# Puzzles, Ice, and Grothendieck Polynomials

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# Outline

# Partitions

## Definition

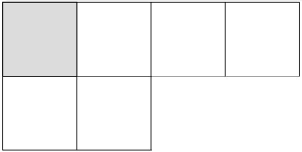
A **partition** $\lambda$ is a string of weakly decreasing nonnegative integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

## Definition

A **skew partition** $\lambda/\mu$ is a set of two partitions $\lambda, \mu$ such that $\forall i,\ \lambda_i \geq \mu_i$.

## Example

A skew partition diagram of shape $(4,2)/(1,0)$



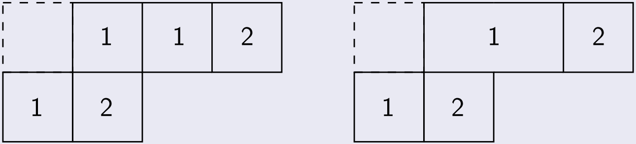
# Tableaux

## Definition

A **semistandard tableau** of shape $\lambda/\mu$ is a filling of the Young/Ferrers diagram from $[n] = 1, ..., n$, with weakly increasing rows and strictly increasing columns

## Definition

Two **valued set tableaux** of shape $(4, 2)/(1, 0)$



# $j_{\lambda/\mu}$

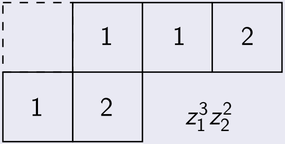
## Definition

The dual weak symmetric Grothendieck polynomial for $\lambda/\mu$

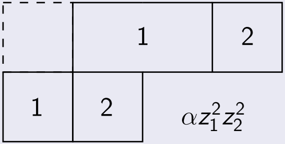
$$j_{\lambda/\mu}(\mathbf{z},\alpha) = \sum_{T \in VST_{\lambda/\mu}} \alpha^{|\lambda/\mu| - |T|} \mathbf{z}^{wt(T)}.$$

## Example

$\alpha^{|\lambda/\mu| - |T|} \mathbf{z}^{wt(T)}$ for two valued set tableaux $T$




$z_1^3 z_2^2$




$\alpha z_1^2 z_2^2$ .

# What is a Lattice Model?

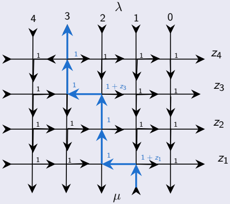


- Model particle interactions within thin sheets of matter.
- Model classes of polynomials.
- We want to model Grothendieck polynomials to demonstrate certain polynomial identities.
  - Cauchy identities with families of dual polynomials
  - Littlewood-Richardson rule
  - Pieri/branching rules

# What is a Lattice Model?



- Boundary conditions are fixed by skew partition $\lambda/\mu$.
- Vertices have a choice of weights.
- Edges are labeled with arrows/orientations, ICE.
- The **state** is a choice of orientation for each edge.
- The **system** is the set of all states.



# Partition Functions



**Definition**

The partition function of a lattice model over partition $\lambda/\mu$ is

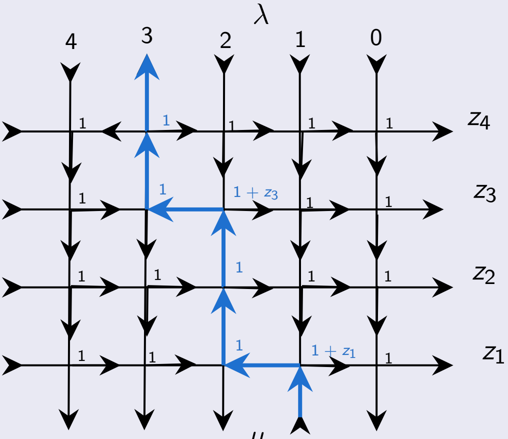
$$Z\left(\mathfrak{S}_{\lambda/\mu}(\mathbf{z})\right) = \sum_{S \in \mathfrak{S}_{\lambda/\mu}} wt(S)$$

where wt(S) is the product of weights of each vertex in the lattice model.

# Partition Functions



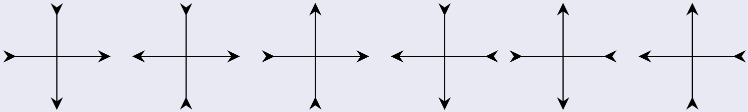
## Example




$\lambda = (3), \mu = (1)$

# A Model for $j_{\lambda/\mu}$

## Definition (Our Boltzmann Weights)



$1 \qquad \alpha + z_i \qquad 1 \qquad z_i \qquad 1 \qquad 0$

## Theorem (Bullock-Caplinger-C.-D.-Shemy)

*Under our choice of Boltzmann weights, for skew partition $\lambda/\mu$,*

$$j_{\lambda/\mu}(z,\alpha) = Z\left(\mathfrak{S}_{\lambda/\mu}(\mathrm{z})\right).$$

# Finding a Compatible Model for $G_\lambda$

## Definition

The stable symmetric Grothendieck polynomial for $\lambda$

$$G_\lambda(\mathbf{z}) = \sum_{T \in SVT_\lambda} \mathbf{z}^{wt(T)}$$

## Proposition (BCCDS)

*There are no top-bottom lattice models for $G_\lambda$ satisfying the following conditions:*

- *Horizontal lattice lines are in direct correspondence with variables $z_1, \ldots, z_n$.*
- *ICE holds, with a 5-vertex model.*
- *There is a bijection between SSYTs and states in the lattice model.*

# Schur Polynomials



**Definition**

The **Schur polynomials** can be defined as,

$$s_\lambda(\mathbf{z}) = \sum_{T \in SSYT(\lambda)} \mathbf{z}^{wt(T)}.$$

These give a vector space basis for the symmetric polynomials in $z_1, z_2, \dots, z_n$.

# Littlewood-Richardson



**Littlewood-Richardson Rule**

There exists some unique expansion,

$$s_\lambda \cdot s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu.$$

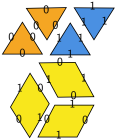
**Theorem (Knutson, Tao, Woodward)**

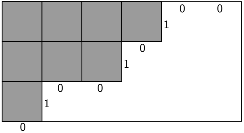
*The* Littlewood-Richardson coefficients, $c^\nu_{\lambda\mu}$, *count puzzle tilings with boundaries determined by* $\lambda$, $\mu$, *and* $\nu$.

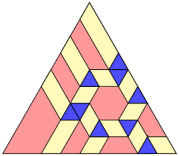
# Puzzles

## Definition

A **puzzle** of size *n* is a filling of an equilateral triangle of side length *n* with KTW tiles such that adjacent edge labels match.







## Example

For $\lambda = (2, 1, 0)$, $\mu = (3, 2, 0)$, and $\nu = (4, 3, 1)$
binary string of $\nu$ and puzzle tiling with boundary $\Delta^{\nu}_{\lambda\mu}$.

# The Connection

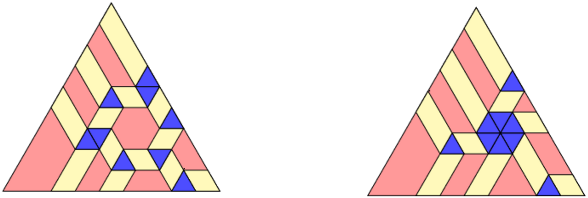


**Theorem (KTW '04)**

*For $\lambda, \mu, \nu$ that fit in a $k \times (n-k)$ ambient rectangle and $|\nu| = |\lambda| + |\mu|$, the number of possible tilings of a puzzle with fixed boundary $\Delta^{\nu}_{\lambda\mu}$ is $c^{\nu}_{\lambda\mu}$.*

**Example**

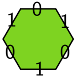
For $\lambda = (2,1,0)$, $\mu = (3,2,0)$, $\nu = (4,3,1)$, we get $c^{\nu}_{\lambda,\mu} = 2$.



# Green Hexagons and $j^{\nu}_{\lambda\mu}$

**Theorem (Pylyvaskyy, Yang '18)**

*For $\lambda, \mu, \nu$ that fit in a $k \times (n-k-1)$ ambient rectangle and $|\nu| \leq |\lambda| + |\mu|$, the number of green hexagon tilings with boundary $\Delta^{\nu}_{\lambda\mu}$ is $d^{\nu}_{\lambda,\mu}$.*




**Littlewood-Richard Rule, $j^{\nu}_{\lambda\mu}$ specific**

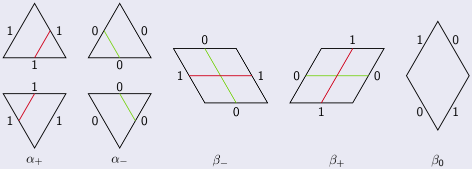
There exists some unique expansion

$$j_\lambda \cdot j_\mu = \sum_{\nu} (-1)^{|\nu|-|\lambda|-|\mu|} d^{\nu}_{\lambda\mu} j_\nu.$$

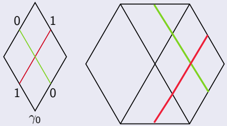
# Path Model

## Path Tiles inspired by Zinn-Justin

We set $+ = 1$ and $- = 0$. These correspond to KTW tiles.




This additional Z-J tile is needed to draw paths through green hexagons.

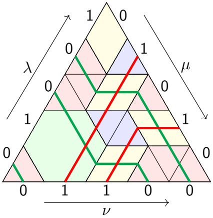


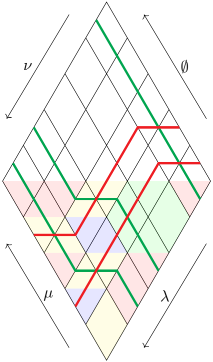
# Puzzle to Lattice Model

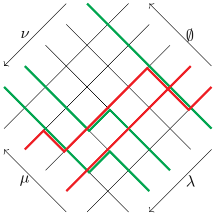


## Example

For $\lambda = (2, 0)$, $\mu = (2, 1)$, and $\nu = (2, 2)$.



# Future Work

- Given a $\lambda$, $\mu$, $\nu$ we want a choice of Boltzmann weights which gives us the corresponding $d^{\nu}_{\lambda\mu}$.
- Attach puzzle lattice model to our $j^{\nu}_{\lambda\mu}$ lattice model so that it satisfies the Yang-Baxter equation.