Cyclic Group Invariants and Free Resolutions

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Introduction

The main overarching theme for this problem involved invariants of the polynomial ring over the action by the cyclic group C_n.

This presentation is split up into three parts. As a rough overview, we will discuss:

- the Hilbert series of the coinvariant algebra,
- free resolutions of isotypic components, and
- some progress on the Garsia-Stanton method.

Basic Definitions

Fix some $n \geq 1$, and define $S := \mathbb{C}[y_0, \ldots, y_{n-1}]$. Let the cyclic group C_n act **diagonally** on S, i.e., if $C_n = \langle g \rangle$, then

$$g \cdot y_k = \zeta_n^k y_k,$$

where ζ_n is a primitive n^{th} root of unity. The polynomials of S fixed by this action forms the C_n-**invariant subring** S^{C_n}, from whence we may define the C_n-**coinvariant algebra** is the quotient ring $T = S/(S_+^{C_n})$.

Basic Definitions

We can view S as an S^{C_n}-graded module. Under the cyclic action, we have the decomposition

$$S = S^{\chi_0} \oplus S^{\chi_1} \oplus \cdots \oplus S^{\chi_{n-1}},$$

where S^{χ_k} is the k^{th} **isotypic component** defined as

$$S^{\chi_k} := \{f(\underline{y}) \in S : g \cdot f = \zeta_n^k f, \text{ where } C_n = \langle g \rangle\}.$$

Coefficients of the Hilbert Series

The coinvariant algebra T is graded (by degree), so we can construct its **Hilbert series**, given by

$$\mathrm{Hilb}(T) := \sum_{d \in \mathbb{N}} t^d \cdot \dim_{\mathbb{C}} T_d.$$

Note that in our case, there will always be a way to write the Hilbert series as a rational function.

Coefficients of the Hilbert Series

Our main question about this Hilbert Series is to compute the values of $\dim_{\mathbb{C}} T_d$ and/or determine a basis for this space.

It turns out that we can do this for sufficiently large d. For instance, we have the following result:

Proposition (Garg-L.-Ren-S.)

T vanishes at degrees n and higher. In other words, the t^i coefficient of $\mathrm{Hilb}(T, t)$ is 0 for $i \geq n$.

Coefficients of the Hilbert Series

For $d \geq \frac{n+1}{2}$, we are able to describe the dimension of our space, as well as give a basis:

Theorem (Garg-L.-Ren-S.)

The coefficient of t^{n-i} in $\mathrm{Hilb}(S/(S_+^{C_n}), t)$ for $1 \leq i \leq \frac{n-1}{2}$, is

$$\phi(n) \sum_{j=0}^{i-1} p(j),$$

where $p(i)$ is the number of partitions of i.

Coefficients of the Hilbert Series

Theorem (Garg-L.-Ren-S.)

(cont.) One basis for T_d are elements of the form

$$y_s^{\,n-i-\sum_{j=1}^{n-1}\lambda_i} y_{2s}^{\lambda_1} y_{3s}^{\lambda_2} \cdots y_{(n-1)s}^{\lambda_{n-2}},$$

where $(\lambda_1, \lambda_2, \ldots, \lambda_{n-1})$ is a partition of some integer $0 \leq k < i$ and $\gcd(s, n) = 1$, taking indices modulo n.

For instance, when $n = 6$, the above theorem generates the following elements in the basis. Note here that we can take $d = 5$ and $d = 4$:

$$y_1^5, y_5^5, y_1^4, y_1^3 y_2, y_5^4, y_5^3 y_4.$$

Coefficients of the Hilbert Series

It turns out that for $d < \frac{n+1}{2}$ that this problem is significantly harder. We can give an explicit formula for the dimension of the spaces when $d = 0, 1, 2, 3$:

- $\dim_{\mathbb{C}} T_0 = 1$,
- $\dim_{\mathbb{C}} T_1 = n - 1$,
- $\dim_{\mathbb{C}} T_2 = 2\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$,
- $\dim_{\mathbb{C}} T_3 = \binom{n+1}{3} - (n-1)\lfloor \frac{n}{2} \rfloor - 2\sum_{j=0}^{\lfloor n/3 \rfloor} (\lfloor \frac{n-3j}{2} \rfloor + 1)$,

but already at $d = 3$ this is rather unwieldy. For more, confer a paper by Zeng and Li.

Free Resolutions

We will now discuss free resolutions of S over S^{C_n}. Given a finitely-generated module M over Noetherian ring R, a **free resolution** of M is an exact sequence

$$\cdots \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} M \longrightarrow 0,$$

where $F_0, F_1, \ldots$ are free $R-$modules.

We can also give a grading to our free resolution if the module and ring are graded.

Throughout, we will be interested in the **minimal free resolution**, where the matrices for ϕ_i have no scalars.

Free Resolutions

For general C_n, we are interested in computing a graded free resolution of S^{χ_k, C_n} as an S^{C_n}-module, i.e.,

$$\cdots \xrightarrow{\phi_2} \bigoplus_{i=1}^{t'} S^{C_n}(-d_i') \xrightarrow{\phi_1} \bigoplus_{i=1}^{t} S^{C_n}(-d_i) \xrightarrow{\phi_0} S^{\chi_k, C_n} \longrightarrow 0.$$

We can think of our work with the Hilbert series of the coinvariant algebra as giving a description of ϕ_0.

The spaces in our free resolution can be described by the **Betti numbers** $\beta_{i,j}$, the dimension of the component of degree j in F_i.

Free Resolution in the C_4 Case

The cases for when $n = 2$ and $n = 3$ are relatively simple and already well understood. In particular, the $n = 3$ free resolution is 2-**periodic**.

For $n = 4$, we were able to obtain a concrete description of the free resolutions of S^{χ_1, C_4} and S^{χ_2, C_4} as S^{C_4}-modules. The action of the **nontrivial element** of $(\mathbb{Z}/4\mathbb{Z})^\times$ allows us to obtain the S^{χ_3, C_4} free resolution from that for S^{χ_1, C_4}.

Free Resolution in the C_4 case

It turns out that for $i \geq 4$, we have a **recursive structure** to our maps. In particular, we can write

$$\phi_i^1 = \begin{pmatrix} \phi_{i-1}^3 & 0 & 0 & 0 & 0 \\ 0 & \phi_{i-1}^2 & 0 & 0 & (-1)^i y_1 y_3 I \\ 0 & 0 & \phi_{i-2}^1 & Y_i & 0 \\ 0 & 0 & 0 & \phi_{i-2}^3 & 0 \\ 0 & 0 & 0 & 0 & \phi_{i-2}^2 \end{pmatrix},$$

$$\phi_i^2 = \begin{pmatrix} \phi_{i-1}^2 & 0 & 0 & 0 & 0 \\ 0 & \phi_{i-1}^2 & 0 & (-1)^i y_1^4 I & 0 \\ 0 & 0 & \phi_{i-2}^2 & 0 & Y_i' \\ 0 & 0 & 0 & \phi_{i-2}^2 & 0 \\ 0 & 0 & 0 & 0 & \phi_{i-2}^2 \end{pmatrix}.$$

Free Resolution in the C_4 case

As a result, we also can explicitly state what the ranks of our free modules in the free resolution are:

Corollary (Garg-L.-Ren-S.)

For $i \geq 2$, where $(i,j) \neq (2,4)$, the following recurrence holds for the minimal free resolutions of S^{χ_1} and S^{χ_2} :

$$\beta_{i,j} = 2\beta_{i-1,j-3} + \beta_{i-1,j-4}.$$

Asymptotics for the General Case

Although we don't yet have a specific description for each n, we still can make some statements about asymptotic behavior of the free resolutions.

For a lower bound, we have the following:

Proposition (Garg-L.-Ren-S.)

In the resolution of S^{χ_k, C_n} as a S^{C_n} module, suppose k is relatively prime to n. Then, $\beta_{i,j} \neq 0$ for all $i \geq 0, j \in [3i+1, ni+n-1]$.

This gives us that

$$\operatorname{rank} F_i \geq \dim_{\mathbb{C}}(T^{\chi_k})(\dim_{\mathbb{C}}(T^{\chi_k}) - 1)^i.$$

Asymptotics for the General Case

As for an upper bound, we can say the following:

Proposition (Garg-L.-Ren-S.)

If $\beta_{i,j} \neq 0$*, then we require* $j \leq (i+1)(n^2 - n)$*.*

This yields us with the bound

$$\operatorname{rank} F_i \leq n^{(i+1)n}((i+1)!)^n.$$

Asymptotics for the General Case

Neither of these bounds are particularly sharp. From `Macaulay2` data we formulated the following (stronger) conjecture:

Conjecture

In the resolution of S^{χ_k, C_n}, for a given level i, we have that the set of j so that $\beta_{i,j}$ is nonzero lies within the interval

$$[3i + 1, ni + n - 1].$$

The Garsia–Stanton Method

Garsia and Stanton suggested in their paper [1] a way to formulate a basis for S as an S^G−module, for a finite group G, by considering the partitioning of a simplicial complex.

Reiner and White proposed in their preprint [2] a partitioning that would allow us to explicitly state such a basis with $G = C_n$, when n is prime. In order for this to be satisfied, two statements needed to be proven. The first has been proven:

Proposition (Garg-L.-Ren-S.)

The formula $\Delta_n/G = \sqcup[w|_{Des(w)}, w]$ given as Question 6.1 in [2] is indeed a partition.

The Garsia–Stanton Method

The second statement requires us to show that a certain incidence matrix is invertible. Some data was computed, however examples were sparse as our matrix quickly became too large at $n = 11$, demanding a $10! \times 10!$ matrix to be inverted.

Future Directions

- It would be illuminating to understand the earlier coefficients in the Hilbert series.
- Mastery over the free resolutions of C_n where $n \geq 5$ is required.
- Swapnil proclaimed that he would invert the incidence matrix from Garsia–Stanton by the end of the REU. We await his results.

References

A. M. Garsia and D. Stanton, *Group actions of Stanley-Reisner rings and invariants of permutation groups*, Adv. in Math. **51** (1984), no. 2, 107–201. MR 736732

V. Reiner and D. White, *Some notes on polya's theorem, kostka numbers and the rsk correspondence*, 2012; available at `http://www-users.math.umn.edu/∼reiner/Papers/Unfinishec`

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