# Virtual Resolutions of Points in $\mathbb{P}^n \times \mathbb{P}^m$

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# Projective space and their products

Definition

Projective space $\mathbb{P}^n$ is defined as the quotient $\mathbb{A}^{n+1}/\sim$, where $x \sim y$ if $y = \lambda x$ for some $\lambda \neq 0$.

We are interested in finite sets of points in $\mathbb{P}^n \times \mathbb{P}^m$.
But! we cannot have points of the form

$$[0 : \ldots : 0] \times [b_0 : \ldots : b_m] \text{ or } [a_0 : \ldots : a_n] \times [0 : \ldots : 0].$$

# Defining ideals

### Definition

Let $X$ be a subset of $\mathbb{P}^n \times \mathbb{P}^m$. The **Cox ring** of $\mathbb{P}^n \times \mathbb{P}^m$ is $S = k[x_0, \ldots, x_n, y_0, \ldots, y_m]$ and is $\mathbb{Z}^2$-graded, where $\deg(x_i) = (1, 0)$ and $\deg(y_j) = (0, 1)$.
Then

$$I(X) = \{f \in S \mid f(x) = 0 \text{ for all } x \in X\}$$

is the bihomogeneous **defining ideal** of $X$.

We also have the **irrelevant ideal**

$$B = \langle x_0, \ldots, x_n \rangle \cap \langle y_0, \ldots, y_m \rangle.$$

# Cox ring and vanishing ideals

Let $S = k[x_0, \dots, x_n, y_0, \dots, y_m]$.

A finite set of points

$$X = \{P_1, P_2, \dots, P_s\}$$

in $\mathbb{P}^n \times \mathbb{P}^m$ has defining ideal

$$I(X) = I(P_1) \cap I(P_2) \cap \dots \cap I(P_s).$$

We call $S/I(X)$ the **Cox ring** of $X$.

## Example

In $\mathbb{P}^2 \times \mathbb{P}^1$, consider the points $P_1 = [1:0:0] \times [1:0]$, and $P_2 = [2:1:0] \times [1:2]$. Then

$$I(P_1) = \langle x_1, x_2, y_1 \rangle,$$

$$I(P_2) = \langle x_1 - 2x_0, x_2, y_1 - 2y_0 \rangle.$$

For $X = P_1 \cup P_2$, then

$$\begin{aligned} I(X) &= I(P_1) \cap I(P_2) \\ &= \langle x_2,\ 2y_0y_1 - y_1^2, 2x_0y_1 - x_1y_1, 2x_1y_0 - x_1y_1, 2x_0x_1 - x_1^2 \rangle \end{aligned}$$

| Degree $\underline{d}$ | Monomial Basis of $(S/I)_{\underline{d}}$ | Dimension |
|---|---|---|
| (0,0) | 1 | 1 |
| (1,0) | $x_0, x_1$ | 2 |
| (0,1) | $y_0, y_1$ | 2 |
| (1,1) | $x_0y_0, x_1y_1$ | 2 |

# Hilbert Function

**Definition**

The **Hilbert function** of $S/I(X)$ is the function $H_{S/I(X)} : \mathbb{N}^2 \to \mathbb{N}$ defined by

$$\begin{aligned} H_{S/I(X)}(i,j) &= \dim_k (S/I(X))_{i,j} \\ &= \dim_k S_{i,j} - \dim_k I(X)_{i,j} \end{aligned}$$

**Example**

$X = \{[1:0:0] \times [1:0], [2:1:0] \times [1:2]\}$ as before.
$I(X) = \langle x_2,\, 2y_0y_1 - y_1^2, 2x_0y_1 - x_1y_1, 2x_1y_0 - x_1y_1, 2x_0x_1 - x_1^2 \rangle$

$$H_{S/I(X)}(i,j) = \begin{cases} 1 & (i,j) = (0,0) \\ 2 & \text{otherwise} \end{cases}, \quad H_{S/I(X)} = \begin{bmatrix} 1 & 2 & \cdots \\ 2 & 2 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

# Free resolutions

**Definition**

A **graded free resolution** of $S/I(X)$ is an exact sequence of free $S$-modules

$$0 \leftarrow S/I(X) \leftarrow \bigoplus_{\underline{d} \in \mathbb{N}^2} S(-\underline{d})^{\beta_{0,\underline{d}}} \leftarrow \bigoplus_{\underline{d} \in \mathbb{N}^2} S(-\underline{d})^{\beta_{1,\underline{d}}} \leftarrow \cdots$$

A free resolution is **minimal** (MFR) if each free module has the minimal number of generators. The $\beta_{i,\underline{d}}$ are the **Betti numbers** of $S/I(X)$.

## Example

$X = \{[1:0:0] \times [1:0], [2:1:0] \times [1:2]\}$ as before. A graded MFR of $S/I(X)$ is given by

$$S \leftarrow \begin{array}{c} S(-1,0) \\ \oplus \\ S(-2,0) \\ \oplus \\ S(-1,-1)^2 \\ \oplus \\ S(0,-2) \end{array} \leftarrow \begin{array}{c} S(-3,0) \\ \oplus \\ S(-2,-1)^4 \\ \oplus \\ S(-1,-2)^3 \end{array} \leftarrow \begin{array}{c} S(-3,-1)^2 \\ \oplus \\ S(-2,-2)^3 \end{array} \leftarrow S(-3,-2) \leftarrow 0$$

## Theorem (Hilbert's Syzygy Theorem, 1890)

*The minimal free resolution of any module over a polynomial ring has finite length, and this length is bounded by the number of variables.*

# Virtual Resolutions

## Definition

**Virtual resolutions** (VR) are complexes of free $S$-modules which are not necessarily exact:

$$0 \leftarrow S/I(X) \xleftarrow{\phi_0} \bigoplus_{\underline{d}\in\mathbb{N}^2} S(-\underline{d})^{\beta_{0,\underline{d}}} \xleftarrow{\phi_1} \bigoplus_{\underline{d}\in\mathbb{N}^2} S(-\underline{d})^{\beta_{1,\underline{d}}} \xleftarrow{\phi_2} \cdots$$

The modules $\operatorname{Ker}(\phi_{i-1})/\operatorname{Im}(\phi_i)$ are allowed to have support in the irrelevant ideal

$$B = \langle x_0, \ldots, x_n \rangle \cap \langle y_0, \ldots, y_m \rangle.$$

Note:
(1) Every MFR is a VR;
(2) In $\mathbb{P}^n \times \mathbb{P}^m$, while MFRs have length bounded by $n + m + 2$, VRs can have length bounded by $n + m$

# Why study resolutions?

MFRs tell us about the module:

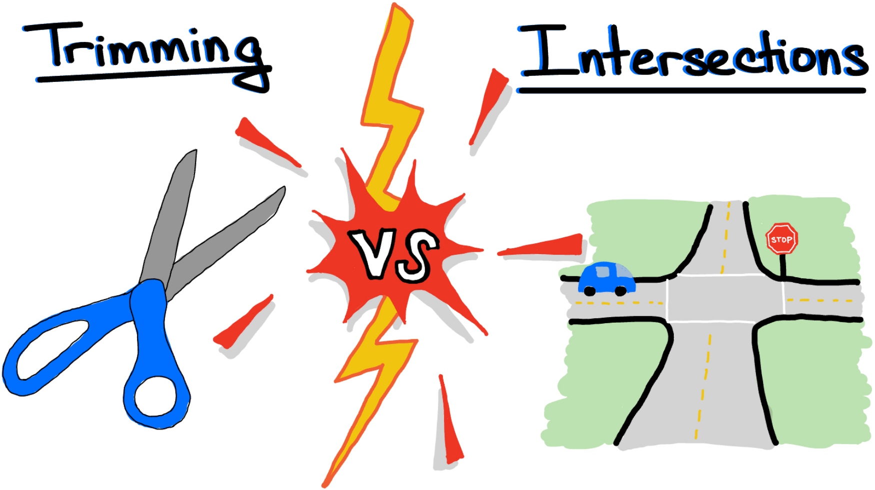
- Hilbert function
- Dimension
- Degree
- Vanishing of cohomology
- Embedded deformation theory
- Smoothness for curves
- Compactness
- Complete intersections
- Intersection theory
- Positivity/ampleness
- and more!

Eisenbud's *Geometry of Syzygies* book summarizes some of these stories for $\mathbb{P}^n$.

BUT! In products of projective space MFRs are "too long"

- VRs are shorter and still give useful geometric information
- Looking at multiple VRs can show *even more* geometry

# Two Approaches Towards Virtual Resolutions



# First Approach: Trimming

Let $X$ be a finite set of points in $\mathbb{P}^n \times \mathbb{P}^m$.

Theorem (Maclagan–Smith 2004)

*The multigraded regularity of $S/I(X)$ is*

$$\operatorname{reg}(S/I(X)) = \{\underline{d} \in \mathbb{Z}^2 \mid H_X(\underline{d}) = |X|\}.$$

Example

$X = \{[1:0:0] \times [1:0], [2:1:0] \times [1:2]\}$ as before.

Hilbert matrix: $H_X = \begin{bmatrix} 1 & 2 & \cdots \\ 2 & 2 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$

# Trimming: First Example

### Definition

Trimming at $\underline{d}$: keep the free summands in the MFR of $X \subseteq \mathbb{P}^n \times \mathbb{P}^m$ generated in degree $\leq \underline{d} + (n, m)$.

### Theorem (Berkesch–Erman–Smith 2020)

*Trimming the MFR of $X$ at $\underline{d} \in \operatorname{reg}(S/I(X))$ always yields virtual resolutions.*

### Example

$X = \{[1:0:0] \times [1:0], [2:1:0] \times [1:2]\}$ as before.

MFR & VR (trimming at $(1,0) + (2,1) = (3,1)$):

$$\color{red}{S} \leftarrow \begin{matrix} \color{red}{S(-1,0)} \\ \oplus \\ \color{red}{S(-2,0)} \\ \oplus \\ \color{red}{S(-1,-1)^2} \\ \oplus \\ S(0,-2) \end{matrix} \leftarrow \begin{matrix} \color{red}{S(-3,0)} \\ \oplus \\ \color{red}{S(-2,-1)^4} \\ \oplus \\ S(-1,-2)^3 \end{matrix} \leftarrow \begin{matrix} \color{red}{S(-3,-1)^2} \\ \oplus \\ S(-2,-2)^3 \end{matrix} \leftarrow S(-3,-2) \leftarrow 0$$

# Trimming: Generic Hilbert Matrix

## Conjecture

When the points in $X \subseteq \mathbb{P}^n \times \mathbb{P}^m$ are in sufficiently general position, the Hilbert matrix should have a fixed form; namely, we should have

$$H_X(i,j) = \min\left\{\binom{i+n}{n}\binom{j+m}{m}, |X|\right\}$$

## Example

$X$ = set of 12 random points in $\mathbb{P}^1 \times \mathbb{P}^2$ generated in Macaulay2

$$H_X = \begin{bmatrix} 1 & 3 & 6 & 10 & 12 & 12 & \cdots \\ 2 & 6 & 12 & 12 & 12 & 12 & \cdots \\ 3 & 9 & 12 & 12 & 12 & 12 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 11 & 12 & 12 & 12 & 12 & 12 & \cdots \\ \color{red}{12} & 12 & 12 & 12 & 12 & 12 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

# From Difference Matrix to Betti Numbers

Example (continued)

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & -8 & \cdots \\ 0 & 0 & -6 & 16 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & -3 & 9 & -9 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \\ \mathbf{0} & 0 & 0 & 0 & \cdots \\ -1 & 3 & -3 & 1 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \end{bmatrix}$$

A certain difference matrix of $H_X$

| Hom. degree | Degree | Betti number |
|---|---|---|
| 1 | $(1,3)$ | 8 |
| 1 | $(2,2)$ | 6 |
| 1 | $(4,1)$ | 3 |
| 1 | $(12,0)$ | 1 |
| 2 | $(2,3)$ | 16 |
| 2 | $(4,2)$ | 9 |
| 2 | $(12,1)$ | 3 |
| 3 | $(4,3)$ | 9 |
| 3 | $(12,2)$ | 3 |
| 4 | $(12,3)$ | 1 |

(Some of the) Betti numbers of $X$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & -8 & \cdots \\ 0 & 0 & -6 & 16 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & -3 & 9 & -9 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \\ \mathbf{0} & 0 & 0 & 0 & \cdots \\ -1 & 3 & -3 & 1 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \end{bmatrix}$$

| Hom. degree | Degree | Betti number |
|---|---|---|
| 1 | $(1,3)$ | 8 |
| 1 | $(2,2)$ | 6 |
| 1 | $(4,1)$ | 3 |
| 1 | $(12,0)$ | 1 |
| 2 | $(2,3)$ | 16 |
| 2 | $(4,2)$ | 9 |
| 2 | $(12,1)$ | 3 |
| 3 | $(4,3)$ | 9 |
| 3 | $(12,2)$ | 3 |
| 4 | $(12,3)$ | 1 |

A virtual resolution of $X$ by trimming at $(11,0)+(1,2)=(12,3)$:

$$S \leftarrow \begin{matrix} S(-2,-2)^{6} \\ \oplus \\ S(-4,-1)^{3} \\ \oplus \\ S(-12,0)^{1} \end{matrix} \leftarrow \begin{matrix} S(-4,-2)^{9} \\ \oplus \\ S(-12,-1)^{3} \end{matrix} \leftarrow S(-12,-2)^{3} \leftarrow 0$$

Here we used a version of the Minimal Resolution Conjecture.

# Trimming: Result

Assuming the two conjectures, we have:

**Theorem (B-D-G-S-S 2023+)**

*For $X \subseteq \mathbb{P}^1 \times \mathbb{P}^2$ in sufficiently general position, when $|X| \geq 12$, doing "trimming" at $(|X|-1, 0) \in \operatorname{reg}(S/I(X))$ will always give us a virtual resolution of length 3 of the form:*

$$S \leftarrow \begin{matrix} S(-m,-2)^{6-r} \\ \oplus \\ S(-m-1,-2)^{r} \\ \oplus \\ S(-m',-1)^{3-r'} \\ \oplus \\ S(-m'-1,-1)^{r'} \\ \oplus \\ S(-n,0) \end{matrix} \leftarrow \begin{matrix} S(-m',-2)^{9-3r'} \\ \oplus \\ S(-m'-1,-2)^{3r'} \\ \oplus \\ S(-n,-1)^{3} \end{matrix} \leftarrow S(-n,-2)^3 \leftarrow 0$$

*where $n = 6m + r = 3m' + r'$.*

# Second Approach: Intersection with $\langle \underline{x} \rangle^a$

**Theorem (Harada–Nowroozi–Van Tuyl 2022)**

*Let $X$ be a finite set of points in $\mathbb{P}^1 \times \mathbb{P}^1$. Let $t$ denote the number of unique first coordinates. Then for all $a \geq t-1$, the MFR of $S/(I(X) \cap \langle x_0, x_1 \rangle^a)$ is a VR of $S/I(X)$ of length two.*

Our result:

**Theorem (B-D-G-S-S 2023+)**

*Let $X$ be a set of points in $\mathbb{P}^n \times \mathbb{P}^1$. Let $t$ denote the number of first coordinates. For all $a \geq t-1$, the MFR of $S/(I(X) \cap \langle x_0, \ldots, x_n \rangle^a)$ is a VR of $S/I(X)$ of length $n+1$.*

## Example

Let $X = \{[1:0:0]\times[1:0],[2:1:0]\times[1:2]\} \subseteq \mathbb{P}^2 \times \mathbb{P}^1$.
Then $t = 2$ and

$$I(X) = \langle x_1, x_2, y_1\rangle \cap \langle x_1 - 2x_0, x_2, y_1 - 2y_0\rangle.$$

The MFR of $S/(I(X) \cap \langle x_0, x_1, x_2\rangle^a)$ has length 3 for all $a \geq 2 - 1 = 1$.

This MFR (for $a = 1$) is a VR of $S/I(X)$.

$$S \leftarrow \begin{matrix} S(-1,0) \\ \oplus \\ S(-2,0) \\ \oplus \\ S(-1,-1)^2 \end{matrix} \leftarrow \begin{matrix} S(-2,-1)^4 \\ \oplus \\ S(-3,0) \end{matrix} \leftarrow S(-3,-1)^2 \leftarrow 0$$

Recall the MFR of $S/I(X)$ is length 4.

# But wait, there's more!

**Theorem (B-D-G-S-S 2023+)**

*Let $X$ be a set of points in $\mathbb{P}^n \times \mathbb{P}^m$. Let $t$ denote the number of distinct first coordinates. For all $a \geq t-1$, the MFR of $S/(I(X) \cap \langle x_0, \ldots, x_n \rangle^a)$ is a VR of $S/I(X)$ of length at most $n+m$.*

Tools we used:

- Auslander–Buchsbaum
- Primary decomposition
- Short exact sequences and additivity of the Hilbert Function

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