PROBLEM 1: INVESTIGATING $\mathfrak{S}_n$-EQUIVARIANT KOSZUL ALGEBRAS FROM THE BOOLEAN LATTICE

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Abstract. Koszul algebras are a class of quadratic algebras with particularly rich structure. To any Koszul algebra we can associate to it a (generally noncommutative) *Koszul dual* algebra which inherits any group action on the original algebra. We study two $\mathfrak{S}_n$-equivariant Koszul algebras arising from the Boolean lattice: the Chow ring of the Boolean matroid and the "colorful ring" associated to the barycentric subdivision of a simplex. The dimensions of the graded pieces of both of these algebras correspond to the Eulerian numbers, but the bases and representations for the graded pieces of the two algebras are vastly different. Here we compare and contrast the bases and representations for each of these algebras and their Koszul duals. Along the way, we investigate (sometimes noncommutative) Gröbner bases for these algebras and their Koszul duals, and we prove branching rules for representations of the symmetric group which categorify a recursion on the Eulerian numbers.

Contents

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1. Introduction

Koszul algebras were introduced by Priddy in [Pri70] to study the homological properties of algebras arising in algebraic topology. An algebra A is *Koszul* if the residue field $\Bbbk$ has a linear minimal free resolution. Priddy constructed a canonical resolution for $\Bbbk$ over a Koszul algebra A in terms of the graded pieces of A and its associated *Koszul dual*, denoted $A^!$ ("A-shriek"). In this paper, we study two $\mathfrak{S}_n$-equivariant Koszul algebras that are quotients of the Stanley-Reisner ring of the Boolean lattice $\mathcal{B}_n$ as well as their Koszul duals.

The first of these is the Chow ring associated to the Boolean matroid, $\mathrm{Chow}(\mathcal{B}_n)$. Chow rings of atomic lattices were introduced by Feichtner and Yuzvinsky [FY04], who found a Gröbner basis for the defining ideals of Chow rings described a monomial basis for its graded components which we refer to as the *Feichtner-Yuzvinsky basis*. Adiprasito, Huh, and Katz used Chow rings of matroids to resolve the long-standing Heron-Rota-Welsh conjecture, and showed that these rings satisfy the *Kahler package*: Poincaré Duality, the hard Lefschetz theorem, and the Hodge-Riemann relations [AHK18]. Recently, Mastroeni and McCullough proved that the Chow ring of a matroid is a Koszul Algebra in [MM2211], which permits us to study their Koszul duals.

The second Koszul algebra we will study is what we call the *colorful ring* for the barycentric subdivision of a simplex. This ring translates combinatorial information on the colorings of a simplicial complex into algebraic and homological information. Study of this ring began in the 1980s by Stanley [Sta82], as well as Garsia and Stanton [GS84].

We first study the representations of the Chow ring of the Boolean lattice and characterize the irreducible representations that show up in each graded piece of $\mathrm{Chow}(\mathcal{B}_n)$. We also show the existence of a branching rule for representations of specific graded components of $\mathrm{Chow}(\mathcal{B}_n)$ that mirror the the recurrence of Eulerian numbers, since the Eulerian numbers show up as the Hilbert series coefficients of $\mathrm{Chow}(\mathcal{B}_n)$. The representations of the colorful ring also are shown to satisfy a similar branching rule. We then study the Koszul dual and give a non-commutative Gröbner basis for the dual's ideal of relations for $n = 3, 4$, and conjecture a basis for general n. We also provide data on the representations showing up in the graded pieces of the Koszul duals of these rings.

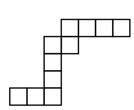
1.1. **Main Results and Organization.** In Section 2, we give background on the representation theory of the symmetric group in characteristic 0, Koszul algebras, and Gröbner bases. In Section 3, we describe the two Koszul algebras of interest that come from $\mathcal{B}_n$, namely the Chow ring of $\mathcal{B}_n$ and the colorful ring associated to $\mathcal{B}_n$. Here, we discuss a basis for the different graded pieces (theorem 3.19). We then discuss the $\mathfrak{S}_n$-representations appearing in the different graded pieces of Chow($\mathcal{B}_n$) in Section 4, and describe the multiplicity of certain hook-shaped irreducible representations in each graded component (theorem 4.2). Section 5 describes branching rules for $\mathfrak{S}_n$-representations between the different graded pieces of the Chow ring (proposition 5.2, proposition 5.3) and the colorful ring (theorem 5.11), including a branching rule for ribbon tableaux (theorem 5.9). Section 6 discusses a basis for Chow$(\mathcal{B}_n)^!$ for $n = 3, 4$ (theorem 6.5, theorem 6.7), and ideas towards basis of the graded components of Chow($\mathcal{B}_n$)! in general (conjecture 6.8) by finding a Gröbner bases for the quotient ideals of Chow$(\mathcal{B}_n)^!$ (theorem 6.6). Finally, Section 7 describes the $\mathfrak{S}_n$-representations in the different graded pieces of Chow$(\mathcal{B}_n)^!$ and $\mathcal{C}(\mathcal{B}_n)^!$.

2. Background

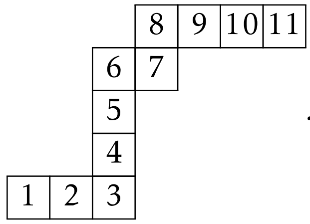
2.1. **Representation theory of the symmetric group in characteristic 0.** Here we provide some background on the representation theory of $\mathfrak{S}_n$ in characteristic 0. We discuss the irreducible representations of $\mathfrak{S}_n$, symmetric functions, restriction, and branching rules for representations of the symmetric group.

The irreducible representations of $\mathfrak{S}_n$ are called *Specht modules* and are canonically indexed by Young diagrams with n boxes. In this paper, we will also be considering *skew Specht modules*, which are reducible representations of $\mathfrak{S}_n$ indexed by skew Young diagrams. We will not supply definitions of these here – more information regarding these can be found in [Wac06]. We will frequently refer to a representation of $\mathfrak{S}_n$ by its corresponding Young diagram of shape λ, where λ is an integer partition of n. The *length* of a Young diagram is the number of rows.

2.1.1. *Ribbon representations.* Certain skew diagrams play an important role in this work, namely the *ribbon* (= *skew/rim hook* = *border strip*) diagrams that we denote $\nu(\alpha)$, whose row sizes from bottom to top are specified by an (ordered) *composition* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$, with one column of overlap between consecutive rows. For example, $\nu((3,1,1,2,4))$ is this diagram:



Let $\nu(\alpha)$ be a ribbon diagram with n cells, so α is a composition of n. Label the cells of $\nu(\alpha)$ with numbers 1 through n, starting at the southwestern end cell, moving through the adjacent cells, and ending at the northeastern cell. For example, we have the labeled ribbon



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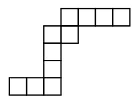
If the cell labeled $i+1$ is above the cell labeled i, then we say that the ribbon has a *descent* at i. For each subset R of $[n-1]$, there is exactly one ribbon with n cells and descent set R. If $R \subseteq [n-1]$, denote by $\rho(R)$ the unique ribbon with descent set R. For example, the ribbon



is the unique ribbon with 11 cells and descent set $\{3,4,5,7\}$.

2.1.2. *Symmetric functions.* We will now move on to discuss symmetric functions. We first define the $\mathfrak{S}_n$-action on a function.

Definition 2.1. Let $f(x_1, x_2, \ldots, x_n)$ be a function on n variables. We define the $\mathfrak{S}_n$-action on f by $\sigma \cdot f(x_1, x_2, \ldots, x_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$ for some $\sigma \in \mathfrak{S}_n$.

We say that a function is symmetric if it is fixed by $\mathfrak{S}_n$.

Example 2.2. Consider the function

$$f(x_1, x_2, x_3) := x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_2x_3 + x_1x_3.$$

Note that any permutation $\sigma \in \mathfrak{S}_3$ will not change f. Thus, f is fixed under the action of $\mathfrak{S}_3$ and hence, is symmetric.

We are especially interested in two kinds of symmetric functions. The first of these are the skew Schur functions.

Definition 2.3 (Skew Schur polynomials)**.** Let λ/μ be a skew partition and let T be the set of semi-standard Young tableau of shape λ/μ. For $t \in T$ and $i \in \mathbb{N}$ let $\alpha_i(T)$ be the number of cells in T filled with i. The skew Schur function on , $s_{\lambda/\mu}(x_1, x_2, \ldots, x_n)$ is

$$\sum_{t \in T} \prod_{i=1}^{n} x_1^{\alpha_i(T)}.$$

The fact that skew Schur functions are symmetric is non-trivial – a proof can be found in [SF99]. A useful fact about skew Schur functions is that the skew Schur function $s_{\lambda/\mu}$ is the character of the skew Specht module $\mathcal{S}^{\lambda/\mu}$ [Wac06]. In addition to the Schur functions, we also use the homogeneous symmetric functions.

Definition 2.4 (Homogeneous symmetric polynomials)**.** Let M_k be the set of monomials in $\Bbbk[x_1, x_2, \ldots, x_n]$ of degree k and let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ be an integer partition of n. The set of homogeneous symmetric functions on n variables are those of the form

$$h_k = \sum_{m \in M_k} m.$$

We extend this to partitions in general by defining $h_\lambda = \prod_{i=1}^{m} h_{\lambda_i}$.

Notice that the function in Example 2.2 is a homogeneous symmetric function on 3 variables.

Finally, we conclude our discussion of symmetric functions with an introduction of the Jacobi-Trudi identity, which allows us to explicitly compute skew Schur polynomials in terms of homogeneous symmetric functions:

Theorem 2.5 (Jacobi-Trudi identity)**.** *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_n) \subseteq \lambda$ be integer partitions. Then, $s_{\lambda/\mu} = \det(h_{\lambda_i - \mu_j - i + j})_{i,j=1}^{n}$.*

A proof of this theorem can be found in [SF99].

2.2. **Restrictions of representations.** We will now move on to a discussion of branching rules and restriction of symmetric group representations:

Definition 2.6 (Restriction to a Subgroup)**.** Let ρ be a representation of a group G and let H be a subgroup of G. The restriction of ρ to H, $\rho|_H$, is the representation of H where for any $h \in H$ we have

$$\rho|_H(h) = \rho(h).$$

A well-known branching rule for the restriction of $\mathfrak{S}_n$ to $\mathfrak{S}_{n-1}$ is given below. [Wac06].

Theorem 2.7. *Consider the $\mathfrak{S}_n$-representation $\mathcal{S}^{\lambda/\mu}$ and let T be the set of Young diagrams which can be obtained from λ by removing a cell from λ. Let S be the set of valid skew partitions of the form t/μ for $t \in T$. Then*

$$\mathcal{S}^{\lambda/\mu}|_{\mathfrak{S}_{n-1}} \cong \bigoplus_{\mu \in S} V_\mu.$$

Later in this paper we present other branching rules which branch from $\mathfrak{S}_n$ to $\mathfrak{S}_{n-1}$.

2.3. **Koszul algebras.** Here we introduce some background on Koszul algebras along with some propositions which we use later in the paper. We direct the reader to look at [Pri70] and [MP15] for more information.

First, we introduce minimal free resolutions:

Definition 2.8 (Graded Minimal Free Resolution)**.** A (left) graded minimal free resolution of a field $\Bbbk$ over a $\Bbbk$-algebra A is an exact complex

$$\cdots \xrightarrow{\partial_n} F_{n-1} \xrightarrow{\partial_{n-1}} F_{n-2} \xrightarrow{\partial_{n-2}} \cdots \xrightarrow{\partial_1} A \xrightarrow{\partial_0} \Bbbk$$

where:

- each F_i is a graded free algebra,
- each $\partial_i : F_i \to F_{i-1}$ is a homogeneous map, i.e. it maps homogenous elements of F_i to homogenous elements of F_{i-1} of the same degree,
- the matrix corresponding to each ∂_i contains no units.

Example 2.9. Let S be the polynomial ring $\Bbbk[x_1, x_2, x_3]$ and let $\bigwedge^i = \bigwedge^i\{e_1, e_2, e_3\}$. Then the *Koszul complex* for the residue field of S is

$$0 \to S \otimes \bigwedge^3 \xrightarrow{\partial_3} S \otimes \bigwedge^2 \xrightarrow{\partial_2} S \otimes \bigwedge^1 \xrightarrow{\partial_1} S \otimes \bigwedge^0 \to \Bbbk \to 0$$

Note that the maps are all linear:

$$\partial_3 = \begin{pmatrix} x_3 \\ -x_2 \\ x_1 \end{pmatrix} \quad \partial_2 = \begin{pmatrix} -x_2 & -x_3 & 0 \\ x_1 & 0 & -x_3 \\ 0 & x_1 & x_2 \end{pmatrix} \quad \partial_1 = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}$$

The next example shows that free resolutions need not be of finite length. In fact, it is generally true that the free resolution of A will be infinite.

Example 2.10. Consider the minimal free resolution of $\Bbbk$ over the ring $A = \Bbbk[x]/(x^2)$. We start by noting that $\Bbbk = A/(x)$. Then, we get the following resolution of $\Bbbk \cong A/(x)$:

$$\cdots \xrightarrow{(x)} A \xrightarrow{(x)} A \xrightarrow{(x)} \Bbbk \to 0$$

Now that we have defined graded minimal free resolutions, we may now provide the following definition of a Koszul algebra from [Pri70].

Definition 2.11 (Koszul algebra)**.** A graded algebra A over $\Bbbk$ is a Koszul algebra if there is a graded minimal free resolution of $\Bbbk$ over A such that for each i, the non-zero entries in the matrix of ∂_i are of degree 1.

Koszul algebras are quadratic algebras [Pri70]. Thus, Koszul algebras have presentation $T(V)/\langle S\rangle$, where S is a set of quadratic relations and $T(V)$ is the tensor algebra over a vector space. We are also able to define the quadratic dual of a Koszul algebra:

Definition 2.12 (Koszul dual)**.** Given a Koszul algebra $A := T(V)/\langle S\rangle$, where S is a set of quadratic relations, let R be the set

$$R := \{r \in V^* \otimes V^* \mid r(S) = 0\}.$$

The Koszul dual A is then $A^! := T(V^*)/\langle R\rangle$.

Remark 2.13. For the remainder of this section, we write the Koszul algebra A and its Koszul dual $A^!$ as

$$A = \frac{\Bbbk[x_1, \ldots, x_n]}{I} \qquad \text{and} \qquad A^! = \frac{\Bbbk\langle z_1, \cdots, z_n\rangle}{J}.$$

We use the following algorithm, defined in [MP15], to compute the generators of J when given a quadratic generating set of I.

Lemma 2.14. *Let G_I be a quadratic generating set of I. We write*

$$G_I = \{f_i \mid 1 \leq i \leq m\} \qquad \text{where} \quad f_i = \sum_{j \leq k} a_{ijk} x_j x_k \quad \text{for} \quad 1 \leq j \leq k \leq n.$$

Next, we consider the system of linear equations

$$\left\{ \sum_{j \leq k} a_{ijk} X_{jk} = 0 \mid 1 \leq i \leq m \right\} \qquad \text{and let its basis of solutions be} \qquad \{\vec{v}_i \mid 1 \leq i \leq l\}.$$

Since there are $n(n+1)/2$ variables X_{jk} for $1 \leq j \leq k \leq n$, it follows that $\vec{v}_i \in \mathbb{Z}^{n(n+1)/2}$. We write v_{ijk} for the entry corresponding to X_{jk} in $\vec{v}_i$. The ideal J is generated by

$$\left\langle \sum_{j \leq k} v_{ijk}[z_j, z_k] \mid 1 \leq i \leq l \right\rangle = J$$

where $[z_j, z_j] = z_j^2$ and $[z_j, z_k] = z_j z_k + z_k z_j$ for $j \neq k$.

The Koszul dual now allows us to define a resolution of the field $\Bbbk$ over A in terms of our algebra A and the graded components of its Koszul dual $A^!$. This resolution is called the Priddy complex.

Definition 2.15 (Priddy complex)**.** The *Priddy complex* is a linear minimal free resolution of $\Bbbk$ over A defined in terms of A and the graded components $A^!$ shown below:

$$\ldots \to (A^!)_2^* \otimes_{\Bbbk} A \to \ldots (A^!)_1^* \otimes_{\Bbbk} A \to (A^!)_0^* \otimes_{\Bbbk} A \to \Bbbk$$

The fact that the Priddy complex is always a resolution of $\Bbbk$ over A is non-trivial, but proven in [Pri70]. Using the Priddy complex we are able to prove the following proposition:

Proposition 2.16. *Let A be a Koszul algebra. Then for each $n \in \mathbb{N}$, there exist short exact sequences of the form*

$$0 \to (A^!)_n^* \to (A^!)_{n-1}^* \otimes_k A_1 \to (A^!)_{n-2}^* \otimes A_2 \to \ldots \to A_n \to 0.$$

Proof. If we take the n-graded components of the Priddy complex for any $n \in \mathbb{N}$, we observe that they are precisely the short exact sequences in the proposition. □

Later in the paper we will consider A and $A^!$, as well as their graded components as $\mathfrak{S}_n$-representations. In particular, we will use Proposition 2.16 to compute $\mathfrak{S}_n$-representations of the graded pieces $A^!$, when given representations of A and its graded components.

Another proposition we can derive from the Priddy complex is shown below:

Proposition 2.17. *Suppose that A is a Koszul Algebra and let* $\mathrm{Hilb}(A,t)$ *be the Hilbert series of A. Then*

$$\mathrm{Hilb}(A^!,t) = \frac{1}{\mathrm{Hilb}(A,-t)}.$$

This proposition is useful when attempting to find bases for the graded components of $A^!$, and it follows as a corollary of Proposition 2.16. We can use this identity to derive a useful recurrence on the dimensions of the graded components of $A^!$, stated in the proposition below.

Proposition 2.18. *Let A be a Koszul algebra. Let* $\mathrm{Hilb}(A,t) = \sum_{i\geq 0} a_i t^i$ *and* $\mathrm{Hilb}(A^!,t) = \sum_{i\geq 0} f_i t^i$. *Then, we have that*

$$f_i = \begin{cases} 1 & \text{if } i = 0 \\ \sum_{j=1}^{i} a_j f_{i-j}(-1)^{j+1} & \text{if } 1 \leq i \leq n-1 \\ \sum_{j=1}^{n} a_j f_{i-j}(-1)^{j+1} & \text{if } n \leq i. \end{cases}$$

Proof. We begin by using proposition 2.17 to obtain the equality

$$\frac{1}{\sum_{i\geq 0} f_i(-t)^i} = \sum_{i\geq 0} a_i t^i.$$

Next, we use that

$$1 = \sum_{i\geq 0} f_i(-t)^i \sum_{j\geq 0} a_j t^j,$$

and observe that we may rewrite this equality as

$$1 = \sum_{i\geq 0} f_i t^i \sum_{j\geq 0} a_j t^j (-1)^j.$$

Now, with the fact that $f_0 = 1$ we rearrange this equality to write

$$1 + \sum_{i\geq 1} f_i t^i \sum_{j\geq 0} a_j t^j (-1)^{j+1} = \sum_{j\geq 0} a_j t^j (-1)^j.$$

After expanding $\sum_{i\geq 1} f_i t^i \sum_{j\geq 0} a_j t^j (-1)^{j+1}$ we can see that

$$f_i = \begin{cases} 1 & \text{if } i = 0 \\ \sum_{j=1}^{i} a_j f_{i-j} (-1)^{j+1} & \text{if } 1 \leq i \leq n-1 \\ \sum_{j=1}^{n} a_j f_{i-j} (-1)^{j+1} & \text{if } n \leq i. \end{cases}$$

□

2.4. **Gröbner Bases.** Here we provide some background on Gröbner bases in commutative and non-commutative rings. We direct the reader to consult [CLO15] and [Mor94] for more background.

Given an ideal I in a polynomial ring $S = \Bbbk[x_1, \ldots, x_n]$, a Gröbner basis $\mathcal{G}$ of $I \subseteq S$ allows division of an element S on $\mathcal{G}$ to be independent of the order in which division is performed. This also holds for a Gröbner basis of an ideal in a non-commutative algebra, with the appropriate division (on the left or right). In this section, we will denote the polynomial ring $\Bbbk[x_1, \ldots, x_n]$ by S, and the free algebra $\Bbbk\langle x_1, \ldots, x_n\rangle$ by R.

A Gröbner basis depends on the monomial order chosen for elements in the ambient ring, which in turn depends on how the generators of S (or R) are ordered. For the purposes of this section, we order the generators $x_1 > x_2 > \cdots > x_n$.

We first consider the commutative case, and then adjust definitions for the non-commutative case as necessary.

Definition 2.19 (Monomial Order, Commutative Case)**.** [CLO15] A *monomial ordering* $<$ on S is a relation on the set of monomials x^α, where $\alpha \in \mathbb{Z}^n_{\geq 0}$, such that

(i) $<$ is a total order.

(ii) If $\alpha < \beta$ and $\gamma \in \mathbb{Z}^n_{\geq 0}$ then $\alpha + \gamma < \beta + \gamma$.

(iii) $<$ is a well-ordering on $\mathbb{Z}^n_{\geq 0}$.

One commonly used monomial order is the lexicographic order. In Section 6.1, we will use the graded lexicographic order, both of which we introduce here.

Definition 2.20 (Lex Order)**.** Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$. We say $x^\alpha <_{\mathsf{lex}} x^\beta$ if and only if there is some index $\ell \in [n]$ such that $\alpha_j = \beta_j$ for all $j < \ell$ and $\alpha_\ell < \beta_\ell$.

Definition 2.21 (Graded Lex Order)**.** Let $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$. We say $\alpha <_{\text{glex}} \beta$ if

$$|\alpha| = \sum_{i=1}^{n} \alpha_i < \sum_{i=1}^{n} \beta_i = |\beta| \quad \text{or} \quad |\alpha| = |\beta| \quad \text{and} \quad x^\alpha <_{\text{lex}} x^\beta$$

Given a monomial order $<$ on S, we can define the initial terms of an element in S, as well as the initial ideal of an ideal I in S.

Definition 2.22. Let $f = \sum_\alpha a_\alpha x^\alpha$ be a polynomial in S, and let $<$ be a monomial order. The *multidegree* of f is

$$\text{multideg}(f) = \max_{<}(\alpha \in \mathbb{Z}_{\geq 0} : a_\alpha \neq 0)$$

The *initial monomial* of f is

$$\text{in}(f) = x^{\text{multideg}(f)},$$

and the *leading term* of f is

$$\text{LT}(f) = a_{\text{multideg}(f)} x^{\text{multideg}(f)}.$$

Let I be an ideal in S. The *initial ideal* of I is $\text{in}(I) := \langle \text{in}(f) \mid f \in I \rangle$.

We can now define a Gröbner basis for S:

Definition 2.23 (Gröbner Basis)**.** Let $<$ be a monomial order on S, and let I b an ideal in S. The set $\mathcal{G} \subset I$ is said to be a *Gröbner basis* of I if for each nonzero $f \in I$, there is some $g \in \mathcal{G}$ such that $\text{in}(g)$ divides $\text{in}(f)$.

In the non-commutative case, we first define the standard basis of R:

$$\mathcal{B} = \left\{1, x_{a_1} x_{a_2} \cdots x_{a_t} \; : \; x_{a_i} \in \{x_1, \dots x_n\}\right\}$$

We define the initial monomial for an element in R according to a total order $\prec$ on $\mathcal{B}$.

Definition 2.24. Let f be an element in R such that

$$f = \sum_{i=1}^{s} \lambda_i u_i \quad \text{where } \lambda_i \in \mathbb{k}^*, \; u_i \in \mathcal{B}, \; u_1 \prec u_2 \prec \cdots \prec u_s$$

The *initial monomial of* f is $\text{in}(f) = u_s$, and the *leading term* of f is $\text{LT}(f) = \lambda_s u_s$.

Definition 2.25 (Monomial Order, non-commutative Case)**.** A *monomial ordering* $<$ on R is a relation on $\mathcal{B}$ such that

(i) $<$ is a well-ordering

(ii) If $w, u, v, s \in \mathcal{B}$ such that $w < u$ then $vws < vus$.

(iii) For $w, u \in \mathcal{B}$, if $u = vws$ for some $v, s \in \mathcal{B}$ where $v \neq 1$ or $s \neq 1$, then $w < u$.

The lexicographic and graded lexicographic orders are defined as in the commutative case, and the definition of a Gröbner basis for an ideal J in R can also be transferred from the commutative case with one modification: for elements f and g in R, we say that $\mathrm{in}(g)$ divides $\mathrm{in}(f)$ if there exists some $w, s \in \mathcal{B}$ such that $\mathrm{in}(f) = \mathrm{in}(w\,\mathrm{in}(g)s)$.

When computing and verifying that a subset $\mathcal{G}$ of an ideal is a Gröbner basis, it is useful to consider S-polynomials and S-elements. These are the elements that result when the leading terms of two elements in S, (resp. R) cancel.

Definition 2.26 (S-polynomial)**.** Let f, g be nonzero polynomials in the commutative ring S. Then the S-*polynomial* of f and g is

$$S(f,g) = \frac{\mathrm{lcm}(\mathrm{LT}(f), \mathrm{LT}(g))}{\mathrm{LT}(f)} f - \frac{\mathrm{lcm}(\mathrm{LT}(f), \mathrm{LT}(g))}{\mathrm{LT}(g)} g$$

If $\mathrm{lcm}(\mathrm{LT}(f), \mathrm{LT}(g)) = \mathrm{LT}(f)\mathrm{LT}(g)$, we say that $S(f,g)$ is *trivial*.

Definition 2.27 (S-elements)**.** Let f, g be elements in the non-commutative ring R, and let a, b, c, d be in $\mathcal{B}$. Define the S-element of (f,g)

$$S(f,g;a,b;c,d) := afb - cgd$$

An S-element is *trivial* if there is some $w \in \mathcal{B}$ such that $a = c \cdot \mathrm{in}(g)w$ and $d = w \cdot \mathrm{in}(f)b$ (or equivalently $c = a \cdot \mathrm{in}(f)w$ and $b = w \cdot \mathrm{in}(g)d$). It is often convenient to abbreviate an S–element by $S(f,g)$ when it is clear that we are working in a non-commutative ring and a choice of a, b, c, d has been established.

In both the commutative and non-commutative cases, one method of verifying that a subset $\mathcal{G}$ of an ideal is a Gröbner basis is to check that the S-polynomials (resp. S-elements) have a *weak Gröbner representation*, [Mor94]. In other words, it suffices to check that each S-polynomial (resp. S-element) has zero remainder on division by $\mathcal{G}$.

For the commutative case, a weak Gröbner representation of the S-polynomial $S(f,g)$ is of the form

$$S(f,g) = \sum_k h_k g_k$$

where $h_k \in S$, $g_k \in \mathcal{G}$, and $\mathrm{in}(h_k g_k) < \mathrm{in}(S(f,g))$ for all k.

In the non-commutative case we have the following analog for an S-element $S(f,g)$:

$$S(f,g) = \sum_{t,\ell} \lambda_{t,\ell} a_{t,\ell} g_t b_{t,\ell}$$

where $\lambda_{t,\ell} \in \mathbb{k}^*$, $a_{t,\ell}, b_{t,\ell} \in \mathcal{B}$, and $a_{t,\ell}\,\mathrm{in}(g_t) b_{t,\ell} \leq \mathrm{in}(S(f,g))$.

Proposition 2.28 (Proposition 3.9, Theorem 5.9,[Mor94])**.** *Let $\mathcal{G} = (g_1, , \ldots, g_s)$ be a basis of $I \subset S$, where $\mathrm{LT}(g_i) = \mathrm{in}(g_i)$ for each i. $\mathcal{G}$ is a Gröbner basis of I if and only if each $S(g_i, g_j)$ has a weak Gröbner representation.*

The following proposition allows us to consider only nontrivial S-polynomials (resp. S-elements), when verifying that a set $\mathcal{G}$ is a Gröbner basis.

Proposition 2.29 (Lemmata 3.8 and 5.7, [Mor94])**.** *A trivial S-polynomial (resp. S-element) always has a weak Gröbner representation.*

When considering the Koszul dual of the Chow ring, finding a Gröbner basis for the algebra's ideal of relations allows us to find a basis for each graded piece of the dual. We undertake this for some specific cases in Section 6.1. We also apply the theory introduced here to find a quadratic Gröbner basis for the Colorful ring.

3. Koszul algebras from the Boolean lattice

3.1. **The Chow ring for the Boolean matroid.** In this section, we introduce the Chow ring and the Colorful ring of the Boolean lattice. We will study these two algebras and their Koszul duals.

Definition 3.1 (Boolean lattice)**.** The Boolean lattice on $[n]$, denoted $\mathcal{B}_n$, is the poset of all subsets of $[n] := \{1, \dots, n\}$, ordered by inclusion.

Note that the order complex of the Boolean lattice, $\Delta(\mathcal{B}_n)$ is a simplicial complex; additionally, the Boolean lattice is a lattice of flats of a matroid. Hence, in this paper, we discuss rings defined for any simplicial complexes or matroids, but describe them specifically in the case of the Boolean lattice.

Definition 3.2. The Stanley-Reisner ring of the order complex of the Boolean lattice is defined as

$$\Bbbk[\Delta\mathcal{B}_n] := \frac{\Bbbk[x_F : \emptyset \neq F \in \mathcal{B}_n]}{\langle x_F x_G : F, G \text{ incomparable in } \mathcal{B}_n \rangle}.$$

We will denote the ideal of $\Bbbk[\Delta(\mathcal{B}_n)]$ as I_Δ.

Traditionally, the Stanley-Reisner ring is defined for any abstract simplicial complex, but here we define it only for the order complex of $\mathcal{B}_n - \{\varnothing\}$, which is always the barycentric subdivision of the $(n-1)$-simplex. For example, consider $\Bbbk[\Delta\mathcal{B}_3]$ and the 2-simplex below:

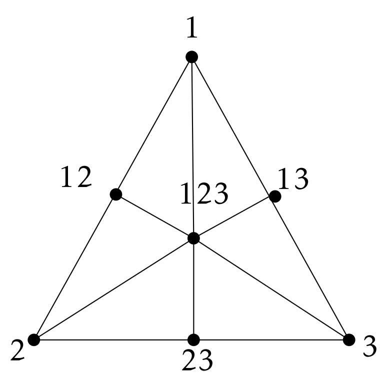


Figure 1. The barycentric subdivision of a 2-simplex, corresponding to $\mathcal{B}_3$.

In this paper, we are interested in the structure of the Chow ring and the colorful ring of the Boolean lattice, both of which are quotients of $\Bbbk[\Delta\mathcal{B}_n]$ by a chosen set of parameters.

Definition 3.3. The Chow ring of the Boolean matroid is defined as

$$A(n) = \mathrm{Chow}(\mathcal{B}_n) = \frac{\Bbbk[\Delta\mathcal{B}_n]}{\langle \Theta_{\mathrm{CH}} \rangle}$$

where $\Theta_{\mathrm{CH}} = \{\theta_e := \sum_{F \supseteq e} x_F\}$ for every $e \in [n]$ are the Chow ring parameters.

The Chow ring is standard graded, and a basis for each graded piece is given by monomials

$$\left\{ \prod_{i=1}^{k} x_{F_i}^{a_i} : \emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subseteq F_k \subset [n] \text{ and } a_i \leq |F_i| - |F_{i-1}| - 1 \right\}$$

called the Feichtner-Yuzvinsky monomials [FY04].

Example 3.4. Consider $\mathcal{B}_3$, the Boolean lattice on three elements. The Chow ring of $\mathcal{B}_3$, denoted $A(3)$, is given by

$$A(3) = \frac{\Bbbk[x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123}]}{I_\Delta + \Theta_{\mathrm{CH}}}$$

where the Stanley Reisner ideal is

$$I_\Delta = \langle x_1x_2, x_1x_3, x_2x_3, x_1x_{23}, x_2x_{13}, x_3x_{12} \rangle$$

and the Chow ring parameters are

$$\Theta_{\mathrm{CH}} = \langle x_1 + x_{12} + x_{13} + x_{123}, x_2 + x_{12} + x_{23} + x_{123}, x_3 + x_{13} + x_{23} + x_{123} \rangle$$

The basis of for each graded piece of $A(3)$ is given by the following monomials:

degree	basis elements
0	1
1	$x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123}$
2	x_{123}^2

We are interested in studying the Koszul dual of $\mathrm{Chow}(\mathcal{B}_n)$, which is only a reasonable endeavor if $\mathrm{Chow}(\mathcal{B}_n)$ is, in fact, Koszul. Maestroni and McCullough show that the Chow ring of any simple matroid is Koszul by constructing a Koszul filtration [MM2211]. As such, $\mathrm{Chow}(\mathcal{B}_n)$ is Koszul and it is possible to consider its Koszul dual, which can be found using the algorithm in lemma 2.14.

Since $\mathrm{Chow}(\mathcal{B}_n)$ is Koszul, it must have a quadratic presentation. This can be be found by replacing the variable corresponding to each $e \in [n]$, the atoms of $\mathcal{B}_n$, using the linear parameter θ_e. Specifically, we write

$$x_e \longleftrightarrow -\sum_{e \subsetneq F} x_F.$$

This results in the *atom-free presentation* of $\mathrm{Chow}(\mathcal{B}_n)$:

Lemma 3.5.

$$A(n) = \mathrm{Chow}(\mathcal{B}_n) = \frac{\Bbbk[x_F : |F| \geq 2]}{I}$$

where the atom-free ideal is

$$I = \langle x_F x_G : |F|, |G| \geq 2, F, G \text{ incomparable } \rangle + \left\langle x_F \sum_{F \vee i \subseteq G} x_G \colon |F| \geq 2, i \in E \setminus F \right\rangle + \left\langle \sum_{i \vee j \subseteq F} x_F^2 + \sum_{i \vee j \subseteq F \subseteq F'} 2x_F x_F' : i, j \in E, i \neq j \right\rangle$$

Example 3.6. We can write the Chow ring of $\mathcal{B}_3$, seen in Example 3.4, in an atom-free way in the following

$$A(3) = \frac{\Bbbk[x_{12}, x_{13}, x_{23}, x_{123}]}{I}$$

where

$$I = \langle x_{12}x_{13},\, x_{12}x_{23},\, x_{13}x_{23} \rangle + \langle x_{12}x_{123},\, x_{13}x_{123},\, x_{23}x_{123} \rangle + \langle x_{12}^2 + x_{123}^2,\, x_{13}^2 + x_{123}^2,\, x_{23}^2 + x_{123}^2 \rangle$$

Throughout this paper, we will use the atom-free presentation for $A(n)$.

We are also interested in the $\mathfrak{S}_n$-representations of $\mathrm{Chow}(\mathcal{B}_n)$. The recent work of [Lia22] constructs a bijection that, when combines with results of Stembridge [Ste92], gives us a generating function for the representations appearing in the degree j graded piece of $\mathrm{Chow}(\mathcal{B}_n)$:

$$\sum_{i=0}^{n-1} \Bbbk[\Delta \mathcal{B}_n]_j t^j z^n = \frac{(1-t)H(z)}{H(tz) - tH(z)}$$

Where $H(z) = \sum_{i \geq 0} h_i(\mathbf{x}) z^i$ is a sum of complete homogeneous symmetric functions.

3.2. **The colorful ring for the Boolean lattice.**

Definition 3.7 (Colorful ring)**.** The *colorful ring* of the Boolean lattice is defined as

$$\mathcal{C}(\mathcal{B}_n) = \frac{\Bbbk[\Delta \mathcal{B}_n]}{\langle \Theta_{\text{color}} \rangle}$$

where we write

$$\Theta_{\text{color}} = \{\theta_i := \sum_{|F|=i} x_F \colon 1 \leq i \leq n,\, F \subseteq [n]\}$$

are the colorful parameters.

The name for this ring comes from the fact that coloring subsets of $[n]$ in a way such that two subsets are given the same color if their cardinalities are equal induces a coloring of the barycentric subdivision of the the standard $(n-1)$-simplex.

Example 3.8. When $n = 3$, we have

$$\mathcal{C}(\mathcal{B}_3) = \frac{\Bbbk[\Delta\mathcal{B}_3]}{\langle x_1 + x_2 + x_3, x_{12} + x_{13} + x_{23}, x_{123}\rangle}.$$

When observing the figure below, one sees that the generators in the quotient are given by adding together the subsets of $\mathcal{B}_n$ corresponding to similarly-colored vertices in the barycentric subdivision of the 2-simplex.

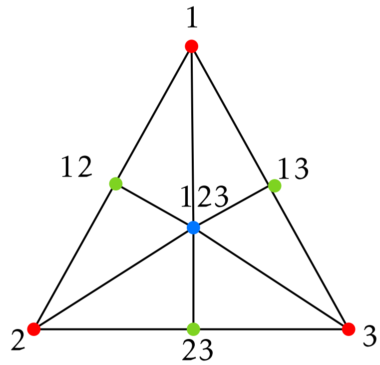


Figure: A 3-coloring of the barycentric subdivision of a 2-simplex

Let $\Delta = \Delta(\mathcal{B}_n)$ be the order complex of the Boolean lattice $\mathcal{B}_n$. Consider the $\mathbb{N}^n$-multigrading given by $\deg(x_F) = \epsilon_{\mathrm{rank}(F)}$ for any $F \in \mathcal{B}_n$, where ϵ_i is the i'th unit vector in $\mathbb{N}^n$. Because the action of the symmetric group $\mathfrak{S}_n$ on $\mathcal{B}_n$ is rank-preserving, it also preserves the $\mathbb{N}^n$-grading on $\Bbbk[\Delta\mathcal{B}_n]$. Therefore, for a fixed multidegree $\mathbf{b} \in \mathbb{N}^n$, the $\mathbf{b}$-homogeneous component of $\Bbbk[\Delta]$, denoted $\Bbbk[\Delta]_{\mathbf{b}}$, is also a representation of the symmetric group $\mathfrak{S}_n$. Denote the representation corresponding to $\Bbbk[\Delta]_{\mathbf{b}}$ as $[\Bbbk[\Delta]_{\mathbf{b}}]$. We can describe the $\mathbb{N}^n$-graded Hilbert series of $\Bbbk[\Delta]$, which lies in $\mathbb{Z}[[\mathbf{t}]] = \mathbb{Z}[[t_1, \ldots, t_n]]$ equivariantly:

$$\mathrm{Hilb}_{\mathrm{eq}}(\Bbbk[\Delta], \mathbf{t}) = \sum_{\mathbf{b}\in\mathbb{N}^n} [\Bbbk[\Delta]_{\mathbf{b}}] \cdot \mathbf{t}^{\mathbf{b}}.$$

If $R \subseteq [n]$, define the *rank-selected subcomplex* of Δ to be

$$\Delta\,|_R := \{F \in \Delta : \mathrm{rank}(F) \subseteq R\}.$$

The following proposition expresses the equivariant $\mathbb{N}^n$-graded Hilbert series of $\Bbbk[\Delta]$ in terms of cohomology of rank-selected subcomplexes of Δ.

Proposition 3.9. *Let $\Delta := \Delta(\mathcal{B}_n)$ be the order complex of the Boolean lattice $\mathcal{B}_n$. Then we have the following expression for the $\mathbb{N}^n$-graded equivariant Hilbert series:*

$$\mathrm{Hilb}_{eq}(\Bbbk[\Delta], \mathbf{t}) = \frac{1}{\prod_{j=1}^{n}(1 - t_j)} \sum_{R\subseteq[n]} \mathcal{S}^{\rho(R)} \cdot \mathbf{t}^R$$

where $\rho(R)$ is the unique ribbon with descent set R, as defined in Section 2.1.

Proof. By [AR23, Prop 2.5], one may write

$$\mathrm{Hilb}_{\mathrm{eq}}(\Bbbk[\Delta],\mathbf{t}) = \frac{1}{\prod_{j=1}^{n}(1-t_j)} \sum_{R\subseteq[n]} [h_R(\Delta)]\cdot \mathbf{t}^R$$

where $[h_R(\Delta)]$ is the $\mathfrak{S}_n$-representation corresponding to the flag h-vector of the simplicial complex Δ, defined as

$$[h_R(\Delta)] := (-1)^{|R|-1}\tilde{\chi}_{\mathrm{eq}}(\Delta\,|_R)$$

where $\tilde{\chi}_{\mathrm{eq}}$ is the *equivariant reduced Euler characteristic*

$$\tilde{\chi}_{\mathrm{eq}} = \sum_{i\geq -1}(-1)^i[\tilde{H}^i(\Delta\,|_R,\Bbbk)]$$

of the rank-selected subcomplex $\Delta\,|_R$. Moreover, because Δ is Cohen-Macaulay over $\Bbbk$, the equivariant reduced Euler characteristic $\tilde{\chi}(\Delta\,|_R)$ has only one nonvanishing term, implying that

$$[h_R(\Delta)] = [\tilde{H}^{|R|-1}(\Delta\,|_R,\Bbbk)].$$

Finally, we can reinterpret a result of Solomon (see [Wac06, Theorem 3.4.4]) to see that

$$[\tilde{H}^{|R|-1}(\Delta\,|_R,\Bbbk)] \cong \mathcal{S}^{\rho(R)}$$

where $\rho(R)$ is the ribbon diagram with descent set R. □

By rewriting the theorem above in the standard grading and using the fact that Θ_{color} is a linear system of parameters for $\Bbbk[\Delta]$ yields the following.

Corollary 3.10. *Let $Q := \mathcal{C}(\mathcal{B}_n)$. The graded pieces Q_i of Q are equivariantly isomorphic to*

$$Q_i \cong \bigoplus_{\substack{\alpha\models n\\ |\alpha|=i}} \mathcal{S}^{\nu(\alpha)},$$

That is, Q_i is a direct sum of all Specht modules corresponding to ribbon diagrams of length i.

This gives us that $\mathcal{C}(\mathcal{B}_n)$ has the same Hilbert series as the Chow ring of $\mathcal{B}_n$. That is,

$$\mathrm{Hilb}(\mathcal{C}(\mathcal{B}_n,t)) = \sum_{k=0}^{n-1}\left\langle {n \atop k}\right\rangle t^k.$$

In order to understand this ring, we would like to know a basis for it. We shall show that *descent monomials* as described in [GS84] provide a suitable basis for $\mathcal{C}(\mathcal{B}_n)$.

Definition 3.11. Given a permutation $\sigma\in\mathfrak{S}_n$, the *descent monomial* associated to σ is given by

$$\eta(\sigma) = \prod_{\sigma(i+1)<\sigma(i)} x_{\sigma(1)\ldots\sigma(i)}.$$

It is an immediate consequence of this definition that the number of descents in the original permutation is the same as the order of its descent monomial. Furthermore, the index of each term in this product is a proper subset of the subsequent index and $[n]$ can never be an index of any term.

Example 3.12. For the permutation $17832465 \in \mathfrak{S}_8$ with 3 descents, we get

$$17832465 \mapsto x_{178}x_{1378}x_{1234678},$$

an order-3 monomial.

3.3. **A basis for $\mathcal{C}(\mathcal{B}_n)$.** We now give two proofs of the basis for the colorful ring of $\mathcal{B}_n$. We first give a purely combinatorial and algebraic proof of the descent monomials as a basis for the colorful ring of $\mathcal{B}_n$. The steps of the proof below are split into four lemmas: first, we give a description of those monomials which are the descent monomial of some perumation, as well as a description of those which are *not*; second, we show that squares of variables in the colorful ring are zero; third, we show that all order-2 monomials in the colorful ring can be rewritten as a linear combination of descent monomials; finally, we show that all monomials can be written as a linear combination of descent monomials. With this plus the known dimension of the colorful ring, we will have a proof that the descent monomials give a basis.

We then present a quadratic Gröbner basis $\mathcal{G}_I$ for I where $\mathcal{C}(\mathcal{B}_n) = \dfrac{\Bbbk[x_F : F \subset [n]]}{I}$, and by [Mor94], we have the following theorem:

Theorem 3.13. *For $S = \Bbbk[x_1, x_2, \cdots, x_n]$, if $\mathcal{G}$ is a Grönber basis for an ideal I, and $O \subseteq S$ is the set of monomials not divisible by* $\mathrm{in}(\mathcal{G})$*, we have*

$$S = I \oplus \mathrm{span}_{\Bbbk}(O), \qquad \frac{S}{I} = \mathrm{span}_{\Bbbk}(O).$$

Therefore, one basis for the d-th graded component of $\mathcal{C}(\mathcal{B}_n)$ is all degree d monomials not divisible by $\mathrm{in}(\mathcal{G}_I)$.

Now we start the first proof.

Lemma 3.14. *Suppose that $m \geq 1$ and that $\emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_m \subsetneq F_{m+1} = [n]$. Then $\prod_{i\in[m]} x_{F_i}$ is a descent monomial for some $\sigma \in \mathfrak{S}_n$ if and only if for all $i \in [m]$, there exists some $\ell \in F_{i+1} \setminus F_i$ such that $\ell < \max F_i \setminus F_{i-1}$.*

Proof. Suppose $\prod_{i\in[m]} x_{F_i}$ is a descent monomial. Let $\sigma \in \mathfrak{S}_n$ denote the permutation such that $\eta(\sigma) = \prod_{i\in[m]} x_{F_i}$. Furthermore, let $i \in [m]$ and k_i denote the i-th value k for which $\sigma(k+1) < \sigma(k)$. That is, the i-th descent appears after the k_i-th position in the permutation. Then $\sigma(k_i+1) \in F_{i+1} \setminus F_i$ and $\sigma(k_i+1) < \sigma(k) = \max F_i \setminus F_{i-1}$.

Conversely, suppose that for all $i \in [m]$, there exists some $\ell \in F_{i+1} \setminus F_i$ such that $\ell < \max F_i \setminus F_{i-1}$. For $j \in [n]$, let $k_j \in [m]$ be the largest value of k for which $j - |F_k| > 0$, and let $r_j = j - |F_{k_j}|$. Let f_{k_j,r_j} denote the r_j-th element of $F_{k_j+1} \setminus F_{k_j}$ when ordered with index starting at 1. Then define $\sigma : [n] \to [n]$ by $\sigma(j) = f_{k_j, r_j}$. By construction, this will be a permutation such that $\eta(\sigma) = \prod_{i \in [m]} x_{F_i}$ as desired. $\square$

This lemma allows us, with some manipulation, to describe those monomials which are *not* descent monomials:

Corollary 3.15. *Suppose that $\emptyset \neq F \subsetneq [n]$ and that $F \notin \{[j] : j \in [n]\}$. For sets of integers X and $i \leq |X|$, let $S_{X,i}$ denote the set of the i smallest elements of X and $L_{X,i}$ denote the set of the i largest elements of X.*

(1) Suppose that $|F| > 1$ and let $i \in [|F| - 1]$. Suppose further that $\emptyset \neq G \subsetneq F$, with $|G| = i$, and $G \notin \{[j] : j \in [n]\}$. Additionally, we consider $x_G x_F \notin \eta(\mathfrak{S}_n)$. If

$$F \setminus L_{F,i} \in \{[j] : j \in [n]\},$$

then $G \in \{S_{F,i}, L_{F,i}\}$. Otherwise, $G = S_{F,i}$.

(2) Suppose that $|F| < n - 1$ and let $i \in [n-1] \setminus [|F|]$. Suppose further that $F \subsetneq G \subsetneq [n]$, with $|G| = i$, and $x_F x_G \notin \eta(\mathfrak{S}_n)$. Then

$$G \in \left\{ F \cup H : H = S_{[n] \setminus F, i - |F|} \text{ or } H \subsetneq [n], |H| = i - |F|, \text{ and } a \in F, b \in H, a < b \right\}.$$

From here, we observe that all nonzero-monomials in the colorful ring must be the product of terms with indices that form a sequence of subsets of $[n]$ which are proper at each step. This is a direct consequence of the fact that the product of incomparable terms is 0 in the Stanley-Reisner ring and thus also the colorful ring as well as the following lemma:

Lemma 3.16. *For all $F \subseteq [n]$, $x_F^2 = 0$ in $\mathcal{C}(\mathcal{B}_n)$.*

Proof. Let $F \subseteq [n]$. We know that

$$\sum_{|G|=|F|} x_G \in \langle \Theta_{\text{color}} \rangle$$

and is thus 0 in the colorful ring. So we see that

$$0 = 0 \cdot x_F = \sum_{|G|=|F|} x_G x_F = x_F^2$$

since if $F \neq G$ and $|F| = |G|$, then F and G are incomparable and thus $x_F x_G = 0$ in the colorful ring. $\square$

With this, we begin to shift towards observing non-square monomials. In order to get the descent monomials, we shall choose to observe that in the colorful ring,

$$x_{[i]} = -\sum_{\substack{|F|=i \\ F \neq [i]}} x_F. \tag{1}$$

Given this fact, we may rewrite the $x_{[i]} x_G$ terms from the Stanley-Reisner ring's quotient in therms of the above summation that comes from the colorful parameters. We know from lemma 3.16 that we need only consider those terms where $|G| \neq i$, since when $|G| = i$, we are left with $x_{[i]}x_G = x_G^2 = 0$ after removing incomparable summands. The remaining terms from the quotient coming from $\Bbbk[\Delta\mathcal{B}_n]$ cannot be rewritten and hence no attention must be paid to them for the rest of the proof.

Lemma 3.17. *All monomials consisting of the product of two terms can be written as the sum of descent monomials in the colorful ring.*

Proof. We start by fixing a total ordering on the subsets of $[n]$. We say that if $F, G \subseteq [n]$, then

$$F < G \quad \text{if} \begin{cases} |F| < |G| & \text{or} \\ |F| = |G| & \text{and } \min F \setminus G < \min G \setminus F. \end{cases}$$

Upon direct verification from the definition of a strict total ordering, one sees that this relation does have the desired properties. Furthermore, we define a monomial order on the monomials we consider by saying that

$$\prod_{F\subseteq[n]} x_F^{\alpha_F} < \prod_{F\subseteq[n]} x_F^{\beta_F} \quad \text{if} \quad \sum_{F\subseteq[n]} \alpha_F < \sum_{F\subseteq[n]} \beta_F \quad \text{or if} \quad \sum_{F\subseteq[n]} \alpha_F = \sum_{F\subseteq[n]} \beta_F \text{ and } \alpha_{F_0} < \beta_{F_0},$$

where F_0 denotes the smallest set under the subset ordering for which $\alpha_F \neq \beta_F$.

First notice that all of the order-2 monomials in the colorful ring that are nonzero are those of the for $x_F x_G$ where $\emptyset \neq F \subsetneq G \subsetneq [n]$. If $F = [i]$ or $G = [i]$ for some $1 \leq i \leq n$, then we can rewrite the term $x_F x_G$ by eq. (1). Therefore, to prove that the descent monomials generate the second graded component of $\mathcal{C}(\mathcal{B}_n)$, it suffices to show that the degree 2 monomials which are neither descent monomials nor contain an index of the form $[i]$ can be written as a linear combination of descent monomials. Let M be the set of all such monomials.

We now show that elements in M are bijective with monomials $x_F x_G$ where F, G incomparable and $F = [i]$. Then by rewriting $x_{[i]}$ as in eq. (1), the smallest monomial in each $-\sum_{|H|=i, H\neq[i]} x_H x_G$ is a distinct nonzero monomial in M, and this let us to write it as a linear combination of descent monomials.

The total number of nonzero monomials is calculated as follows:

$$\begin{aligned} \sum_{i=1}^{n-2} \sum_{j=1}^{n-i-1} \binom{n}{i}\binom{n-i}{j} &= \sum_{i=1}^{n-2} \binom{n}{i}(2^{n-i}-2) \\ &= \sum_{i=0}^{n} \binom{n}{i}(2^{n-i}-2) - \binom{n}{0}(2^n-2) - \binom{n}{n-1}(2^1-2) - \binom{n}{n}(2^0-2) \\ &= 3^n - 3\cdot 2^n + 3. \end{aligned}$$

We shall show these are in bijection with the union of the following three disjoint sets: the set A descent monomials of order 2; the set B of monomials of the form $x_F x_G$ where F and G are *incomparable* and $F = [i]$ for some $i \in [n-1]$; and the set C of monomials of the form $x_F x_G$ where F and G are *comparable* and $F = [i]$ for some $i \in [n-1]$. We count the cardinalities of these sets as follows:

$$|A| = \left\langle {n \atop 2} \right\rangle = 3^n - (n+1)2^n + \frac{n(n+1)}{2}$$
$$|B| = \sum_{i=1}^{n-1}\left(2^n - 2^{n-i} - 2^i - \binom{n}{i} + 2\right)$$
$$|C| = \sum_{i=1}^{n-1}(2^{n-i} + 2^i - 4) - \frac{(n-1)(n-2)}{2}.$$

The formula for $|A|$ can be found as a specific case of a theorem on page 243 of [Com74]. Since these sets are disjoint, we see that their union has the cardinality below:

$$\begin{aligned}|A \sqcup B \sqcup C| &= |A| + |B| + |C| \\ &= 3^n - (n+1)2^n + \frac{n(n+1)}{2} + \sum_{i=1}^{n-1}\left(2^n - \binom{n}{i} - 2\right) - \frac{(n-1)(n-2)}{2} \\ &= 3^n - 3\cdot 2^n + 3.\end{aligned}$$

Because $A \sqcup C$ contains nonzero monomials, this means $|M| = |B|$.

We now begin rewriting the elements in B, i.e. monomials of the form $x_{[i]}x_F$, where $|F| \neq i$ and $[i]$ and F are incomparable, which are 0 in the colorful ring. First consider when $|F| > i$. We have

$$0 = x_{[i]}x_F = -\sum_{|G|=i} x_G x_F = -\sum_{\substack{|G|=i \\ G\subset F}} x_G x_F.$$

With this, we know from corollary 3.15 that there are at most 2 non-descent monomials in this sum. The one that must be in this sum is $x_{S_{F,i}}x_F$, the smallest monomial under our ordering. If $F \setminus L_{F,i} \notin \{[j] : j \in [n]\}$, then this shows that $x_{S_{F,i}}x_F$ is a linear combination of descent monomials. Otherwise, it is a linear combination of descent monomials and $x_{L_{F,i}}x_F$. However in this case, $x_{L_{F,i}}x_F$ will be the smallest monomial in the sum for $x_{L_{F,i}}x_{[|F|]}$, which we will show is a linear combination of descent mononials after considering the next case.

Consider when $|F| < i$. We see that

$$0 = x_F x_{[i]} = -\sum_{|G|=i} x_F x_G = -\sum_{\substack{|G|=i \\ G\supset F}} x_G x_F.$$

The smallest monomial in this sum is $x_F x_{F\sqcup S_{[n]\setminus F, i-|F|}} \in M$. So we may rewrite $x_F x_{F\sqcup S_{[n]\setminus F, i-|F|}}$ as a linear combination of descent monomials and monomials of the form $x_F x_{F\sqcup H}$ where $H \subsetneq [n]$, $|H| = i - |F|$, and $\forall a \in F, b \in H$, $a < b$. For any monomial of this form, we see that it is the smallest term in the expansion of $x_{[|F|]}x_{F\sqcup H}$, which brings us back to the previous case. So it remains to show that the monomials of the form $x_{L_{F,i}}x_F$ for $i < |F|$ can be written as a linear combination of descent monomials.

When using the previous case to observe $x_{L_{F,i}}x_{[|F|]}$, we see that there may again be more than one element of the sum which is not a descent monomial. However, notice that we only further rewrite the terms $x_F x_{F\sqcup H}$ where elements in H are all larger than those in F. So, we may move back and forth through these cases, but since the elements of the indices are necessarily increasing and cannot be larger than n, we must have a terminating process of rewriting elements of M, which ends with a linear combination of descent monomials as we desired. $\square$

Lemma 3.18. *All monomials in the colorful ring can be written as a linear combination of descent monomials of the same degree.*

Proof. Suppose that $\emptyset \neq F_1 \subsetneq \ldots \subsetneq F_m \subsetneq [n]$ and consider $p = \prod_{i=1}^{m} x_{F_i}$. We proceed by induction on m. In the case of $m = 0$, then $p = 1$, the descent monomial of the trivial permutation. When $m = 1$, either $p = x_{[}i] = -\sum_{|F|=i} x_F$ or $p = x_G$ for some $G \subsetneq [n]$, $G \neq [i]$ for any i. When $m = 2$, see lemma 3.17.

Now suppose $m > 2$ and that for all $0 \leq k < m$ the proposition holds. Then $p = \left(\prod_{i=0}^{m-1} x_{F_i}\right) x_{F_m}$. We know that $\prod_{i=0}^{m-1} x_{F_i}$ is a linear combination of descent monomials, say $\sum_{\sigma\in S_{m-1}} \alpha_\sigma \eta(\sigma)$, where S_{m-1} is the set of permutations of $[n]$ with $m-1$ descents. So after distribution, we have $p = \sum_{\sigma\in S_{m-1}} \alpha_\sigma \eta(\sigma) x_{F_m}$. For each $\sigma \in S_{m-1}$ for which $\alpha_\sigma \neq 0$, we see that $\eta(\sigma)x_{F_m} = \left(\prod_{\sigma(i+1)<\sigma(i)} x_{\sigma(1)\ldots\sigma(i)}\right) x_{F_m}$. Letting $i_1, \ldots, i_{m-1}$ denote the positions after which σ has a descent, we can inductively write each $x_{\sigma(1)\ldots\sigma(i_j)} x_{F_m}$ as either 0 if their indices are incomparable or otherwise as a linear combination of descent monomials, and subsequently may distribute over the sum. At the end of this process we are left with a linear combination of descent monomials. The lemma follows by induction. $\square$

Combining these lemmas together, we get the following theorem:

Theorem 3.19. *The descent monomials of permutations in $\mathfrak{S}_n$ are a basis for $\mathcal{C}(\mathcal{B}_n)$.*

Proof. By lemma 3.18, we see that the descent monomials span $\mathcal{C}(\mathcal{B}_n)$. Furthermore,

$$\dim(\mathcal{C}(\mathcal{B}_n)) = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle = n! = |\mathfrak{S}_n| = |\eta(\mathfrak{S}_n)|.$$

So, the descent monomials are a basis of $\mathcal{C}(\mathcal{B}_n)$. $\square$

Now we give a proof of the basis by giving a quadratic Grönber basis for the quotient ideal.

Let $S = \{[i] \mid 1 \leq i \leq n\}$.

We first recall the form of the quotient ideal I_0 in definition 3.7: $\mathcal{C}(\mathcal{B}_n) = \frac{\Bbbk[x_F : F\subset[n]]}{I_0}$ where

$$I_0 = \langle x_F x_G : F, G \text{ incomparable}\rangle + \left\langle \sum_{|F|=i} x_F : 1 \leq i \leq n \right\rangle.$$

Similar to lemma 3.5, we replace $x_{[i]}$ by $x_{[i]} - \sum_{|F|=i, F\neq[i]} x_F$ to get a quadratic presentation for $\mathcal{C}(\mathcal{B}_n)$:

$$\mathcal{C}(\mathcal{B}_n) = \frac{\Bbbk[x_F : F \subset [n], F \notin S]}{I}$$

$$I = \langle x_F x_G \mid X, G \text{ incomparable}, X, G \notin S\rangle + \langle x_F^2 \mid F \notin S\rangle$$

$$+\langle x_G \sum_{|F|=i, F\subset G} x_F \mid [i] \not\subset G, |G| > i, 1 \leq i \leq n\rangle + \langle x_G \sum_{|F|=i, G\subset F} x_F \mid G \not\subset [i], |G| < i, 1 \leq i \leq n\rangle,$$

where we denote the four generating sets I_1, I_2, I_3, I_4 respectively.

Theorem 3.20. *$\mathcal{G} := I_1 \cup I_2 \cup I_3 \cup I_4$ forms a quadratic Gröbner basis of I.*

Proof. We first fix the monomial order: we say $x_F < x_G$ iff $|F| < |G|$ or $|F| = |G|$ and $\min F \backslash G > \min G \backslash F$, and we use the induced graded lex order as defined in definition 2.21.

For example, when $n = 4$ we have the following total order on degree one monomials:

$$x_{124} > x_{134} > x_{234} > x_{13} > x_{14} > x_{23} > x_{24} > x_{34}.$$

We show that all non-trivial S-polynomials have a weak Gröbner representation, and by proposition 2.28 this suggests $\mathcal{G}$ forms a Gröbner basis.

(1) Consider the S-polynomials between two elements of $I_1 \cup I_2$, because they are monomials, the S-polynomial of two monomial terms is always 0.

(2) Consider the S-polynomials between $x_F x_G \in I_1 \cup I_2$ and $x_G \sum_H x_H \in I_3 \cup I_4$, then it's a multiply of $x_F x_G$.

(3) Consider the S-polynomial of $x_F x_G \in I_1 \cup I_2$ and $x_H \sum\limits_{|L|=i, L\subset H} x_L \in I_3$ where $x_F = \mathrm{in}(\sum\limits_{|L|=i, L\subset H} x_L)$. If G is incomparable with H, then the S-polynomial is $x_H x_G \sum\limits_{|L|=i, F\neq L\subset H} x_L$ with $x_H x_G \in \mathcal{G}$.

    Otherwise, we may assume $G \subset H$, and after subtracting incomparable $x_L x_G$'s, the S-polynomial becomes $x_H x_G \sum\limits_{|L|=i, F\neq L\subset H} x_L = x_H(x_G \sum\limits_{|L|=i, L\subset G} x_L)$ where $x_G \sum\limits_{|L|=i, L\subset G} x_L \in \mathcal{G}$.

(4) The S-polynomial of $x_F x_G \in I_1 \cup I_2$ and $x_H \sum\limits_{|H|=i, H\subset L} x_L \in I_4$ where $x_F = \mathrm{in}(\sum\limits_{|L|=i, H\subset L} x_L)$ has a weak Gröbner representation similar to the case above.

(5) Consider the S-polynomial of $x_G \sum\limits_{|F|=i, F\subset G} x_F, x_{G'} \sum\limits_{|F|=i, F\subset G'} x_F \in I_3$ where $\mathrm{in}(\sum\limits_{|F|=i, F\subset G} x_F) = \mathrm{in}(\sum\limits_{|F|=i, F\subset G'} x_F)$. If G, G' incomparable, then the S-polynomial is $x_G x'_G P \in I_1$ for some polynomial P.

    Otherwise, suppose $G \subset G'$. Then the S-polynomial is $x_{G'}(\sum\limits_{|F|=i, F\subset G', F\not\subset G} x_F) x_G$. Because $i < |G|$, this means F, G are incomparable.

(6) Consider the S-polynomial of $x_G \sum_{|F|=i, F\subset G} x_F, x_{G'} \sum_{|H|=|G|, H\subset G'} x_H \in I_3$ where $\mathrm{in}(\sum_{|H|=|G|, H\subset G'} x_H) = x_G$ and $LT := \mathrm{in}(\sum_{|F|=i, F\subset G} x_F)$.

For x_F, G where $F, G \subset [n]$, here we abuse notation to write $x_F = G$ for $F = G$.

The S-polynomial can be written as

$$
\begin{aligned}
&x_G x_{G'} \sum_{LT\neq F\subset G} x_F - LT x_{G'} \sum_{G\neq H\subset G'} x_H \\
&= \left(x_{G'} \sum_{H\subset G'} x_H\right)\left(\sum_{LT\neq F\subset G} x_F\right) - x_{G'}\left(\sum_{G\neq H\subset G'} x_H\right)\left(\sum_{LT\neq F\subset G} x_F\right) - LT x_{G'} \sum_{G\neq H\subset G'} x_H \\
&= \left(x_{G'} \sum_{H\subset G'} x_H\right)\left(\sum_{LT\neq F\subset G} x_F\right) - \left(\sum_{G\neq H\subset G'} x_H\right)\left(x_{G'} \sum_{F\subset G} x_F\right),
\end{aligned}
$$

and we can write

$$
\begin{aligned}
&\left(\sum_{G\neq H\subset G'} x_H\right)\left(x_{G'} \sum_{F\subset G} x_F\right) \\
&= x_{G'}\left(\sum_{G\neq H\subset G'} x_H\right)\left(\sum_{F\subset G'} x_F\right) - x_{G'}\left(\sum_{G\neq H\subset G'} x_H\right)\left(\sum_{F\subset G', F\not\subset G} x_F\right) \\
&= \left(\sum_{G\neq H\subset G'} x_H\right)\left(x_{G'} \sum_{F\subset G'} x_F\right) - x_{G'} \sum_{F\subset G', F\not\subset G} x_F \left(\sum_{F\subset H, H\subset G'} x_H\right) \\
&= \left(\sum_{G\neq H\subset G'} x_H\right)\left(x_{G'} \sum_{F\subset G'} x_F\right) - x_{G'} \sum_{F\subset G', F\not\subset G} x_F \left(\sum_{F\subset H} x_H\right)
\end{aligned}
$$

where the last equal sign follows from the fact that if $H \not\subset G'$, then H is incomparable with G', which yields a term that a multiply of $x_{G'}x_H \in I_1$. Because $x_{G'} \sum_{F\subset G'} x_F, x_F \sum_{F\subset H} x_H \in I_3 \cup I_4$, this is a weak Gröbner representation.

(7) Consider the S-polynomial of $x_G \sum_{|F|=i, F\subset G} x_F, x_G \sum_{|F'|=j, F'\subset G} x_{F'} \in I_3$, where $LT_1 := \mathrm{in}(\sum_{|F|=i, F\subset G} x_F)$ and $LT_2 := \mathrm{in}(\sum_{|F'|=j, F'\subset G} x_{F'})$. Then the S=pair can be written as

$$
\begin{aligned}
&LT_2 x_G \sum_{LT_1\neq F\subset G} x_F - LT_1 x_G \sum_{LT_2\neq F'\subset G} x_{F'} \\
&= \left(x_G \sum_{F'\subset G} x_{F'}\right)\left(\sum_{LT_1\neq F\subset G} x_F\right) - \left(x_G \sum_{LT_2\neq F'\subset G} x_{F'}\right)\left(\sum_{LT_1\neq F\subset G} x_F\right) - LT_1 x_G \sum_{LT_2\neq F'\subset G} x_{F'} \\
&= \left(x_G \sum_{F'\subset G} x_{F'}\right)\left(\sum_{LT_1\neq F\subset G} x_F\right) - \left(\sum_{LT_2\neq F'\subset G} x_{F'}\right)\left(x_G \sum_{F\subset G} x_F\right)
\end{aligned}
$$

where $x_G \sum_{|F|=i, F\subset G} x_F, x_G \sum_{|F'|=j, F'\subset G} x_{F'} \in I_3$.

(8) Consider the S-polynomial of $x_G \sum_{|F|=i, F\subset G} x_F \in I_3$ and $x_{G'} \sum_{|F|=j, G'\subset F} x_F \in I_4$ where $x_G = \mathrm{in}(\sum_{|F|=j, G'\subset F} x_F)$. In this case, $\mathrm{in}(\sum_{|F|=i, F\subset G} x_F)$ is incomparable with G', so the S-polynomial is a multiply of $x_{G'} \mathrm{in}(\sum_{|F|=i, F\subset G} x_F) \in I_1$.

(9) The S-polynomial of $x_G \sum_{|F|=i, F\subset G} x_F \in I_3$ and $x_{G'} \sum_{|F|=|G|, G'\subset F} x_F \in I_4$ where $x_{G'} = \mathrm{in}(\sum_{|F|=i, F\subset G} x_F)$ has a weak Grönber representation similar to the case above.

(10) The S-polynomial of $x_G \sum_{|F|=i, F\subset G} x_F \in I_3$ and $x_G \sum_{|F|=j, G\subset F} x_F \in I_4$ has a weak Grönber representation similar to item 7.

(11) The S-polynomial of $x_G \sum_{|F|=i, F\subset G} x_F \in I_3$ and $x_{G'} \sum_{|F|=i, G'\subset F} x_F \in I_4$ where $LT := \mathrm{in}(\sum_{|F|=i, F\subset G} x_F) = \mathrm{in}(\sum_{|F|=i, G'\subset F} x_F)$. Then the S-polynomial can be written as

$$x_{G'} x_G \sum_{LT\neq F\subset G} x_F - x_G x_{G'} \sum_{LT\neq F\supset G'} x_F.$$

After removing all terms which are multiply of some $x_F x_H \in I_1$, the S-polynomial becomes

$$x_{G'} x_G \sum_{LT\neq F, G'\subset F\subset G} x_F - x_G x_{G'} \sum_{LT\neq F, G'\subset F\subset G} x_F = 0.$$

(12) The S-polynomials of two elements in I_4 has a weak Gröbner representation similar to item 5, item 6, and item 7.

□

Notice that the initial monomials of $\mathcal{G}$ are exactly the smallest monomials in the sum described in lemma 3.17, which are all nonzero monomials in $\mathcal{C}(\mathcal{B}_n)$ neither descent monomials nor contain an index of the form $[i]$. Therefore, together with theorem 3.13, this gives an alternative proof for theorem 3.19. Furthermore, this gives an alternative proof for the Koszulity of the colorful ring, because an algebra with a quadratic Grönber basis is Koszul [Con00].

4. Representations for the Chow ring of the Boolean matroid

We may describe the representations comprising the different graded components of $A(n) = \mathrm{Chow}(\mathcal{B}_n)$ using the FY-basis.

Lemma 4.1. *For every increasing sequence of integers less than n of the form $(f_1, \ldots, f_k)$ with $f_i - f_{i-1} - 1 \geq 1$, and fixed exponents $(a_1, \ldots, a_k)$ such that $1 \leq a_i \leq f_i - f_{i-1} - 1$, all FY-basis elements $\prod_{i=1}^k x_{F_i}^{a_i}$ corresponding to flags of the form $\varnothing = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_k \subseteq [n]$ with $|F_i| = f_i$ and exponents $(a_1, \ldots, a_k)$ are a basis for the representation given by $\bigotimes_{i=1}^k (f_i - f_{i-1}) \otimes (n - f_k)$.*

Proof. Consider the construction of a basis for this representation as a Specht module, i.e. via SYTs of this skew shape. Note that since this skew shape has no column stabilizers, the SYTs can be thought of as a basis for this vector space. For any such SYT T in the basis, with rows $T_1, T_2, \ldots, T_{k+1}$ reading from bottom to top, we can construct subsets $F_i = \bigcup_{j=1}^{i} T_j$ with $|F_i| = f_i$ that form such a flag. For a fixed choice of the $(a_1, \ldots, a_k)$, this corresponds to a unique subset of the FY-basis corresponding to this representation. $\square$

Using this information, we can compute the multiplicities of some irreducible representations in each graded component.

Theorem 4.2. *The multiplicity of the irreducible representation $(n-k, 1, 1, \cdots, 1)$ in the degree d component of $A(n)$ is*

$$\binom{n-k-1}{k}\binom{n-2k-1}{d-k}.$$

In particular, when $k = 0$, the multiplicity of (n) in the degree d component is $\binom{n-1}{d}$. We first prove this special case separately, although it's covered by the general proof below, because it gives us a useful bijection between basis elements of the degree d component and certain d-tuples.

Lemma 4.3. *The multiplicity of the trivial representation in the degree d component of $A(n)$ is $\binom{n-1}{d}$.*

Proof. We first show that there are $\binom{n-1}{d}$ forms of generators in the d-th graded component, then since each corresponding skew diagram contains the trivial representation (n) exactly once (a proof of this is included in the general proof below), we have the desired result.

By a form of generator, we mean an increasing sequence and a fixed component described in lemma 4.1. Notice that each form of generator corresponds to a skew tableau in the representation.

Let $a(n,d)$ be the number of forms of generators of the d-th graded component of $A(n)$, we show that $a(n,d) = a(n-1,d) + a(n-1,d-1)$, then $a(n,d) = \binom{n-1}{d}$ follows from the relation $\binom{n-1}{d} = \binom{n-2}{d} + \binom{n-2}{d-1}$.

We consider the Gröbner basis for I given by [Wan21]:

$$\{x_F x_G : F, G \text{ incomparable}\} \cup \left\{ \left(\sum_{G \subset H} x_H \right)^{|G|} \right\}$$
$$\cup \left\{ x_F \left(\sum_{G \subset H} x_H \right)^{|G|-|F|} : F \subsetneq G \right\}$$

with the leading monomials

$$\mathrm{in}(A(n)) = \{x_F x_G : F, G \text{ incomparable}\} \cup \{x_G^{|G|}\} \cup \{x_F x_G^{|G|-|F|} : F \subsetneq G\},$$

so the d-th graded component of $A(n)$ is generated by degree d monomials not in the ideal $\langle \mathrm{in}(A(n))\rangle$, i.e. $x_{F_1}x_{F_2}\cdots x_{F_d}$ s.t. $F_1 \subset F_2 \subset \cdots \subset F_d$ and contains no $x_G^{|G|}$ or $x_F x_G^{|G|-|F|}$. Then each form of generator can be written as a d-tuple: $(|F_1|, |F_2|-|F_1|, \cdots, |F_{i+1}|-|F_i|, \cdots, |F_d|-|F_{d-1}|)$. For example, $x_{ij}x_{ijkl}$ corresponds to the tuple $(2,2)$, and x_{ijk}^2 corresponds to the tuple $(3,0)$.

The legal tuples are exactly those with coordinates ≥ 2 or 0, coordinates sum up to no more than n, and each coordinate k is followed by no more than $k-2$ zero's.

Proof. If $|F|-|G| = 1$, then $x_F x_G \in \mathrm{in}(A(n))$, so the d-tuple cannot have 1 as coordinate. If the first coordinate k is followed by $k-1$ zeros, this means the corresponding monomial contains $x_G^{|G|}$; if some other coordinate k is follows by $k-1$ zeros, this means the monomial contains $x_F x_G^{|G|-|F|}$. Conversely, these are the only cases where the monomial is in $\langle \mathrm{in}(A(n))\rangle$. □

For example, in the second graded component of $A(5)$, we have

form of generator	legal tuple
$x_{ij}x_{ijkl}$	$(2,2)$
$x_{ij}x_{12345}$	$(2,3)$
x_{ijk}^2	$(3,0)$
$x_{ijk}x_{12345}$	$(3,2)$
x_{ijkl}^2	$(4,0)$
x_{12345}^2	$(5,0)$

For $a(n,d)$, first notice that the d-tuples of $A(n)$ that sum up to $< n$ are exactly d-tuples of $A(n-1)$. For each d-tuple of $A(n)$ that sums up to n, we remove the first coordinate to get a $(d-1)$-tuple of $A(n)$. The only problem with this is when we have $(k, 0, \cdots)$, in this case we map it to $(k-1, \cdots)$.

Let $(k_1, k_2, \cdots, k_{d-1})$ be a $(d-1)$-tuple of $A(n-1)$. If it sums to less than $n-1$, then $(n - \sum_i k_i, k_1, \cdots, k_{d-1})$ is the unique d-tuple of $A(n)$ that maps to $(k_1, k_2, \cdots, k_{d-1})$. If it sums to $n-1$, then $(k_1+1, 0, k_2, \cdots, k_{d-1})$ is the unique legal d-tuple of $A(n)$ that maps to $(k_1, k_2, \cdots, k_{d-1})$. So this correspondence is bijective, and we have $a(n,d) = a(n-1,d) + a(n-1,d-1)$. □

Now we are ready to prove this for any k.

Proof of theorem 4.2. First notice that any skew diagram $(\lambda_1, \lambda_2, \cdots, \lambda_n)/(\lambda_2, \cdots, \lambda_n)$ where $\lambda_1 > \lambda_2 > \cdots > \lambda_n$, corresponds to the same representation as $(\lambda_2 - \lambda_1) \otimes (\lambda_3 - \lambda_2) \otimes \cdots \otimes (\lambda_n - \lambda_{n-1})$, and the coefficient of the hook $(n-k, 1, \cdots, 1)$ in the decomposition of the skew diagram $(\mu_1) \otimes (\mu_2) \otimes \cdots \otimes (\mu_l)$ is $\binom{l-1}{k}$. We call l the length of the skew diagram $(\lambda_1, \lambda_2, \cdots, \lambda_l)/(\lambda_2, \cdots, \lambda_l)$.

We prove the coefficient of the hook $(n-k, 1, \cdots, 1)$ in $(\mu_1) \otimes (\mu_2) \otimes \cdots \otimes (\mu_l)$ is $\binom{l-1}{k}$.

Proof. By the property of tensor product of Young diagrams, for any partition $\lambda = (\lambda_1, \cdots, \lambda_n)$ and the partition (ρ), we get exactly one copy of the partition $(|\lambda| + \rho)$ in $\lambda \otimes (\rho)$ iff $\lambda = (\lambda_1)$.

Therefore, the coefficient of the one partition in $(\mu_1) \otimes (\mu_2) \otimes \cdots \otimes (\mu_l)$ is the same as the coefficient of the one partition in $(\mu_1) \otimes (\mu_2) \otimes \cdots \otimes (\mu_{l-1})$, i.e. each skew diagram contains $s[n]$ exactly once.

The hook $(|\lambda| - \rho - k, 1, 1, \cdots, 1)$ appears exactly once in $\lambda \otimes (\rho)$ iff $\lambda = (\lambda_1 - k + 1, k - 1)$ or $(\lambda_1 - k, k)$, so the coefficient of $s[n-k, 1, \cdots, 1]$ in $(\mu_1) \otimes (\mu_2) \otimes \cdots \otimes (\mu_l)$ is the sum of its coefficient and the coefficient of $s[n-k+1, 1, \cdots, 1]$ in $(\mu_1) \otimes (\mu_2) \otimes \cdots \otimes (\mu_{l-1})$. From this, we know it's $\binom{l-1}{k}$ by the relation $\binom{l-1}{k} = \binom{l-2}{k} + \binom{l-2}{k-1}$. □

We define $k_l(d, n)$ to be the number of skew tableaux with length l in the representation of the d-th graded component of $A(n)$, and $j_l(d, n)$ to be the number of such tableaux corresponding to tuples that do not sum up to n. We claim that

$$k_l(d, n) = \binom{d}{l-1}\binom{n-d-1}{l-1} \qquad \text{and} \qquad j_l(d, n) = \binom{d-1}{l-2}\binom{n-d-1}{l-1}.$$

Proof. To see this, notice that they satisfy the relation

$$j_l(n, d) = j_l(n-1, d) + k_{l-1}(n-2, d-1)$$
$$k_l(n, d) = k_l(n-1, d-1) + j_l(n, d).$$

We first consider the d-tuples of $A(n)$ that sum up to n, i.e. those given by $(d-1)$-tuples of $A(n-1)$. If the $(d-1)$-tuple $(k_1, \cdots, k_{d-1})$ of $A(n-1)$ sums to $< n-1$, the corresponding d-tuple of $A(n)$ is $(k_0, k_1, \cdots, k_{d-1})$, where the length of its corresponding partition is the number of nonzero entries in $(k_0, k_1, \cdots, k_{d-1})$; and the length of the partition corresponding to $(k_1, \cdots, k_{d-1})$ is the number of nonzero entries in it plus 1, so the two partitions have the same length. If the $(d-1)$-tuple of $A(n-1)$ sums to n, the corresponding d-tuple of $A(n)$ is $(k_1 + 1, 0, k_2, \cdots, k_{d-1})$, and the partitions corresponding to both tuples is the number of nonzero entries in them, which is the same.

This means the number of d-tuples of $A(n)$ sum up to n with length l is the same as the number of $(d-1)$-tuples of $A(n-1)$ with length l. Thus, summing this with $j_l(n, d)$, the number of d-tuples of $A(n)$ with length l that sum up to $< n$, we have $k_l(n, d) = k_l(n-1, d-1) + j_l(n, d)$.

Next we consider the d-tuples of $A(n)$ counted by $j_l(n, d)$, i.e. those that sum to $< n$ and have length l. These are exactly the d-tuples of $A(n-1)$. If the d-tuple sums to $< n-1$, the corresponding partition in $A(n-1)$ and $A(n)$ both has length l equals the number of nonzero coordinates in the tuple plus 1, and there are $j_l(n-1, d)$ of them. If the d-tuple sum up to $n-1$, its corresponding partition in $A(n)$ has length one longer than the partition in $A(n-1)$, which by the argument above has the same number as $(d-1)$-tuples of $A(n-2)$ with length $l-1$. Summing them up gives us $j_l(n, d) = j_l(n-1, d) + k_{l-1}(n-2, d-1)$.

Then the claim follows from the relation

$$\binom{d-1}{l-2}\binom{n-d-1}{l-1} = \binom{d-1}{l-2}\binom{n-d-2}{l-1} + \binom{d-1}{l-2}\binom{n-d-2}{l-2},$$

$$\binom{d}{l-1}\binom{n-d-1}{l-1} = \binom{d-1}{l-1}\binom{n-d-1}{l-1} + \binom{d-1}{l-2}\binom{n-d-1}{l-1}.$$

□

Then the coefficient of $s[n-k, 1, \cdots, 1]$ in the decomposition of the representation of the d-th graded component in $A(n)$ is

$$\begin{aligned}
\sum_l k_l(n,d)\binom{l-1}{k} &= \sum_l \binom{d}{l}\binom{n-d-1}{l}\binom{l}{k} = \sum_l \binom{d}{l}\frac{(n-d-1)!l!}{l!k!(l-k)!(n-d-l-1)!} \\
&= \frac{(n-d-1)!}{k!}\sum_l \binom{d}{l}\frac{1}{(n-d-l-1)!(l-k)!} \\
&= \frac{(n-d-1)!}{k!(n-d-k-1)!}\sum_l \binom{d}{l}\frac{(n-d-k-1)!}{(n-d-l-1)!(l-k)!} \\
&= \frac{(n-d-1)!}{k!(n-d-k-1)!}\sum_l \binom{d}{d-l}\binom{n-d-k-1}{l-k} && (2)\\
&= \frac{(n-d-1)!}{k!(n-d-k-1)!}\binom{n-k-1}{d-k} = \frac{(n-d-1)!}{k!(n-d-k-1)!}\frac{(n-k-1)!}{(n-d-1)!(d-k)!} && (3)\\
&= \frac{(n-k-1)!}{k!(d-k)!(n-k-d-1)!} = \binom{n-k-1}{k}\binom{n-2k-1}{d-k},
\end{aligned}$$

where we go from (2) to (3) by Vandermonde's identity

$$\sum_l \binom{m}{l}\binom{n}{k-l} = \binom{m+n}{k}.$$

This gives us the desired result. □

5. Branching rules categorifying a recursion on the Eulerian numbers

Recall that the Eulerian numbers satisfy the recursion

$$\left\langle {n \atop k} \right\rangle = (n-k)\left\langle {n-1 \atop k-1} \right\rangle + (k+1)\left\langle {n-1 \atop k} \right\rangle.$$

Since the different graded components of $A(n)$ admit a $\mathfrak{S}_n$ action, each graded component can be thought of as a $\mathfrak{S}_n$-representation. This inspires us to consider the following question:

Question 5.1. Can one categorify the recursion on Eulerian numbers to a branching rule at the level of Specht modules of the form:

$$A(n)_k \downarrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} = \chi_1 \otimes A(n-1)_{k-1} \oplus \chi_2 \otimes A(n-1)_k$$

or perhaps even a $\mathfrak{S}_{n-1}$-equivariant short exact sequence of the form

$$0 \to \chi_1 \otimes A(n-1)_k \xrightarrow{i} A(n)_{k+1} \downarrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} \xrightarrow{q} 2\chi_2 \otimes A(n-1)_{k+1} \to 0$$

where

- $A(n)$ is either $\mathrm{Chow}(\mathcal{B}_n)$ or $\mathcal{C}(\mathcal{B}_n)$, and
- χ_1 and χ_2 are $\mathfrak{S}_{n-1}$-representations of dimensions $n-k$ and $k+1$, respectively?

In the cases where $k = 1$, we can indeed find an $\mathfrak{S}_{n-1}$-equivariant short exact sequence involving the low-degree pieces of $\mathrm{Chow}(B_n)$.

Proposition 5.2. *Let $A(n) := \mathrm{Chow}(\mathcal{B}_n)$. Then we have a short exact sequence of $\mathfrak{S}_{n-1}$-representations*

$$0 \to \mathcal{S}^{(n-1,1)/(1)} \otimes A(n-1)_0 \xrightarrow{i} A(n)_1 \downarrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} \xrightarrow{q} 2\,\mathcal{S}^{(n-1)} \otimes A(n-1)_1 \to 0.$$

Proof. We first identify bases of the first and last spaces in our sequence. Identify the basis elements of $\mathcal{S}^{(n-1,1)/(1)}$ as corresponding to the SYTs of $(n-1,1)/(1)$, each of which can be uniquely identified by the number k appearing in the bottom-left box, where $1 \le k \le n-1$. As such, we consider a basis $\{y_k\}_{k=1}^{n-1}$ of $\mathcal{S}^{(n-1,1)/(1)}$, which gives the basis $\{y_k \otimes 1\}_{k=1}^{n-1}$ for $\mathcal{S}^{(n-1,1)/(1)} \otimes A(n-1)_0$.

For the last space, recall that $\mathcal{S}^{(n-1)}$ is 1-dimensional, so $2\mathcal{S}^{(n-1)}$ is 2-dimension with a basis we call $\{b_1, b_2\}$. Moreover, $A(n)_1$ has basis x_F for $F \in \mathcal{B}_n$, $|F| \ge 2$. This gives the basis $\{b_1 \otimes x_F, b_2 \otimes x_F\}_{F \in \mathcal{B}_{n-1}, |F| > 2}$ for this last space.

We can now define our maps i and q on the basis elements as follows:

$$i(y_k \otimes 1) = x_{\{k,n\}} \quad q(x_F) = \begin{cases} 0 & F = \{k,n\}, 1 \le k \le n-1 \\ b_1 \otimes x_F & n \notin F \\ b_2 \otimes x_{F \setminus \{n\}} & n \in F, |F| > 2 \end{cases}$$

We can see that i is an injection since its image has the same dimension as its domain, and we can see the exactness at $A(n)_1 \downarrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}$ by definition. To see that q is surjective, we can show that every basis element is in the image of q. For an element $b_1 \otimes x_F$ in the basis for $F \in \mathcal{B}_{n-1}$, $|F| \ge 2$, $F \in \mathcal{B}_n$ as well, so we see that x_F is a basis element of $A(n)_1 \downarrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}$ with $q(x_F) = b_1 \otimes F$. Otherwise, if we have an element $b_2 \otimes x_F$ for some $F \in \mathcal{B}_{n-1}$, $|F| \ge 2$, then $|F \cup \{n\}| > 2$ and $q(x_{F \cup \{n\}}) = b_2 \otimes x_F$. This shows that the above sequence is exact everywhere.

Since this is a short exact sequence of vector spaces, it splits, yielding $A(n)_1 \downarrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} = \mathcal{S}^{(n-1,1)/(1)} \otimes A(n-1)_0 \oplus 2\,\mathcal{S}^{(n-1)} \otimes A(n-1)_1$.

Finally, to show that this is a short exact sequence of representations, we need to show that i and q are $\mathfrak{S}_{n-1}$-equivariant, noting that $A(n)_1$ is also an $\mathfrak{S}_{n-1}$ representation if we leave n invariant. For i, fix some $\sigma \in \mathfrak{S}_{n-1}$. Then $\sigma \cdot (y_k \otimes 1) = y_{\sigma(k)} \otimes 1$, so $i(\sigma \cdot y_k \otimes 1) = x_{\{\sigma(k),n\}}$. In the other direction, $\sigma \cdot i(y_k \otimes 1) = \sigma \cdot x_{\{k,n\}} = x_{\{\sigma(k),n\}}$, so i is indeed $\mathfrak{S}_{n-1}$-equivariant.

To show the $\mathfrak{S}_{n-1}$-equivariance of q, we have to treat the three cases independently. Again, fix $\sigma \in \mathfrak{S}_{n-1}$, and consider the three cases for x_F:

- If $F = \{k, n\}$ for $1 \le k \le n-1$, then $\sigma \cdot x_{\{k,n\}} = x_{\{\sigma(k),n\}}$, and $q(\sigma \cdot x_F) = 0$ still by construction. Similarly, $\sigma \cdot q(x_F) = \sigma \cdot 0 = 0$, as desired.
- If $n \notin F$, then applying σ pointwise to every element in F, $\sigma \cdot x_F = x_{\sigma(F)}$ where $\sigma(F) = \{\sigma(k) \mid k \in F\}$, and $n \notin \sigma(F)$. Then $q(\sigma \cdot x_F) = b_1 \otimes x_{\sigma(F)}$. Similarly, $\sigma \cdot q(x_F) = \sigma \cdot b_1 \otimes x_F = b_1 \otimes x_{\sigma(F)}$, as desired.
- If $n \in F$, then $\sigma \cdot x_F = x_{\sigma(F \setminus \{n\}) \cup \{n\}}$, and so $q(\sigma \cdot x_F) = q(x_{\sigma(F \setminus \{n\}) \cup \{n\}}) = b_2 \otimes x_{\sigma(F \setminus \{n\})}$. Similarly, $\sigma \cdot q(x_F) = \sigma \cdot b_2 \otimes x_{F \setminus \{n\}} = b_2 \otimes x_{\sigma(F \setminus \{n\})}$, as desired.

Therefore both i and q are $\mathfrak{S}_{n-1}$-equivariant, so this is indeed a short exact sequence of $\mathfrak{S}_{n-1}$-representations. This gives the desired branching rule.

□

We have an analagous short exact sequence for the case where $k = n-2$:

Proposition 5.3. *When $k = n-2$, we can consider the short exact sequence*

$$0 \to \mathcal{S}^{(n-1,1)/(1)} \otimes A(n-1)_{n-2} \xrightarrow{i} A(n)_{n-2} \downarrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} \xrightarrow{q} 2\,\mathcal{S}^{(n-1)} \otimes A(n-1)_{n-3} \to 0$$

Proof. Note that from the FY-basis, we have that $A(n)_{n-1}$ has the single basis element $x_{[n]}^{n-1}$. Less obviously, we can see that $A(n)_{n-2}$ has a basis element for every $F \in \mathcal{B}_n$ with $|F| \ge 2$. In particular, we can take $x_{[n]}^{n-2}$ as the element corresponding to $[n]$, and for any $|F|$ with $2 \le |F| < n$, we get the basis element $x_F^{|F|-1} x_{[n]}^{n-|F|-1}$ which also has degree $n-2$. We can show that this list of basis elements is exhaustive from the definition of the FY-basis.

Again, supposing that $\mathcal{S}^{(n-1,1)/(1)}$ has basis $\{y_k\}_{k=1}^{n-1}$, and that $2\mathcal{S}^{(n-1)}$ has basis $\{b_1, b_2\}$, then we consider the maps i and q on the basis elements of the tensor-product spaces:

$$i\left(y_k \otimes x_{[n-1]}^{n-2}\right) = x_{[n]-k}^{n-2} \; q(x_F^{|F|-1} x_{[n]}^{n-|F|-1}) = \begin{cases} 0 & |F| = n-1,\ n \in F \\ b_1 \otimes x_{[n-1]}^{n-3} & |F| = n-1,\ n \notin F \\ b_2 \otimes x_{[n-1]}^{n-3} & F = [n] \\ b_1 \otimes x_F^{|F|-1} x_{[n-1]}^{n-|F|-2} & 2 \le |F| < n-1,\ n \notin F \\ b_2 \otimes x_{[n]\setminus F}^{n-|F|-1} x_{[n-1]}^{F-2} & 2 \le |F| < n-1,\ n \in F \end{cases}$$

Again, i is an injection as its image has the same dimension as its domain, and the exactness at $A(n)_{n-2} \downarrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}$ is by construction. To see that q is surjective, we see that $b_1 \otimes x_{[n-1]}^{n-3}$ and $b_2 \otimes x_{[n-1]}^{n-3}$ are in the image of q, so we have to check basis elements $b_i \otimes x_F^{|F|-1} x_{[n-1]}^{n-|F|-2}$ lie in the image. For a basis element $b_1 \otimes x_F^{|F|-1} x_{[n-1]}^{n-|F|-2}$, this is the image of $x_F^{|F|-1} x_{[n]}^{n-|F|-1}$ under q, and for a basis element $b_2 \otimes x_F^{|F|-1} x_{[n-1]}^{n-|F|-2}$, this is the image of $x_{[n]\setminus F}^{n-|F|-1} x_{[n]}^{|F|-1}$. As such, q is surjective, so this is indeed a short exact sequence of vector spaces.

Finally, to show that this is a short exact sequence of representations, we need to show that i and q are $\mathfrak{S}_{n-1}$-equivariant, noting that $A(n)_{n-2}$ is a $\mathfrak{S}_{n-1}$ representation. For i, fix some $\sigma \in \mathfrak{S}_{n-1}$. Then $\sigma \cdot (y_k \otimes x_{[n-1]}^{n-2}) = y_{\sigma(k)} \otimes x_{[n-1]}^{n-2}$, so $i(\sigma \cdot y_k \otimes x_{[n-1]}^{n-2}) = x_{[n]\setminus\{\sigma(k)\}}^{n-2}$. On the other hand, $\sigma \cdot i(y_k \otimes x_{[n-1]}^{n-2}) = \sigma \cdot x_{[n]\setminus\{k\}}^{n-2} = x_{[n]\setminus\{\sigma(k)\}}^{n-2}$, since n is fixed by σ and the only element missing from the indexing set is now $\sigma(k)$. This shows that i is indeed $\mathfrak{S}_{n-1}$-equivariant.

To show the $\mathfrak{S}_{n-1}$-equivariance of q, we again have to do casework. Fix $\sigma \in \mathfrak{S}_{n-1}$, and look at all the cases for x_F:

- If $|F| = n-1$ and $n \in F$, then $F = [n] \setminus \{k\}$ for some $1 \leq k \leq n-1$. In this case, we have that $\sigma \cdot x_{[n]\setminus\{k\}}^{n-2} = x_{[n]\setminus\{\sigma(k)\}}^{n-2}$, and $q(\sigma \cdot x_{[n]\setminus\{k\}}^{n-2}) = 0$ by construction. Similarly, $\sigma \cdot q(x_{[n]\setminus\{k\}}^{n-2}) = \sigma \cdot 0 = 0$, as desired.

- If $|F| = n-1$ with $n \notin F$, then $F = [n-1]$. As such, $\sigma \cdot x_{[n-1]}^{n-2} = x_{[n-1]}^{n-2}$, and so $q(\sigma \cdot x_{[n-1]}^{n-2}) = b_1 \otimes x_{[n-1]}^{n-3}$ as before. Similarly, $\sigma \cdot q(x_{[n-1]}^{n-2}) = \sigma \cdot b_1 \otimes x_{[n-1]}^{n-3} = b_1 \otimes x_{[n-1]}^{n-3}$, as desired.

- If $F = [n]$, then similarly $\sigma \cdot x_{[n]}^{n-2} = x_{[n]}^{n-2}$ and $q(x_{[n]}^{n-2}) = b_2 \otimes x_{[n]}^{n-3}$, and $\sigma \cdot q(x_{[n]}^{n-2}) = \sigma \cdot b_2 \otimes x_{[n]}^{n-3} = b_2 \otimes x_{[n]}^{n-3}$, as desired.

- If $n \notin F$ and $2 \leq |F| < n-1$, then applying σ pointwise to every element in F, $\sigma \cdot x_F^{|F|-1} x_{[n-1]}^{n-|F|-1} = x_{\sigma(F)}^{|F|-1} x_{[n-1]}^{n-|F|-1}$. Then $q(\sigma \cdot x_F^{|F|-1} x_{[n-1]}^{n-|F|-1}) = b_1 \otimes x_{\sigma(F)}^{|F|-1} x_{[n-1]}^{n-|F|-2}$. Similarly, $\sigma \cdot q(x_F^{|F|-1} x_{[n-1]}^{n-|F|-1}) = \sigma \cdot b_1 \otimes x_F^{|F|-1} x_{[n-1]}^{n-|F|-2} = b_1 \otimes x_{\sigma(F)}^{|F|-1} x_{[n-1]}^{n-|F|-2}$, as desired.

- If $n \in F$, then $\sigma \cdot x_F^{|F|-1} x_{[n-1]}^{n-|F|-1} = x_{\sigma(F\setminus\{n\})\cup\{n\}}^{|F|-1} x_{[n-1]}^{n-|F|-1}$, and so $q(\sigma \cdot x_F^{|F|-1} x_{[n-1]}^{n-|F|-1}) = q(x_{\sigma(F\setminus\{n\})\cup\{n\}}^{|F|-1} x_{[n-1]}^{n-|F|-1}) = b_2 \otimes x_{[n-1]\setminus\sigma(F\setminus\{n\})}^{n-|F|-1} x_{[n-1]}^{|F|-2}$. Similarly, $\sigma \cdot q(x_F^{|F|-1} x_{[n-1]}^{n-|F|-1}) = \sigma \cdot b_2 \otimes x_{[n]\setminus F}^{n-|F|-1} x_{[n-1]}^{|F|-2} = b_2 \otimes x_{[n-1]\setminus\sigma(F\setminus\{n\})}^{n-|F|-1} x_{[n-1]}^{|F|-2}$, as desired.

Therefore both i and q are $\mathfrak{S}_{n-1}$-equivariant, so this is indeed a short exact sequence of $\mathfrak{S}_{n-1}$-representations. This gives the desired branching rule.

□

We don't have much hope of branching rules for higher k – the coefficients become too small and the available irreps become too large to be tensored properly. As an example:

Example 5.4 (Non-example of branching.)**.** When $n = 5, k = 2$, we wish to express

$$A(5)_2 \downarrow_{S_4}^{S_5} = X \otimes A(4)_1 \oplus Y \otimes A(4)_2$$

where X has dimension 3 and Y has dimension 3. Since $A(4)_1 = A(4)_2 = S_{(2,2)} + 2S_{(3,1)} + 3S_{(4)}$, it suffices to find one representation Z of dimension 6 such that $Z \otimes A(4)_1$ is isomorphic to $A(5)_2$. The two options that do not introduce a sign representation in the resulting tensor product are $S_{(3,1)} + 3S_{(4)}$ and $6S_{(4)}$, neither of which work. In the former case, we get rather close, but we have numbers of irreducible representations showing up that are not quite correct.

On the other hand, graded pieces of the colorful ring do satisfy branching rules that mirror the recursion for the Eulerian numbers. We first establish a branching rule for ribbons with a key symmetric function lemma.

Definition 5.5. For a tuple of non-negative integers $\lambda = (a_1, \ldots, a_k)$, define the matrix H_λ that appears in the Jacobi-Trudi identity calculation for λ, where

$$(H_\lambda)_{ij} = \begin{cases} h_{\sum_{l=i}^{j} a_l} & j \geq i \\ 1 & j = i-1 \\ 0 & \text{else} \end{cases}$$

and in particular, if λ represents a ribbon diagram $(a_1, \ldots, a_k)$ (i.e. all of the a_i are strictly positive), then $s_\lambda = \det H_\lambda$.

Definition 5.6. The **squashing** of ribbon $\lambda = (a_1, \ldots, a_k)$ at row i for $1 \leq i < k$ gives the ribbon $\lambda \downarrow_i = (a_1, \ldots, a_{i-1}, a_i + a_{i+1}, a_{i+2}, \ldots, a_k)$.

This is the same as splitting this ribbon from row 1 to row i and from row $i+1$ to row k, and near-concatenating these two tableaux.

Now, a lemma about lowering of ribbons:

Lemma 5.7. *Consider the ribbon diagram $\lambda = (a_1, \ldots, a_k)$. We show that*

$$s_\lambda \downarrow_{S_{n-1}}^{S_n} = \sum_{i=1}^{k} \det H_{\lambda - e_i} + \sum_{i=1}^{k-1} \det H_{\lambda\downarrow_i - e_i}$$

Proof. From the Jacobi-Trudi identity, we see that $s_\lambda = \det H_\lambda$, which we may expand:

$$s_\lambda = \det H_\lambda = \sum_{\sigma \in S_k} \operatorname{sign}(\sigma) \prod_{l=1}^{k} (H_\lambda)_{\sigma(l)l}$$

We now make an observation about the $\sigma \in S_k$ such that the product $\prod_{l=1}^{k} (H_\lambda)_{\sigma(l)l}$ is nonzero.

Claim 5.8. *There is a bijection between compositions of k and such $\sigma \in S_k$.*

Proof. Consider j such that $\sigma(j) = 1$. We show that for all $l < j$, $\sigma(l) = l+1$. This can be seen inductively – since $\sigma(1) = 1, 2$ in order to produce a nonzero result, we see that $\sigma(1) = 2$. Similarly by induction, since $\sigma(l) \in [l+1]$ for all l, but $\sigma(\{j\} \cup [l-1]) = [l]$, we must have that $\sigma(l) = l+1$.

Now, we have that $\sigma([k] \setminus [j]) = [k] \setminus [j]$, and restricting σ to the lower right $(k-j)\times(k-j)$ submatrix of H_λ, we have that σ picks out a product of terms of this submatrix which is also nonzero. By induction on k, we can see that such a σ corresponds to a composition of $k-j$, so σ corresponds to a composition of k.

This correspondence is invertible – given a composition $(b_1,\ldots,b_m)$ of k, we may consider constructing a $\sigma \in S_k$ that produces a nonzero product by letting $\sigma(\sum_{i=1}^{j} b_i) = \sum_{i=1}^{j-1} b_i + 1$ for all $1 \le j \le m$, which forces the rest of the elements $\sigma(l) = l+1$ by the same argument as above. Applying the above algorithm to this permutation recovers the composition $(b_1,\ldots,b_m)$. $\square$

This allows us to reindex this sum by compositions of k. For the sake of notation, define $B_m = \sum_{i=1}^{m} b_i$. Note that in particular, the determinant now becomes

$$s_\lambda = \sum_{(b_1,\ldots,b_r)\vDash k} (-1)^{k-r} \prod_{s=1}^{r} h_{\sum_{l=B_{s-1}+1}^{B_s} a_l}$$

We can now lower this with the branching rule for a product of homogeneous polynomials. In particular,

$$s_\lambda \downarrow_{S_{n-1}}^{S_n} = \sum_{(b_1,\ldots,b_r)\vDash k} (-1)^{k-r} \sum_{i=1}^{r} \prod_{s=1}^{r} h_{\sum_{l=B_{s-1}+1}^{B_s} a_l - \delta_{is}}$$

where δ_{is} is the Kronecker delta (1 if $i = s$, 0 otherwise). This amounts to iterating through subtracting one from the ith term of the product, where i comes from the second inner sum and iterates from 1 to r.

We now interpret this expression combinatorially. For every composition $(b_1,\ldots,b_r) \vDash k$, we assign a sign to the corresponding r terms based on the parity of the number of terms it is missing from the maximum number of terms (n). The r terms represent a way to iteratively "tag" one of the parts of the composition that is being subtracted from in the index, where each of these terms is a product of homogeneous polynomials that is derived from summing corresponding parts of the composition. In summary, we get one term corresponding to a pair of a composition of k and a tag for one of the components of that composition, with the correct sign.

We compare this to the right-hand side and again interpret the terms combinatorially. Considering the first sum, note that

$$\det H_{\lambda-e_i} = \sum_{\sigma\in S_k} \mathrm{sign}(\sigma) \prod_{l=1}^{k} (H_{\lambda-e_i})_{\sigma(l)l} = \sum_{(b_1,\ldots,b_r)\vDash k} (-1)^{k-r} \prod_{s=1}^{r} h_{\sum_{l=B_{s-1}+1}^{B_s} a_l - \delta_{li}}$$

where the decrease at component i in λ propagates to any sum that happens to add the component at i. Interpreting this combinatorially, fixing a composition $(b_1,\ldots,b_r) \vDash k$, we get k terms, which tags the component b_i exactly b_i times, and overcounts $b_i - 1$ times.

To correct for this overcount, we consider the second sum. We rewrite the determinants in the same way, but note that signs and compositions change because these matrices have dimension $(k-1)\times(k-1)$. Suppose that $\lambda\downarrow_i$ has entries $(a_{i,1},\dots,a_{i,k-1})$, where $a_{i,l}=a_l$ for $l<i$, $a_{i,i}=a_i+a_{i+1}$, and $a_{i,l}=a_{l+1}$ for $l>i$. Then

$$\det H_{\lambda\downarrow_i-e_i}=\sum_{\tau\in S_{k-1}}\mathrm{sign}(\tau)\prod_{l=1}^{k-1}(H_{\lambda\downarrow_i-e_i})_{\tau(l)l}=\sum_{(b'_1,\dots,b'_r)\vDash k-1}(-1)^{k-1-r}\prod_{s=1}^{r}h_{\sum_{l=B'_{s-1}+1}^{B'_s}a_{i,l}-\delta_{li}}$$

To match the compositions of $k-1$ appearing here to compositions of k, note that a term of the form $\prod_{s=1}^r h_{\sum_{l=B_{s-1}+1}^{B_s}a_l-\delta_{li}}$ only appears (with the opposite sign) if the composition of $k-1$ has all identical components in the same order except for one, in particular the one where $B_{s-1}\le i<B_s$. This implies that terms corresponding to composition $(b_1,\dots,b_r)$ where the composition is tagged at b_s above appear in this sum with the opposite sign $b'_s=b_s-1$ times when iterating over all i, which exactly corrects for our overcount.

As such, we get a term corresponding to every composition b of k and a selected component of b exactly once on each side, multiplied by the same sign, so these two sums are in fact equal. This completes the proof of the equality. □

Theorem 5.9. *For a ribbon $\lambda=(a_1,\dots,a_k)$, the lowering of s_λ from an S_n-representation to an S_{n-1} representation is computed as*

$$s_\lambda\downarrow_{S_{n-1}}^{S_n}=\sum_{i\mid a_i>1}s_{\lambda-e_i}+\sum_{i\mid a_i,a_{i+1}>1}s_{\lambda\downarrow_i-e_i}+\sum_{\textit{continuous columns of 1s }C\subseteq\lambda,i\in C}s_{\lambda\downarrow_i}$$

Proof. We draw upon the previous lemma. If $a_i>1$, we can interpret the resulting determinant as $s_{\lambda-e_i}$ directly and include it into our sum. Otherwise, $\det H_{\lambda-e_i}$ has its (i,i) component equal to h_0. Using a row operation to subtract the $i+1$st row from the ith row, row i is nearly all 0 except for the (i,i) coordinate which is $h_0=1$. Expanding by minors along this row, we get a positive contribution of $s_{\lambda\downarrow_i}$ for every $a_i=1$.

Looking at the second terms, note that $\lambda\downarrow_i$ is equivalent to omitting a row with one box in it if either $a_i=1$ or $a_{i+1}=1$. Terms not of this form we may include directly in our final result. Otherwise, if we have a continuous column of rows of length one that appears in λ that has length l, then the second sum supplies $l+1$ negated copies of a term that omits a row of length one from this continuous column. These negated copies cancel the l copies of this term that come from the terms in the first sum. This leaves one such term for each continuous column of ones, as desired. □

Corollary 5.10. *For a ribbon $\lambda=(a_1,\dots,a_k)$ with $\lambda^\top=(b_1,\dots,b_l)$ also a ribbon, the lowering of s_λ is*

$$s_\lambda\downarrow_{S_{n-1}}^{S_n}=\sum_{i\mid a_i>1}s_{\lambda-e_i}+\sum_{j\mid b_j>1}s_{(\lambda^\top-e_j)^\top}$$

Proof. Considering the last two sums, we can see that for every column with more than two boxes, we may consider the rows that are contained in this column. If this column contains exactly two boxes and the rows containing these boxes both have more than two boxes, then the result of the squashing operation done to this column is exactly the term obtained from the second contribution from the sum. Otherwise, such a column contains a column of ones, (i.e. the column contains at least one row that has exactly one box), and the removal of exactly one box from this column corresponds to this squashing operation. ☐

This gives us a branching rule for ribbons in terms of other ribbons. Observe that this last rule is very symmetric in that the first sum consists only of ribbons that preserve the height of the original ribbon, and the second sum consists only of ribbons that decrease the height of the ribbon by one.

This rule also extends to graded pieces of the colorful ring:

Theorem 5.11. *Let* $A(n) := \mathcal{C}(\mathcal{B}_n)$*. Then* $A(n)_k \downarrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} \cong (n-k)A(n-1)_{k-1} \oplus (k+1)A(n-1)_k$ *on the level of ribbons.*

Proof. We prove this combinatorially. First, we show that among the ribbons with $n-1$ boxes contained within the result of the lowering, every ribbon with height $k+1$ or k appears at least once.

First, fix an arbitrary ribbon μ of height $k+1$ with $n-1$ boxes, so $\mu = (b_1, \dots, b_{k+1})$. Then this ribbon appears in the lowering of the ribbon $(b_1+1, \dots, b_{k+1})$ that contains n boxes, as a result of the first summation. Similarly, if we fix an arbitrary ribbon $\mu' = (b'_1, \dots, b'_k)$ of height k with $n-1$ boxes, then taking the ribbon $(1, b'_1, \dots, b'_k)$ produces μ' from the second sum as a result of lowering a column with at least two boxes.

To finish, we show that each ribbon with $n-1$ boxes and height $k+1$ or k appears with multiplicity $k+1$ and $n-k$ in the collection of lowered ribbons, respectively. We handle these two cases separately:

- **Case 1.** Consider a ribbon $\mu = (b_1, \dots, b_{k+1})$ with $n-1$ boxes. From the first sum, we see that only the ribbons that produce μ are $\mu + e_i$ for $1 \le i \le k+1$, ribbons with n boxes and height $k+1$, of which there are $k+1$ total. Therefore, μ appears with multiplicity $k+1$ in the lowering of the representationcorresponding to $A(n)_k$.
- **Case 2.** Consider a ribbon $\mu' = (b'_1, \dots, b'_k)$ with $n-1$ boxes. Then, the ribbon $\mu'^{\top}$ is a ribbon with height $n-k$, as all horizontal strings of boxes will be exchanged for vertical boxes and vice versa, and since there are $n-1$ box adjacencies, $k-1$ of which were vertical in μ, this gives $n-k$ vertical box adjacencies in $\mu'^{\top}$. Then $(\mu'^{\top} + e_j)$ produces $\mu'^{\top}$ in its lowering for $1 \le j \le n-k$, and symmetrically $(\mu'^{\top} + e_j)^{\top}$ produces μ' after lowering for $1 \le j \le n-k$. Therefore, μ' appears with multiplicity $n-k$ in the lowering of the representation corresponding to $A(n)_k$.

This gives the exact multiplicities of every ribbon that can show up in the lowering of $A(n)_k$, since we only get ribbons of height $k+1$ or k, and we have established their multiplicities above. This gives the desired identity. ☐

Next, we present a short exact sequence of the graded components of the colorful ring as vector spaces. We have attempted a homological argument, but currently have only found this on the level of vector spaces, not as $\mathfrak{S}_{n-1}$-representations:

Theorem 5.12. *Let $A(n) := \mathcal{C}(\mathcal{B}_n)$. There is a short exact sequence of vector spaces*

$$0 \to (k+1)\mathcal{S}^{(n-1)} \otimes A(n-1)_k \xrightarrow{i} A(n)_k \downarrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} \xrightarrow{q} (n-k)\mathcal{S}^{(n-1)} \otimes A(n-1)_{k-1} \to 0$$

Proof. Recall that the basis elements for $A(n)_k$ are in bijection with the permutations of $\mathfrak{S}_n$ that have k descents. These permutations written in one line are also in bijection with the reading words of the standard Young tableaux of the ribbons with n boxes with height $k+1$, which are obtained by reading the entries in the tableaux from the bottom row to the top row, left to right. As such, we may index the basis elements of the graded pieces by standard Young tableaux of ribbons.

Again, suppose $(k+1)\mathcal{S}^{(n-1)}$ has basis $\left\{y_j\right\}_{j=1}^{k+1}$ (as it is a direct sum of $k+1$ 1-dimensional irreducible representations) and $(n-k)\mathcal{S}^{(n-1)}$ has basis $\left\{z_j\right\}_{j=1}^{n-k}$ for the same reasons. We define the maps i and q.

First, define the jth descent-preserving insertion of a permutation $\pi \in \mathfrak{S}_{n-1}$ with k descents to be the permutation $\pi \uparrow^j \in \mathfrak{S}_n$ with k descents where n was inserted into π in the jth position from the left. This allows us to define i:

$$i(y_j \otimes x_\pi) = x_{\pi\uparrow^j}$$

For the other map, we again define it piece-wise. For $\pi \in \mathfrak{S}_n$, define $\pi \downarrow$ to be the permutation in $\mathfrak{S}_{n-1}$ where we drop n from the one-line string representing π. Then

$$q(x_\pi) = \begin{cases} 0 & \pi \downarrow \text{ has } k \text{ descents} \\ z_j \otimes x_{\pi\downarrow} & \pi \downarrow \text{ has } k-1 \text{ descents and } (\pi \downarrow) \uparrow^j = \pi \text{ for } 1 \le j \le n-k \end{cases}$$

By construction, we have exactness at $A(n) \downarrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}$, so we need to show the injectivity of i and the surjectivity of q. The injectivity of i is easy to see, as insertion of n into a specific position for some $\pi \in \mathfrak{S}_{n-1}$ is unique. For surjectivity, choose some $\sigma \in \mathfrak{S}_{n-1}$ such that σ has $k-1$ descents, and some $1 \le j \le n-k$. Then insert n into σ in the jth position that would create a new descent in the result, to get some new permutation $\sigma' \in \mathfrak{S}_n$. Then $q(x_{\sigma'}) = z_j \otimes x_\sigma$ as desired.

□

6. Algebraic descriptions for Koszul duals associated to the Boolean lattice

6.1. **Koszul duals for Chow rings of matroids.** In this section, we compute the quotient ideal J of $\mathrm{Chow}(\mathcal{B}_n)^!$ explicitly, in turn proving the following theorem:

Theorem 6.1. *We have* $\mathrm{Chow}(\mathcal{B}_n) = A(n)^! = \Bbbk\langle z_F : |F| \ge 2\rangle / J$*, where* $z_i = x_i^*$ *and* J *is the two-sided ideal*

$$\left\langle [z_G, z_G] - \sum_{\{i,j\}\subset G} [z_{ij}, z_{ij}] : |G| > 2 \right\rangle$$

$$+\left\langle [z_G, z_H] - \sum_{i\in H\setminus G} [z_G, z_{G\cup i}] + 2(k-1) \sum_{\{i,j\}\subset G} [z_{ij}, z_{ij}] : G \subset H, |G| - |H| = k > 1, |H| \ge 2 \right\rangle.$$

For easy of notation, we will denote the set $\{i,j\} \subset [n]$ where i,j are distinct, as ij. When we say $F \cup ij$, we mean $F \cup \{i,j\}$ where $i,j \notin F$.

Recall the atom-free presentation of the Chow ring

$$A(n) = \frac{\mathbb{k}[x_F : |F| \geq 2]}{I},$$

where

$$I = \langle x_F x_G : |F|,|G| \geq 2, F, G \text{ incomparable } \rangle + \left\langle x_F \sum_{F \vee i \subseteq G} x_G : |F| \geq 2, i \in E \setminus F \right\rangle$$

$$+ \left\langle \sum_{i \vee j \subseteq F} x_F^2 + \sum_{i \vee j \subseteq F \subsetneq F'} 2x_F x_F' : i,j \in E, i \neq j \right\rangle.$$

We begin by massaging one of the equations in the atom-free presentation of $A(n)$ to a form more amenable for our purposes.

Lemma 6.2. *In the atom-free presentation for $A(n)$, one may rewrite*

$$\left\langle \sum_{ij \subset F} x_F^2 + 2 \sum_{ij \subset F \subsetneq G} x_F x_G : i \neq j \right\rangle$$

instead as

$$\left\langle \sum_{ij \subset F} x_F^2 - 2 \sum_{k \geq 2} \Big(k-1) \sum_{\substack{ij \subset F \subset G \\ |G| = |F| + k}} x_F z_G \Big) : i \neq j \right\rangle$$

where when k is large enough, there is no F,G s.t. $|G| - |F| = k$.

Proof. For an ideal I, if

$$\langle g_1, \cdots, g_n \rangle + \langle f_1, \cdots, f_m \rangle = I,$$

then we also have

$$\langle g_1, \cdots, g_n \rangle + \langle f_1 - \sum_{i=1}^{n} c_{1i} g_i, \cdots, f_n - \sum_{i=1}^{n} c_{ni} g_i \rangle = I.$$

In this case, we are going to subtract terms in

$$A := \left\langle x_F \sum_{i \cup F \subset G} x_G : |F| \geq 2, i \notin F \right\rangle \quad \text{from} \quad B := \left\langle \sum_{ij \subset F} x_F^2 + 2 \sum_{ij \subset F \subsetneq G} x_F x_G : i \neq j \right\rangle$$

to get the desired form.

We fix $i \neq j$. For each F, G such that $\{i,j\} \subset F$ and $G = F \cup k$ for some $k \notin F$, we consider the term $x_F \sum_{G \subset H} x_H$ in A, and subtract it from

$$\sum_{ij \subset F} x_F^2 + 2 \sum_{ij \subset F \subsetneq G} x_F x_G \in B.$$

We have

$$\sum_{ij\subset F\subsetneq G} x_F x_G - \sum_{\substack{ij\subset F\\ G=F\cup k, k\notin F}} x_F \sum_{G\subset H} x_H = -\sum_{h=2}(h-1) \sum_{\substack{ij\subset F\\ |G|=|F|+h}} x_F x_G.$$

To see this, note that if $\{i,j\}\subset F$, and $G = F\cup k$ for some $k\notin F$, then the term $x_F x_G$ appears exactly once in

$$\sum_{\substack{ij\subset F\\ G=F\cup k}} x_F \sum_{G\subset H} x_H,$$

so it must cancel.

In the case that $\{i,j\}\subset F\subset G$ where $|G|-|F| = h > 1$, there is exactly h number of sets H such that $H = F\cup k$ for some $k\in G\setminus F$. Hence, the coefficient of $x_F x_G$ is $-(h-1)$.

Therefore, we can rewrite each generator of B as

$$\sum_{ij\subset F} x_E^2 + 2\sum_{ij\subset F\subsetneq G} x_F x_G \quad\longrightarrow\quad \sum_{ij\subset F} x_F^2 - 2\sum_{k=2}(k-1) \sum_{\substack{ij\subset F\\ |G|=|F|+k}} x_F x_G,$$

which gives us the desired form. □

With this lemma in hand, we may now prove our theorem.

Proof of Theorem 6.1. Next, notice that for F,G incomparable, by the algorithm, the term $[x_F, x_G]$ does not appear in J, so we only need to consider terms $[x_F, x_G]$ s.t. $F\subset G$ (or $F\supset G$).

For any linear function $y_1 + \sum_{i=2}^n c_i y_i = 0$, we can choose $y_i, i > 1$ freely, and then $y_1 = -\sum_{i=2}^n c_i y_i$, i.e. the solution is generated by

$$\begin{bmatrix} -c_2\\ 1\\ \vdots\\ 0\end{bmatrix}, \begin{bmatrix} -c_3\\ 0\\ 1\\ \vdots\end{bmatrix}, \dots, \begin{bmatrix} -c_n\\ \vdots\\ 0\\ 1\end{bmatrix}.$$

We define the set

$$U := \{[x_F, x_G] : F = G, |F| = 2\} \cup \{[x_F, x_G] : G = F\cup k, \text{ for } k\notin F\}.$$

Then each linear function given by I contains exactly one $[x_F, x_G]\in U$. The function has only one term if and only if G is the whole ground set and $F = G\setminus k$, so these terms also don't appear in J.

Otherwise, for each $[x_F, x_G]\notin U$, so pairs F,G such that either they are equivalent, $|F| > 2$, $F\subset G$, or $|G|-|F| > 1$, we can assign value to it freely, and the values of $[x_F, x_G]\in U$ are determined uniquely.

Therefore, if $F = G, |F| > 2$, from I we have the term

$$[x_F, x_F] - \sum_{ij\subset F} [z_{ij}, z_{ij}];$$

and if $F \subset G, |G| - |F| > 1$, we have the term

$$[x_F, x_G] - \sum_{i \in G \setminus F} [x_F, z_{F \cup i}] + 2(k-1) \sum_{ij \subset F} [z_{ij}, z_{ij}],$$

which gives us the desired result. □

6.2. **A basis for** $\mathrm{Chow}(\mathcal{B}_n)^!$**.** In this section, we find a basis for each graded component of $A^!(n)$ by computing a Gröbner basis for J when $n = 3$ and $n = 4$. We also include a conjecture for a Gröbner basis for J for general n.

Since we are considering the Koszul dual of the Boolean lattice, we will work in the free algebra $\Bbbk\langle x_F : |F| \geq 2\rangle$. It will also be useful to refer to the standard basis for this ring, denoted by $\mathcal{B}$:

$$\mathcal{B} = \left\{1, z_{F_1} z_{F_2} \cdots z_{F_n} : |F_i| \geq 2\right\}$$

Remark 6.3. In this section, whenever we say ijk, we assume $i < j < k$.

Remark 6.4. We specify an order on the generators of $\Bbbk\langle x_F : |F| \geq 2\rangle$. Sort the generators x_F based on the rank of the flat F, where $x_G > x_F$ when $|G| > |F|$. Among the generators of same flat size, order the generators lexicographically.

We impose the graded lexicographic order on monomials: meaning if $x^a = z_{a_1} \cdots z_{a_n}$ and $x^b = z_{b_1} \cdots z_{b_m}$ then $x^a < x^b$ if and only if there is some $1 \leq p \leq \min\{n, m\}$ such that $z_{a_k} = z_{b_k}$ for $k < p$ and $z_{a_p} < z_{b_p}$.

6.2.1. *A basis for* $\mathrm{Chow}(\mathcal{B}_3)^!$. In this section, we give describe a basis for $\mathrm{Chow}(\mathcal{B}_3)^!$ and prove this result using two methods. The first is an exclusion argument that considers which monomials cannot appear in a basis for any graded component of $A(\mathcal{B}_3)^!$ and uses the recurrence relation for $Hilb(A^!, t)$ outlined in 2.3. The second finds a Gröbner basis for J when $n = 3$. By the same dimension argument as in the first proof, this gives gives the same basis for $A(\mathcal{B}_3)^!$.

Theorem 6.5 (Basis for $A(3)^!_d$)**.** *Define the set M_d to be the set of all monomials of degree d that do not contain z_{123}^2 nor $z_{123} z_{23}^2$. Explicitly, define M_d as follows:*

$$M_d = \{z_{a1} z_{a2} \cdots z_{an} \mid z_{ai} z_{a(i+1)} \neq z_{123}^2, z_{a(j-1)} z_{aj} z_{a(j+1)} \neq z_{123} z_{23}^2, \ \forall i, j\},$$

Then M_d is a degree d basis for $A^!(\mathcal{B}_3)$.

Proof. We first show M_d is a spanning set.

For any z_{123}^2, we can write it as $z_{12}^2 + z_{13}^2 + z_{23}^2$, then eventually we can write any monomial containing z_{123}^2 as a finite linear combination of monomials without z_{123}^2.

For monomials containing $z_{123} z_{23}^2$, we prove it by induction on the position of where $z_{123} z_{23}^2$ first appear. If we have $\cdots z_{123} z_{23}^2$, we can write it as

$$\cdots (z_{12}^2 + z_{13}^2 + z_{23}^2) z_{123} - \cdots z_{123} (z_{12}^2 + z_{13}^2),$$

where each term contains neither z_{123}^2 nor $z_{123}z_{23}^2$.

Then inductively, if we have $\cdots z_{123}z_{23}^2\cdots$, we write it as

$$\cdots(z_{12}^2+z_{13}^2+z_{23}^2)z_{123}\cdots - \cdots z_{123}(z_{12}^2+z_{13}^2)\cdots.$$

If $\cdots(z_{12}^2+z_{13}^2+z_{23}^2)z_{123}\cdots$ is actually $\cdots(z_{12}^2+z_{13}^2+z_{23}^2)z_{123}^2\cdots$, we replace

$$z_{123}^2 \longrightarrow z_{12}^2+z_{13}^2+z_{23}^2.$$

Then each term in the linear combination does not contain z_{123}^2 and either does not contain $z_{123}z_{23}^2$ or $z_{123}z_{23}^2$ appears later, where by induction hypothesis is a linear combination of our basis.

Then we show $k_n := \#M_d$ is the same as the dimension of the degree n graded component, and this means A_n is a basis for the degree n component.

We first show k_n satisfies the recurrence relation $k_{n+1} = 4k_n - (4k_{n-2} - k_{n-3})$.

Consider M_d, and we multiply it by $Z := \{z_{12}, z_{13}, z_{23}, z_{123}\}$ on the left to get ZM_d with $|ZM_d| = 4k_d$, and $k_{d+1} = 4k_d - \#\{y \in M_d \mid y = z_{23}^2\cdots \text{ or } y = z_{123}\cdots\}$.

If $y = z_{23}^2\cdots$, because $\cdots$ can be anything in M_{d-2}, we have $\#\{y \in M_d \mid y = z_{23}^2\} = k_{d-2}$, similarly if $y = z_{123}z_{12}\cdots$ or $y = z_{123}z_{13}\cdots$. If $y = z_{123}z_{23}\cdots$, $\cdots$ can be anything in M_{d-2} not begin with z_{23}, which is $h_{d-2} - h_{d-3}$, so we have

$$\#\{y \in M_d \mid y = z_{23}^2\cdots \text{ or } y = z_{123}\cdots\} = 4k_{d-2} - k_{d-3}.$$

Then we can show that k_d also satisfies the relation $k_{d+1} = 4k_d - k_{d-1}$ given by $d_0 = 1$, $d_1 = 4$. We verify this is true for the first several d_n's, then for m, by induction hypothesis we have

$$k_{m+1} = 4k_m - (4k_{m-2} - k_{m-3}) = 4k_m - k_{m-1}.$$

Because the Hilbert series of $A^!(\mathbb{B}^3)$ is $\frac{1}{1-4t+t^2}$, and the coefficients, i.e. the dimension of each graded component satisfies this relation with $c_0 = 1, c_1 = 4$ by proposition 2.18, this means $k_d = c_d$.

Therefore, M_d is a basis for the degree d component of $\mathrm{Chow}(\mathcal{B}^3)$. $\square$

We now give a proof by giving a Gröbner basis for J.

Theorem 6.6. *The following is a Gröbner basis for J when $n = 3$:*

$$\mathcal{G} = \{z_{123}^2 - z_{12}^2 - z_{13}^2 - z_{23}^2,\ z_{123}z_{12}^2 + z_{123}z_{13}^2 + z_{123}z_{23}^2 - z_{12}^2z_{123} - z_{13}^2z_{123} - z_{23}^2z_{123}\}$$

Proof. Let g_1 and g_2 be the elements of $\mathcal{G}$, namely

$$\begin{aligned} g_1 &= z_{123}^2 - z_{12}^2 - z_{13}^2 - z_{23}^2 \\ g_2 &= z_{123}z_{12}^2 + z_{123}z_{13}^2 + z_{123}z_{23}^2 - z_{12}^2z_{123} - z_{13}^2z_{123} - z_{23}^2z_{123} \end{aligned}$$

Referring to remark 6.4, we choose the monomial order $z_{123} > z_{23} > z_{13} > z_{12}$.

We will consider all three S-elements and show that each has a weak Gröbner representation, which by proposition 2.28 implies $\mathcal{G}$ is a Gröbner basis for J.

For g_1, the leading term, z_{123}^2, can only be cancelled by multiplying on either side by z_{123}, giving the following S-element:

$$\begin{aligned} S(g_1, g_1; 1, z_{123}; z_{123}, 1) &= g_1 z_{123} - z_{123} g_1 \\ &= z_{123} z_{12}^2 + z_{123} z_{13}^2 + z_{123} z_{23}^2 - z_{12}^2 z_{123} - z_{13}^2 z_{123} - z_{23}^2 z_{123} \end{aligned}$$

The result is defined to be g_2, with leading term $\mathrm{in}(g_2) = z_{123} z_{23}^2$.

The only S-element between g_2 and itself is trivial, so by proposition 2.29, it has a weak Gröbner representation.

Finally, consider $S(g_1, g_2)$:

$$\begin{aligned} S(g_1, g_2; 1, z_{23}^2; z_{123}, 1) &= g_1 z_{23}^2 - z_{123} g_2 \\ &= (-z_{12}^2 - z_{13}^2 - z_{23}^2) z_{23}^2 \\ &\quad - z_{123}(z_{123} z_{12}^2 + z_{123} z_{13}^2 - z_{12}^2 z_{123} - z_{13}^2 z_{123} - z_{23}^2 z_{123}) \\ &= -z_{123}^2(z_{12}^2 + z_{13}^2) + z_{123}(z_{12}^2 + z_{13}^2 + z_{23}^2) z_{123} - (z_{12}^2 + z_{13}^2 + z_{23}^2) z_{23}^2; \end{aligned}$$

$$\begin{aligned} & g_1(z_{12}^2 + z_{13}^2) + g_2 z_{123} = \\ &\quad \left(z_{123}^2 - z_{12}^2 - z_{13}^2 - z_{23}^2\right)(z_{12}^2 + z_{13}^2) - z_{123}(z_{12}^2 + z_{13}^2 + z_{23}^2) z_{123} \\ &\quad + \left(z_{12}^2 + z_{13}^2 + z_{23}^2\right) z_{123}^2 \\ &\quad = (z_{12}^2 + z_{13}^2 + z_{23}^2)(z_{123}^2 - z_{12}^2 - z_{13}^2 - z_{23}^2) - S(g_1, g_2) \\ &\quad = (z_{12}^2 + z_{13}^2 + z_{23}^2) g_1 - S(g_1, g_2), \end{aligned}$$

This gives a weak Gröbner representation for $S(g_1, g_2)$.

Having checked all of the S-elements, the condition given in 2.28 is satisfied, and so $\mathcal{G}$ is indeed a Gröbner basis for J. □

This argument verifies that z_{123}^2 and $z_{123} z_{23}^2$ should not appear in a basis for the graded components of $A(3)^!$ because they are leading terms of g_1 and g_2, and can be replaced by $g_1 - \mathrm{in}(g_1)$ and $g_2 - \mathrm{in}(g_2)$, respectively, as outlined in the previous proof. For larger n, it is difficult to determine which terms to exclude by inspection, but finding a Gröbner basis for J gives us this information: we can exclude the leading terms of the basis elements.

6.2.2. *A basis for $A(\mathcal{B}_4)^!$.* We now find a basis for the graded components of $A(4)^!$. Our line of reasoning will be similar to that of 6.5: we will choose a collection of monomials that are a subset of $\mathrm{in}(J)$ and exclude these terms from all degree d monomials. We then show that this choice gives the correct dimension for each graded component.

First, we adopt the following notation for generators of the ideal J:

$$g_{F^2} := z_F^2 - \sum_{ij \subset F} z_{ij}^2 \qquad g_{G,F} := [z_G z_F + z_F z_G] - \sum_{i \in F \setminus E} [z_{E \cup i} z_E + z_E z_{E \cup i}] + 2(k-1) z_F^2,$$

Additionally, when $|F| = 3$ we will denote g_F^2 by g_{ijk}.

Theorem 6.7. *Let M_d be the set of all degree d monomials not in $\mathrm{in}(G)$, where $G = \cup_{i=1}^4 G_i$ and each G_i is given by:*

$$\begin{aligned}
G_1 &= \{g_{F^2} : |F| \geq 3\}\\
G_2 &= \{z_{ijk} g_{ijk^2} - g_{ijk^2} z_{ijk}\}\\
G_3 &= \{g_{[4],F} : |F| = 2\}\\
G_4 &= \{g_{F\cup ij^2} z_F - z_{F\cup i<j} g_{F\cup ij,F}\}
\end{aligned}$$

Explicitly, $\mathrm{in}(G)$ *is generated by the following set of monomials:*

$$\begin{aligned}
&\{z_F^2 : |F| > 2\}\\
\cup&\{z_{ijk} z_{jk}^2 : i < j < k\}\\
\cup&\{z_{[4]} z_F : |F| = 2\}\\
\cup&\{z_{F\cup ij} z_{F\cup j} z_F : i < j, |F| = 2\}
\end{aligned}$$

Then M_d is a basis for the degree d piece of $A(4)^!$.

Proof. First, we show that M_d spans the degree d piece of $A(4)^!$, denoted $(A(4)^!)_d$, by showing that $\mathrm{in}(G) \subset \mathrm{in}(J)$. Every element in G that is not a generator of J can be found by taking S-elements of the generators. In particular, the elements in G_2 are S-elements of each g_{F^2} with itself, where $|F| = 3$:

$$S(g_{F^2}, g_{F^2}; z_{ijk}, 1; 1, z_{ijk}) = z_{ijk} g_{F^2} - g_{F^2} z_{ijk}$$

by the definition of an S-element. Note that the resulting initial term is $z_{ijk} z_{jk}^2$.

Similarly, we take $F \cup ij = [4]$ with $i < j$ to see that the elements in G_4 are S-elements of $g_{[4],F}$ and $g_{[4]^2}$:

$$S(g_{[4]^2}, g_{[4],F}; 1, 1, z_F; z_{[4]}, 1) = g_{[4]^2} z_F - z_{[4]} g_{F\cup ij,F}$$

where the resulting initial term is $z_{[4]} z_{F\cup j} z_F$.

Since each $g \in G$ is either a generator of J or found by taking the difference of elements in J, every initial term of an element in G is also an initial term of an element in J, and so $\mathrm{in}(G) \subset \mathrm{in}(J)$.

To show that this spanning set is a basis for $(A(4)^!)_d$, we show that M_d satisfies the recurrence for $\mathrm{Hilb}(A(4)^!, t)$.

The leading terms of elements in G are:

$$\begin{aligned}
\mathrm{in}(4) = &\{z_{123}^2, z_{124}^2, z_{134}^2, z_{234}^2, z_{1234}^2\} = \{z_{ijk}^2, z_{1234}^2\}\\
&\cup \{z_{1234}z_{12}, z_{1234}z_{13}, z_{1234}z_{14}, z_{1234}z_{23}, z_{1234}z_{24}, z_{1234}z_{34}\}\\
&\cup \{z_{123}z_{23}^2, z_{124}z_{24}^2, z_{134}z_{34}^2, z_{234}z_{34}^2\} = \{z_{i<j<k} z_{jk}^2\}\\
&\cup \{z_{1234}z_{124}z_{12}, z_{1234}z_{134}z_{13}, z_{1234}z_{134}z_{14}, z_{1234}z_{234}z_{23}, z_{1234}z_{234}z_{24}, z_{1234}z_{234}z_{34}\}
\end{aligned}$$

Let A_n be the set of degree n monomials not in the ideal $\langle \mathrm{in}(4)\rangle$, and let $k_n = \# A_n$. We show that $k_{n+1} = 10k_n - 10k_{n-2} + k_{n-3}$, which by induction means $k_{n+1} = 11k_n - 11k_{n-1} + k_{n-2}$.

Let $A = \{z_{ij}, z_{ijk}, z_{1234}\}$, then we consider AA_n and remove elements in $\langle \mathrm{in}(4)\rangle$.

For $z_{ij}A_n$, all terms are not in $\langle \mathrm{in}(4)\rangle$, so this gives us $6k_n$ elements in A_{n+1}.

For $z_{i<j<k}A_n$, we remove all terms starting with z_{ijk} or z_{jk}^2 in A_n. Let $k_{ijk,n}$ be the elements starting with z_{ijk} in A_n. Elements starting with z_{jk}^2 in A_n are exactly $z_{jk}^2 A_{n-2}$. And there are 4 of $z_{i<j<k}$'s, which gives us $4(k_n - k_{ijk,n} - k_{n-2})$ elements in A_{n+1}.

For $z_{1234}A_n$, we only need to consider $z_{1234}z_{ijk}\cdots$, and there are $4k_{ijk,n}$ of them. Also, we need to remove $z_{1234}z_{124}z_{12}, \cdots, z_{1234}, z_{234}z_{34}$, which are exactly (almost) $z_{1234}z_{ijk}z_{ij}A_{n-2}$, so we subtract $6k_{n-2}$. Furthermore, in the case $z_{1234}z_{234}z_{34}$, we remove $z_{1234}z_{234}z_{34}^2\cdots$ which is not in $z_{1234}z_{234}z_{34}A_{n-2}$, so we add them back, which are $z_{1234}z_{234}z_{34}^2 A_{n-3}$, so has k_{n-3}. Therefore, this step gives us $4k_{ijk,n} - 6k_{n-2} + k_{n-3}$.

Adding them up, we have

$$k_{n+1} = 10k_n - 10k_{n-2} + k_{n-3}.$$

Suppose $k_{m+1} = 11k_m - 11k_{m-1} + k_{m-2}$ for $m < n$, then

$$\begin{aligned} k_{n+1} &= 10k_n - 10k_{n-2} + k_{n-3} \\ &= 11k_n - 10k_{n-2} + k_{n-3} - (11k_{n-1} - 11k_{n-2} + k_{n-3}) \\ &= 11k_n - 11k_{n-1} + k_{n-2}. \end{aligned}$$

Therefore, M_d is a basis for the degree d piece of $A(4)^!$. □

6.2.3. *A basis for $A(\mathcal{B}_n)^!$.* Based on our observations, we have the following conjecture for a Gröbner basis for J for arbitrary n, which leads to a conjecture for the set M_d.

Conjecture 6.8. *We use the notation for generators of J introduced previously:*

$$g_{F^2} := z_F^2 - \sum_{ij\subset F} z_{ij}^2 \qquad g_{G,F} := [z_G z_F + z_F z_G] - \sum_{i\in F\setminus E} [z_{E\cup i} z_E + z_E z_{E\cup i}] + 2(k-1) z_F^2.$$

Let $\mathcal{G} = \cup_{i=1}^4 G_i$, where each G_i is given below. Then $\mathcal{G}$ is a Gröbner basis for J of $A(\mathcal{B}_n)^!$:

$$\begin{aligned} G_1 &= \{g_{F^2} : |F| \geq 3\} \\ G_2 &= \{z_{ijk} g_{ijk^2} - g_{ijk^2} z_{ijk}\} \\ G_3 &= \{g_{G,F}\} \\ G_4 &= \{g_{F\cup ij^2} x_F - z_{F\cup i<j} g_{F\cup ij,F}\} \end{aligned}$$

Let M_d be the set of all degree d monomials not in $\mathrm{in}(\mathcal{G})$*, which is generated by the following monomials:*

$$
\begin{aligned}
&\{x_F^2 : |F| \geq 3\}\\
\cup&\{z_{i<j<k} z_{jk}^2\}\\
\cup&\{x_G x_F : |G| - |F| \geq 2\}\\
\cup&\{z_{F\cup i<j} z_{F\cup j} x_F \mid |F| \leq n-2\}
\end{aligned}
$$

Then M_d is a basis for the degree d component of $A(n)^!$.

One approach to proving these statements would be to confirm that the set $\mathcal{G}$ is indeed a Gröbner basis by considering all S-elements and showing that the condition given in proposition 2.28 is satisfied. This would be an extension of the argument presented in theorem 6.5. Another approach would be to show that M_d spans the degree d component of $A(n)^!$, and then use the recurrence relation on the Hilbert series to show that this gives the correct dimension for each d, as in the proof of theorem 6.7.

7. Representations for Koszul duals of algebras associated to the Boolean lattice

7.1. Representations for the Koszul dual of Chow($\mathcal{B}_n$).

Data tables for $\mathrm{Chow}(B_n)^!$:

$n = 3$:

degree	irreps
0	$\ydiagram{3}$
1	$\ydiagram{2,1} + 2\ydiagram{3}$
2	$\ydiagram{1,1,1} + 5\ydiagram{2,1} + 4\ydiagram{3}$
3	$7\ydiagram{1,1,1} + 19\ydiagram{2,1} + 11\ydiagram{3}$
4	$32\ydiagram{1,1,1} + 70\ydiagram{2,1} + 37\ydiagram{3}$

$n = 4$:

degree	irreps
0	$\ydiagram{4}$
1	$\ydiagram{2,2} + 2\,\ydiagram{3,1} + 3\,\ydiagram{4}$
2	$\ydiagram{1,1,1,1} + 58\,\ydiagram{2,1,1} + 10\,\ydiagram{2,2} + 18\,\ydiagram{3,1} + 11\,\ydiagram{4}$
3	$28\,\ydiagram{1,1,1} + 116\,\ydiagram{2,1,1} + 93\,\ydiagram{2,2} + 154\,\ydiagram{3,1} + 66\,\ydiagram{4}$
4	$380\,\ydiagram{1,1} + 1276\,\ydiagram{2,1,1} + 903\,\ydiagram{2,2} + 1418\,\ydiagram{3,1} + 523\,\ydiagram{4}$

One can observe from the representations (in the $n = 3$ case, for instance) that the graded pieces $\mathrm{Chow}(\mathcal{B}_n)^!$ do not admit a nice $\mathfrak{S}_n$ action as the graded pieces of $\mathrm{Chow}(\mathcal{B}_n)$ do. Note that for this to be the case we would like to have the representation corresponding to each graded piece to be expressible as a product of homogeneous polynomials. Decomposing the possible skew representations into irreducibles for $n = 3$:

$$\ydiagram{2+1,1+1,1} = \ydiagram{1,1,1} + 2\,\ydiagram{2,1} + \ydiagram{3} \qquad \ydiagram{1+2,1} = \ydiagram{2,1} + \ydiagram{3}$$

However, $\mathrm{Chow}(\mathcal{B}_n)^!_3$ is not expressible as a linear combination of $\ydiagram{2+1,1+1,1}$, $\ydiagram{1+2,1}$, and $\ydiagram{3}$. As such, we see that the Koszul dual of $\mathrm{Chow}(\mathcal{B}_n)$ does not have the same nice symmetry that $\mathrm{Chow}(\mathcal{B}_n)$ does.

7.2. **Representations for the Koszul dual of colorful**$(\mathcal{B}_n)$**.** Data tables for $\mathcal{C}(\mathcal{B}_n)^!$:

$n = 3$:

degree	representations
0	$\ydiagram{3}$
1	$\ydiagram{2,1} + \ydiagram{1+1,2}$
2	$3\,\ydiagram{1,1,1} + 2\left(\ydiagram{2,1} + \ydiagram{1+1,2}\right) + 4\,\ydiagram{3}$
3	$8\,\ydiagram{1,1,1} + 10\left(\ydiagram{2,1} + \ydiagram{1+1,2}\right) + 8\,\ydiagram{3}$
4	$36\,\ydiagram{1,1,1} + 34\left(\ydiagram{2,1} + \ydiagram{1+1,2}\right) + 37\,\ydiagram{3}$
5	$128\,\ydiagram{1,1,1} + 131\left(\ydiagram{2,1} + \ydiagram{1+1,2}\right) + 128\,\ydiagram{3}$
6	$487\,\ydiagram{1,1,1} + 484\left(\ydiagram{2,1} + \ydiagram{1+1,2}\right) + 488\,\ydiagram{3}$
7	$1808\,\ydiagram{1,1,1} + 1812\left(\ydiagram{2,1} + \ydiagram{1+1,2}\right) + 1808\,\ydiagram{3}$

$n = 4$:

degree	representations
0	$\ydiagram{4}$
1	$\ydiagram{3,1} + \ydiagram{1+2,2} + \ydiagram{2+1,3}$
2	$\ydiagram{1,1,1,1} + 4\left(\ydiagram{2,1,1} + \ydiagram{1+1,2,1} + \ydiagram{1+1,1+1,2}\right) + 5\left(\ydiagram{3,1} + \ydiagram{1+2,2} + \ydiagram{2+1,3}\right) + 10\,\ydiagram{4}$
3	$36\,\ydiagram{1,1,1,1} + 41\left(\ydiagram{2,1,1} + \ydiagram{1+1,2,1} + \ydiagram{1+1,1+1,2}\right) + 50\left(\ydiagram{3,1} + \ydiagram{1+2,2} + \ydiagram{2+1,3}\right) + 53\,\ydiagram{4}$
4	$406\,\ydiagram{1,1,1,1} + 437\left(\ydiagram{2,1,1} + \ydiagram{1+1,2,1} + \ydiagram{1+1,1+1,2}\right) + 462\left(\ydiagram{3,1} + \ydiagram{1+2,2} + \ydiagram{2+1,3}\right) + 496\,\ydiagram{4}$
5	$4301\,\ydiagram{1,1,1,1} + 4398\left(\ydiagram{2,1,1} + \ydiagram{1+1,2,1} + \ydiagram{1+1,1+1,2}\right) + 4504\left(\ydiagram{3,1} + \ydiagram{1+2,2} + \ydiagram{2+1,3}\right) + 4598\,\ydiagram{4}$

$n = 5$:

degree	representations
0	$A(5)_0$
1	$A(5)_1$
2	$20A(5)_0+9A(5)_1+5A(5)_2+A(5)_3$
3	$204A(5)_0+162A(5)_1+116A(5)_2+79A(5)_3+44A(5)_4$
4	$3606A(5)_0+3152A(5)_1+2736A(5)_2+2336A(5)_3+1965A(5)_4$

As a remark, the different graded pieces of $\mathcal{C}(\mathcal{B}_n)^!$ appear to be expressible as a direct sum of the graded pieces of $\mathcal{C}(\mathcal{B}_n)$. This suggests the following conjecture:

Conjecture 7.1. *For λ_1/μ_1 and λ_2/μ_2 ribbon diagrams, the internal product $s_{\lambda_1/\mu_1} * s_{\lambda_2/\mu_2}$ is expressible as a sum of ribbons.*

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