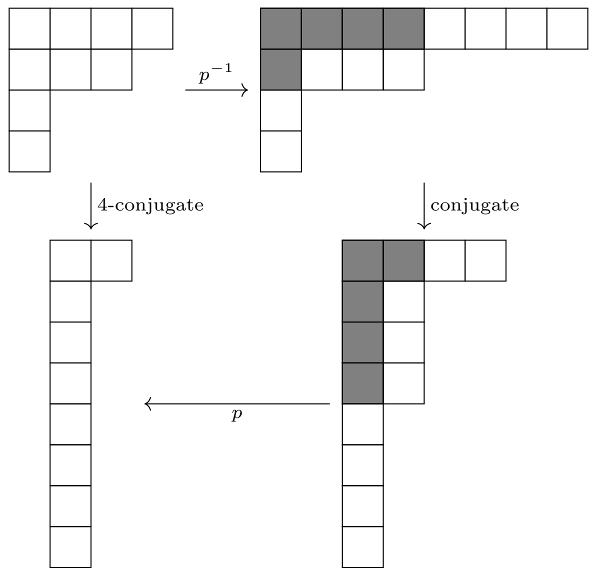
Proof. Following [9, Ch.2 Prop. 1.3], we shall describe the inverse map p^{-1}: Consider a k-bounded partition and work from top to bottom in its diagram; for a given row, calculate the hook-lengths of its boxes; if there is a box with hook-length greater than k, slide this row to the right until all boxes have hook-length less than or equal to k. We omit the rest of the proof as readers can check it themselves or read from [9]. $\square$

This bijection is best described by illustrating an example labelled with hook-lengths. Considering again the 4-bounded partition $\lambda = (4, 3, 1, 1)$ and we apply the map p^{-1} to obtain a 5-core $p^{-1}(\lambda)$.

$$\begin{array}{|c|c|c|c|}\hline 7 & 4 & 3 & 1 \\ \hline 5 & 2 & 1 \\ \cline{1-3} 2 \\ \cline{1-1} 1 \\ \cline{1-1}\end{array} \quad \to \quad \begin{array}{|c|c|c|c|c|c|c|c|}\hline 11 & 8 & 7 & 6 & 4 & 3 & 2 & 1 \\ \hline 6 & 3 & 2 & 1 \\ \cline{1-4} 2 \\ \cline{1-1} 1 \\ \cline{1-1}\end{array}$$

Definition 5.6. Given a k-bounded partition λ, we define its *k-conjugate* $\lambda^{\omega(k)}$ to be $p(p^{-1}(\lambda)')$, i.e. a composition of three maps: first p^{-1}, then the usual conjugation, and finally p.

For example, to obtain the 4-conjugate of $\lambda = (4, 3, 1, 1)$, we do the following operations:



5.4. **k-Schur Functions and Two Conjectural Bases.** There are multiple equivalent definitions of k-Schur functions (with an additional parameter t). However, we shall only use the following definition from [10], which defines "parameterless" k-Schur functions (i.e. $t = 1$).

Definition 5.7. The **k-Schur functions**, indexed by k-bounded partitions, are defined by inverting the unitriangular system:

$$h_\lambda = s_\lambda^{(k)} + \sum_{\mu:\mu \rhd \lambda} K_{\mu\lambda}^{(k)} s_\mu^{(k)} \text{ for all } \lambda_1 \leq k \tag{5.1}$$

where $\mu \rhd \lambda$ is the dominance partial ordering on partitions of a fixed size n defined by the condition $\mu_1 + \cdots + \mu_i > \lambda_1 + \cdots + \lambda_i$ for some i and $\mu_1 + \cdots + \mu_j = \lambda_1 + \cdots + \lambda_j$ for all $j < i$, and $K_{\mu\lambda}^{(k)}$ are the **k-Kostka numbers**, and they are defined as the number of k-tableaux of shape $p^{-1}(\mu)$ and k-weight λ. Precise definitions of these terms can be found in [10].

We will begin with some facts about k-Schur functions.

Fact: The set $\{s_\lambda^{(k)} \mid \lambda_1 \leq k\}$ indexed by all k-bounded partitions is a basis of $\Lambda_k = \mathbb{Q}[h_1, \ldots, h_k]$; Moreover, it induces a basis $\{s_\lambda^{(k)} \mid \lambda \subset (k^\ell)\}$ of $R^{\ell,k} = \mathbb{Q}[e_1, \ldots, e_\ell]/(h_{k+1}, \ldots, h_{k+\ell}) \cong \mathbb{Q}[h_1, \ldots, h_k]/(e_{\ell+1}, \ldots, e_{\ell+k})$ by Proposition 2.17 and [12, Proposition 1] with the following key property:

Proposition 5.8. *The involution in Proposition 2.17, $\omega : \Lambda_k \to \Lambda_k$ that takes any h_λ to e_λ has the following action on the k-Schur basis:*

$$\omega(s_\lambda^{(k)}) = s_{\lambda^{\omega(k)}}^{(k)}$$

That is, for any k-bounded partition λ, the involution ω takes a k-Schur function indexed by λ to a k-Schur function indexed by the k-conjugate of λ. Moreover, if we consider the induced involution ω on $R^{\ell,k}$ and partitions $\lambda \subset (k^\ell)$, the same formula still holds.

Proof. See [10, Theorem 38]. □

To simplify the notation, we denote

$$P^{\ell,k,m} =: \{\lambda \mid \lambda_1 \leq m,\, \lambda^{\omega(k)} \subset (k^\ell)\}$$

and

$$P^{\ell,k} =: \{\lambda \mid \lambda^{\omega(k)} \subset (k^\ell)\}.$$

By proposition 5.8, we see that the set $S_{\ell,k} = \{s_\lambda^{(k)} \mid \lambda \in P^{\ell,k}\}$ is a basis for $R^{\ell,k}$.

Let's recall the following useful definition, which will be used in the proof below.

Definition 5.9. A *k-bounded* partition $\lambda = (\lambda_1, \ldots, \lambda_{\ell(\lambda)})$ is called *m-vacant* if m is the largest integer so that a $((m-1)^m)$ rectangle can fit inside the complement of λ in the $(k^{\ell(\lambda)})$ rectangle.

Now we are ready to give an new combinatorial interpretation of the RHS of the Reiner-Tudose Conjecture. We already know by Propsosition 4.19 that the RHS counts the number of k-bounded m-vacant partitions, we want to show that k-bounded m-vacant partitions are exactly those k-conjugates that are m-bounded: We repeat the definition of an m-vacant Ferrers diagram with respect to the swtich we've made in ℓ and k.

Proposition 5.10. *For any $1 \leq m \leq k$,*

$$\sum_{\lambda \in P^{\ell,k,m}} q^{|\lambda|} = 1 + \sum_{i=1}^{m} q^i \begin{bmatrix} k \\ i \end{bmatrix}_q \left(\sum_{j=0}^{\ell-i} q^{j(k-i+1)} \begin{bmatrix} i+j-1 \\ j \end{bmatrix}_q \right) \tag{5.2}$$

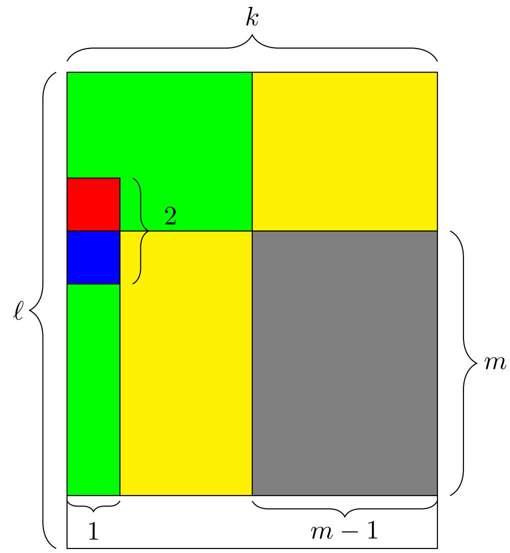
In other words, the RHS of the Reiner-Tudose Conjecture is counting partitions in (k^ℓ) whose k-conjugate is m-bounded.

Proof. Let's first fix m and j, and set $\ell(\lambda)$ to be $m+j$, then the formula

$$q^i \begin{bmatrix} k \\ i \end{bmatrix}_q q^{j(k-m+1)} \begin{bmatrix} m+j-1 \\ j \end{bmatrix}_q$$

counts k-bounded and m-vacant partitions $\lambda = (\lambda_1, ..., \lambda_{\ell(\lambda)})$.

As shown in the Figure below, the white and gray parts are empty. The green-colored part (including the red and blue boxes inside it) indicates the parts that λ must have in order for $((m-1)^m)$ to be the maximal rectangle fitting in the complement. The yellow-colored part is however, an optional part of λ. This correspondence between m-vacant partitions and the combinatorial formula in the conjecture was first observed and explained by Reiner and Tudose in [5, Prop. 8] and is also explained in Proposition 4.19.



ambient rectangle: (k^ℓ)

gray maximal rectangle: $((m-1)^m)$

green part: $((k-m+1)^j) \cup (1^m)$

yellow part: unknown size

Our goal is to show that when we apply the map p^{-1} to such a λ, we will always get a m-bounded but not $(m-1)$-bounded partition, i.e. $(\lambda^{\omega(k)})_1 = m$.

Lemma 5.11. *For a k-bounded partition λ, we have that λ is m-vacant if and only if $\lambda^{\omega(k)}$ is m-bounded, but not $(m-1)$-bounded.*

Proof of Lemma. It suffices to show that the first part of $\lambda^{\omega(k)}$ has exactly m boxes, since $\lambda^{\omega(k)}$ is always a partition, and hence its parts form a weakly decreasing sequence. Recall the construction of p^{-1}: Working from bottom to top in the diagram of λ, for a given row, calculate the hook-lengths of its boxes; if there is a box with hook-length greater than k, slide this row to the right until all boxes have hook-length less than or equal to k.

In the process of constructing $\lambda^{\omega(k)}$, the box blue $b = (j+1, 1)$ will not move because its hook-length is at most $(k-m+1)+m-1 = k$. Meanwhile, the red box $r = (j, 1)$ must slide to the right (for at least 1 unit) because its hook-length is at least $(k-m+1)+(m+1)-1 = k+1$.

Therefore, the first part of $\lambda^{\omega(k)}$ must have exactly m boxes, which is what we claimed. Conversely, if we start with $\lambda^{\omega(k)}$ whose first part has exactly m' boxes where $m' \neq m$ is a positive integer, then by the same argument, λ is m'-vacant. In particular, λ is not m-vacant. □

Lemma 5.11 then completes the proof of this Proposition. □

It seems tempting to conclude that the basis $\{s_\lambda^{(k)} \mid \lambda \in P^{\ell,k}\}$ would be the key ingredient to prove the conjecture. However, this is problematic for the following reason: for a m-bounded $\lambda \in P^{\ell,k}$, the basis element $s_\lambda^{(k)}$ is not necessarily contained in the m-th subalgebra $R^{\ell,k,m}$. This suggests us to consider the following two sets of elements:

(1) $\{h_\lambda \mid \lambda \in P^{\ell,k}\}$
(2) $\{s_\lambda^{(i)} \mid \lambda \in P^{\ell,k}$ and $\lambda_1 = i\}$

Either of these two sets has the property that the element indexed by a m-bounded partition is inside the m-th subalgebra $R^{\ell,k,m}$.

More precisely, we make the following conjectures about these two sets:

Conjecture 5.12. *$P^{\ell,k}$ can be used as the index set for complete homogeneous functions and i-Schur functions where i depends on each individual index:*

(1a) *$\mathcal{H}_{\ell,k} := \{h_\lambda \mid \lambda \in P^{\ell,k}\}$ is a basis of $R^{\ell,k}$.*
(1b) *$\mathcal{H}_{\ell,k,m} := \{h_\lambda \mid \lambda \in P^{\ell,k,m}\}$ is a basis of $R^{\ell,k,m}$.*
(2a) *$\{s_\lambda^{(i)} \mid \lambda \in P^{\ell,k}$ and $\lambda_1 = i\}$ is a basis of $R^{\ell,k}$.*
(2b) *$\{s_\lambda^{(i)} \mid \lambda \in P^{\ell,k,m}$ and $\lambda_1 = i\}$ is a basis of $R^{\ell,k,m}$.*

In both conjectures above, part (b) is stronger than part (a). However, part (a) suffices to prove one side of the inequality in the R-T conjecture. Without loss of generality, we assume that (1a) is true and demonstrate how we could get the R-T conjecture from it:

Theorem 5.13. *If Conj. 5.12 (1a) holds, then in the Reiner-Tudose Conjecture, the left hand side is degree-wise greater than or equal to its right hand side, i.e.:*

$$\mathrm{Hilb}(R^{\ell,k,m}, q) \geq 1 + \sum_{i=1}^{m} q^i \begin{bmatrix} k \\ i \end{bmatrix}_q \left(\sum_{j=0}^{\ell-i} q^{j(k-i+1)} \begin{bmatrix} i+j-1 \\ j \end{bmatrix}_q \right) \tag{5.3}$$

Note that we have reversed the roles of k and ℓ in the original conjecture for convenience, which we can do by Proposition 2.15.

Proof. Since the $\mathsf{span}(\mathcal{H}_{\ell,k,m}) \subset R^{\ell,k,m}$, we clearly have

$$\mathrm{Hilb}(R^{k,\ell,m}, q) \geq \mathrm{Hilb}(\mathsf{span}(\mathcal{H}_{\ell,k,m}), q)$$

where the inequality is interpreted as a degree-wise inequality. But by definition, $\mathrm{Hilb}(\mathsf{span}(\mathcal{H}_{\ell,k,m}), q)$ is just $\sum_{\lambda \in P^{\ell,k,m}} q^{|\lambda|}$. Thus the theorem follows easily from Proposition 5.10. $\square$

5.5. **Correspondence Between Filtered and Monomial Basis.** Initially, we hoped that k-conjugation gives the 1-1 correspondence we were looking for in Subsection 4.8. However, this does not work as we found a counterexample when $k = 6, \ell = 5, m = 3$. Indeed, the degree 13 part of $D^{6,5,3}$ is

$$\{h_3^2 h_2^3 h_1, \quad h_3^3 h_1^4, \quad h_3^3 h_2 h_1^2, \quad h_3^3 h_2^2, \quad h_3^4 h_1\}.$$

However, the k-conjugates of the strong 3-vacant partitions in the (6^5) box is

$$\{h_3 h_1^{10}, \quad h_3 h_2^5, \quad h_3 h_2^4 h_1^2, \quad h_3^3 h_2^2, \quad h_3^2 h_2^3 h_1\}.$$

Note that this does not mean that a correspondence between the doubly filtered basis and the monomial basis does not exist, it merely means that the k-conjugation does not define the 1-1 correspondence we are looking for.

If we manage to demonstrate that the h_λ indexed by λ which are k-conjugates of weak m-vacant partitions form a basis of the m'th filtered $R^{k,\ell,m}$, it amounts to proving the Reiner-Tudose Conjecture. This motivates us to seek this 1-1 correspondence. We initially observed based on Macaulay2 data, that the basis of $R^{k,\ell}$ obtained from indexing the generators h_λ where λ is a k-conjugated Ferrers diagram fitting in an (ℓ^k) box is the same as the monomial basis. However, this is not true in general, as $R^{3,3}$ presents a counterexample; for $m = 2$, not all basis elements of degree 7 agree. Where $h_2 h_1^5$ is a standard monomial given by Gröbner basis, but it is not in $\mathcal{H}_{k,\ell,m}$ (however, $h_2^2 h_1^3 \in \mathcal{H}_{k,\ell,m}$). This is a counterexample that shows that the $\mathbb{Q}$-basis of standard monomials with respect to the lex Gröbner basis and the set $\mathcal{H}_{k,\ell,m}$ are not the same.

6. Implications

In this section we discuss how, if assuming Conj. 5.12 (1a) holds, Theorem 5.13 would lead to a significant simplification of the proof of [5, Thm. 5] (Hoffman's Theorem), and we comment on how our result may lead to a possible k-Schur analogue in other Lie types.

6.1. **Simplifying Hoffman's Theorem.** We re-present here [5, Conj. 2]

Conjecture 6.1. *For $d \geq k\ell - m^2 + m + 1$ we have $R_d^{k,\ell} = R_d^{k,\ell,m-1}$.*

By [5], we know that the Reiner-Tudose Conjecture $\implies$ Conjecture 6.1. However, since in Theorem 5.13 we only prove one side of the inequality, we now want to show why the mere inequality in Theorem 5.13 suffices to imply Conjecture 6.1.

Proposition 6.2. *Theorem 5.13 $\implies$ Conjecture 6.1*

Proof. Since $R^{k,\ell} \supseteq R^{k,\ell,m-1}$ and hence $R_d^{k,\ell} \supseteq R_d^{k,\ell,m-1}$, this means Conjecture 6.1 is equivalent to saying that

$$\dim_{\mathbb{Q}} R_d^{k,\ell} = \dim_{\mathbb{Q}} R_d^{k,\ell,m-1}$$

whenever $d \geq k\ell - m^2 + m + 1$, or equivalently

$$\mathrm{Hilb}(R^{k,\ell}, q) - \mathrm{Hilb}(R^{k,\ell,m-1}, q) = \sum_{d=0}^{k\ell} \left(\dim_{\mathbb{Q}} R_d^{k,\ell} - \dim_{\mathbb{Q}} R_d^{k,\ell,m-1}\right) \cdot q^d$$

has no q^d terms for $d \geq k\ell - m^2 + m + 1$, that is, it has degree in q at most $k\ell - m^2 + m$. However, we can re-express

$$\begin{aligned}\mathrm{Hilb}(R^{k,\ell}, q) - \mathrm{Hilb}(R^{k,\ell,m-1}, q) &= \mathrm{Hilb}(R^{k,\ell,k}, q) - \mathrm{Hilb}(R^{k,\ell,m-1}, q) \\ &= \sum_{p=m}^{k} \left(\mathrm{Hilb}(R^{k,\ell,p}, q) - \mathrm{Hilb}(R^{k,\ell,p-1}, q)\right) \\ &= \sum_{p=m}^{k} \mathrm{Hilb}(R^{k,\ell,p}/R^{k,\ell,p-1}, q).\end{aligned}$$

Since we want to show the left side above has degree in q at most $k\ell - m^2 + m$, it suffices to show that the same is true for every summand on the right side. As done in [5], Theorem 5.13 is equivalent to the following assertion: for $p \geq 1$, the quotient (graded) vector space $R^{k,\ell,p}/R^{k,\ell,p-1}$ has Hilbert Series

$$\mathrm{Hilb}(R^{k,\ell,p}/R^{k,\ell,p-1}, q) \geq \sum_{j=0}^{k-p} q^{j(\ell-p+1)} \begin{bmatrix} p+j-1 \\ j \end{bmatrix}_q q^p \begin{bmatrix} \ell \\ p \end{bmatrix}_q . \tag{6.1}$$

Thus we want to show that the right hand side of equation 6.1 has degree in q at most $k\ell - m^2 + m$ whenever $p \geq m (\geq 1)$. Since the q-binomial coefficient $\begin{bmatrix} r+s \\ r \end{bmatrix}_q$ has degree rs as a polynomial in q, the right hand side of equation 6.1 has degree in q equal to

$$\begin{aligned} \max_{j=0,1,\dots,k-p}\{j(\ell-p+1)+p+(p-1)j+p(\ell-p)\} &= \max_{j=0,1,\dots,k-p}\{\ell(j+p)-p^2+p\} \\ &= k\ell - p^2 + p \end{aligned}$$

and this is bounded above by $k\ell - m^2 + m$ for $p \geq m \geq 1$.

Next, we take the sum of the RHS of equation 6.1 ranging from $p = m$ to $p = k$, that is:

$$\sum_{p=m}^{k} \left(\sum_{j=0}^{k-p} q^{j(\ell-p+1)} \begin{bmatrix} p+j-1 \\ j \end{bmatrix}_q q^p \begin{bmatrix} \ell \\ p \end{bmatrix}_q \right),$$

and we see that the degree of this expression is still bounded above by $k\ell - m^2 + m$.

Now, multiplying the expression in Theorem 5.13, by -1, the inequality is reversed to obtain:

$$-\operatorname{Hilb}(R^{\ell,k,m}, q) \leq -(1 + \sum_{i=1}^{m} q^i \begin{bmatrix} k \\ i \end{bmatrix}_q \left(\sum_{j=0}^{\ell-i} q^{j(k-i+1)} \begin{bmatrix} i+j-1 \\ j \end{bmatrix}_q \right)) \tag{6.2}$$

The key point here is that, by applying equation 6.2, we get:

$$\operatorname{Hilb}(R^{k,\ell}, q) - \operatorname{Hilb}(R^{k,\ell,m-1}, q) \leq \operatorname{Hilb}(R^{k,\ell}, q) - \left(1 + \sum_{i=1}^{m-1} q^i \begin{bmatrix} \ell \\ i \end{bmatrix}_q \left(\sum_{j=0}^{k-i} q^{j(\ell-i+1)} \begin{bmatrix} i+j-1 \\ j \end{bmatrix}_q \right) \right).$$

Also, $Hilb(R^{k,\ell,k}, q) = Hilb(R^{k,\ell}, q)$ and

$$Hilb(R^{k,\ell}, q) = \left(1 + \sum_{i=1}^{k} q^i \begin{bmatrix} \ell \\ i \end{bmatrix}_q \left(\sum_{j=0}^{k-i} q^{j(\ell-i+1)} \begin{bmatrix} i+j-1 \\ j \end{bmatrix}_q \right) \right)$$

since the $m = k$ case has been proven by [5]. Thus, we have

$$\begin{aligned} \operatorname{Hilb}(R^{k,\ell}, q) - \operatorname{Hilb}(R^{k,\ell,m-1}, q) &\leq \operatorname{Hilb}(R^{k,\ell}, q) - \left(1 + \sum_{i=1}^{m-1} q^i \begin{bmatrix} \ell \\ i \end{bmatrix}_q \left(\sum_{j=0}^{k-i} q^{j(\ell-i+1)} \begin{bmatrix} i+j-1 \\ j \end{bmatrix}_q \right) \right) \\ &= \sum_{p=m}^{k} \left(\sum_{j=0}^{k-p} q^{j(\ell-p+1)} \begin{bmatrix} p+j-1 \\ j \end{bmatrix}_q q^p \begin{bmatrix} \ell \\ p \end{bmatrix}_q \right) \qquad (*) \end{aligned}$$

where the inequality in (*) is a degree-wise inequality.

This demonstrates that the degree of (*) is bounded above by $k\ell - m^2 + m$, hence the degree of $\operatorname{Hilb}(R^{k,\ell}, q) - \operatorname{Hilb}(R^{k,\ell,m-1}, q)$ must also be bounded above by $k\ell - m^2 + m$. $\square$

We now briefly review the implication of the above results. Recall that Conjecture 6.1 is the same as [5, Conj. 2], and [5, §3] explains why [5, Conj. 2] $\Longrightarrow$ [5, Conj. 3] $\Longrightarrow$ [5, Conj. 4]. Moreover [5, SS4, 5] demonstrates, using an inductive argument on m and making use of the fact that $R^{k,\ell}$ satisfies the Hard Lefschetz

Theorem that [5, Conj. 4] leads to a significant shortening of the proof of the following theorem of Hoffman. To state it, define for each $\alpha \in \mathbb{Q}^\times$ a ring endomorphism $\phi_\alpha : R \to R$ on a graded $\mathbb{Q}$-algebra R via $\phi_\alpha(x) = \alpha^d x$ for homogeneous elements x in R_d.

Theorem 6.3. *(Hoffman, [5]) For $k \neq \ell$, every graded algebra endomorphism $\phi : R^{k,\ell} \longrightarrow R^{k,\ell}$ which does not annihilate, $R_1^{k,\ell}$ is of the form ϕ_α.*
For $k = \ell$ any such endomorphism is either of the form ϕ_α or $\omega \circ \phi_\alpha$.

Here ω is the involution in Proposition 2.17. Not only does this characterise endomorphisms on $R^{k,\ell}$ that are nonzero on $R_1^{k,\ell}$, but it also constitutes a significant progress towards proving the stronger Hoffman Theorem conjectered in [14]. Furthermore, via the Lefschetz fixed point theorem, we conclude that $Gr(k, \mathbb{C}^{k+\ell})$ has the *fixed point property* if and only if $k\ell$ is odd [5]. This contributes to the identification of manifolds with this important property, which is intriguing as they are relatively scarce [14].

6.2. **k-Schur Analogue for Other Lie Types.** It is worth noting that there exists an analogue of the Reiner-Tudose conjecture in Lie type C, namely the Lagrangian Conjecture 2.3. The argument we used to prove Theorem 5.13 involves tools such as k-Schur functions and k-conjugation. It is not currently clear to us whether we can advance to prove half of Conj. 2.3 using a similar argument as in Conj. 2.1. However, the fact that these two conjectures have strikingly similar forms may strongly suggest that there exists an analogue of k-Schur functions and k-conjugation in the setting of Lie type C. This may be an ending point for this research, but it is an exciting starting point for a future project.

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