Dotying our i's and Carrying our p's

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Background

Our problem

We're interested in $\mathrm{GL}_n(k)$-stable ideals, and their free resolutions and syzygies. We take k to be an algebraically closed field of prime characteristic.

Action of general linear group on polynomial ring

Let k be an algebraically closed field. Consider the k-span of $x_1, \ldots, x_n$, and the general linear group $\mathrm{GL}_n(k)$ acting on this vector space.

The general linear group $\mathrm{GL}_n(k)$ acts on $S = k[x_1, \ldots, x_n]$ by

$$A \cdot f(x_1, \ldots, x_n) := f(Ax_1, \ldots, Ax_n).$$

Example

Let $S = k[x, y]$. Then

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \cdot xy^3 = Ax(Ay)^3 = x(ax + y)^3$$
$$= a^3x^4 + 3a^2x^3y + 3ax^2y^2 + xy^3.$$

Stable ideals

An ideal I in $S = k[x_1, \ldots, x_n]$ is called $\mathrm{GL}_n(k)$-stable ("stable") if $Af \in I$ for all $A \in \mathrm{GL}_n(k)$, $f \in I$.

Example

Let $S = k[x, y]$, where $char(k) = 3$, and let $I = \langle x^3, y^3 \rangle$. We show that I is stable.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x^3 = (ax + cy)^3$$
$$= a^3x^3 + 3a^2cx^2y + 3ac^2xy^2 + c^3y^3 = a^3x^3 + c^3y^3$$

If $char(k) = 2$, the ideal I would not be stable.

More on stable ideals

Fact

Any stable ideal must be homogeneous and generated by monomials.

Each graded component I_d of a $\mathrm{GL}_n(k)$-stable ideal has the structure of a $\mathrm{GL}_n(k)$-submodule of S_d.

We are interested in the graded structure of stable ideals of S, and their free resolutions, when k has positive characteristic.

We are also interested in determining which representations of $\mathrm{GL}_n(k)$ appear in certain modules associated to free resolutions of stable ideals. (Not in this talk!)

Why positive characteristic?

In characteristic 0, the problem is not interesting! When k has characteristic 0, the graded components of a $\mathrm{GL}_n(k)$-stable ideal are either S_d or 0.

The submodule structure of S (and of its graded components S_d) is richer when k has positive characteristic, so the structure of stable ideals becomes more complex.

We briefly outline what is known about the submodule structure of S and S_d.

Carry patterns

Doty found all $\mathrm{GL}_n(k)$-submodules of S_d for k with positive characteristic.

There is a correspondence between the lattice of submodules of S_d and combinatorial objects called **carry patterns**.

A carry pattern is a tuple associated to a monomial, and contains information about its exponent vector and the ambient ring's characteristic.

Example

Take $\mathrm{char}(k) = 2$ and consider the monomial x^3y^2z. Expand the exponents in base 2 and add them together:

$$3 = \mathbf{0} \cdot 2^2 + \mathbf{1} \cdot 2 + \mathbf{1}$$
$$2 = \mathbf{0} \cdot 2^2 + \mathbf{1} \cdot 2 + \mathbf{0}$$
$$1 = \mathbf{0} \cdot 2^2 + \mathbf{0} \cdot 2 + \mathbf{1}$$

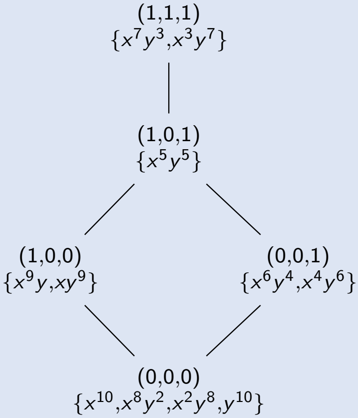
$$\longrightarrow \quad \begin{array}{cccc} & \color{red}{1} & \color{blue}{1} & \\ & 0 & 1 & 1 \\ & 0 & 1 & 0 \\ + & 0 & 0 & 1 \\ \hline & 1 & 1 & 0 \end{array}$$

The carry pattern of x^3y^2z is $(\color{blue}{1}, \color{red}{1})$.

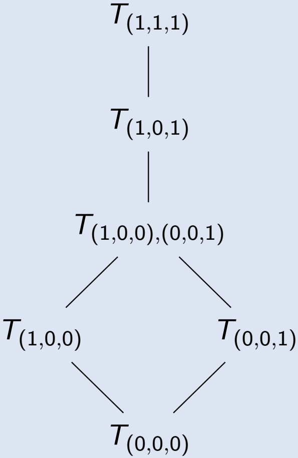
Example

Let $S = k[x, y]$ with $\mathrm{char}(k) = 2$.

We have a poset of carry patterns of monomials of degree 10 in S:



The submodule lattice of S_{10}:

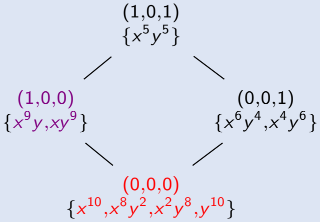


Stable ideals revisited

Given a degree d and carry pattern c, we define the **carry ideal** I_c to be the ideal generated by all monomials in S_d with carry pattern less than or equal to c.

Example

Let $I = \langle x^{10}, x^9y, x^8y^2, x^2y^8, xy^9, y^{10} \rangle$



$\longrightarrow I = I_{(1,0,0)}$

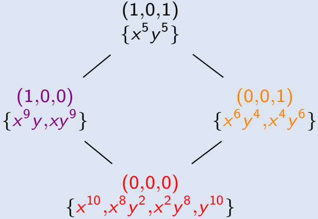
For any degree d, the minimal element in the lattice of carry patterns is always $(0, 0, \dots)$. This corresponds to the **smallest stable ideal**, $I_{(0,0,\dots)}$.

Smallest Stable Ideals

We can show that all stable ideals decompose as a finite sum of carry ideals.

Example

Let $I = \langle x^{10}, x^9y, x^8y^2, x^6y^4, x^4y^6, x^2y^8, xy^9, y^{10}\rangle$



$I_{(1,0,0)} = \langle x^{10}, x^9y, x^8y^2, x^2y^8, xy^9, y^{10}\rangle$

$I_{(0,0,1)} = \langle x^{10}, x^8y^2, x^6y^4, x^4y^6, x^2y^8, y^{10}\rangle$

$$I = I_{(1,0,0)} + I_{(0,0,1)}$$

Minimal free resolutions of stable ideals

Depth of stable ideals

The simplest stable ideals are the ideals whose generators are the monomials of a submodule of S_d for some fixed d.

We found that stable ideals decompose into a finite sum of these simple stable ideals.

A consequence of this combined with the $\mathrm{GL}_n(k)$ action is that for stable ideals I, the module S/I has depth 0.

Equivalently, every element of S is a zero divisor on S/I.

Free resolutions

Definition

A graded **free resolution** of S/I for an ideal I is an exact sequence of free S-modules

$$0 \leftarrow S/I \leftarrow \bigoplus_{d \in \mathbb{N}} S(-d)^{\beta_{0,d}} \leftarrow \bigoplus_{d \in \mathbb{N}} S(-d)^{\beta_{1,d}} \leftarrow \cdots$$

A free resolution is **minimal** (MFR) if each free module has the minimal number of generators.

We are interested in the minimal free resolutions of S/I for stable ideals I.

Length of MFR of stable ideals

Fact

Minimal free resolutions of S/I for any ideal I have length at most n, the number of indeterminates of S. (Hilbert's syzygy theorem)

Fact

The length of a minimal free resolution of S/I is equal to n minus the depth of S/I. (Auslander-Buchsbaum formula)

Since the depth of S/I is zero for stable ideals I, the minimal free resolution of S/I has length exactly n.

Form of MFR when in two variables

In the case that S has two variables, the minimal free resolution of S/I for a stable ideal I has length 2.

In particular, these resolutions are **Hilbert–Burch** resolutions, which are "nice".

Example

Let $S = k[x,y]$ *and* $\operatorname{char}(k) = 2$*, and let* $I = I_{\{(0,0)\}} = \langle x^5, x^4y, xy^4, y^5 \rangle$*. Then the MFR of* S/I *is:*

$$0 \leftarrow S/I \leftarrow S \xleftarrow{\begin{pmatrix} x^5 & x^4y & xy^4 & y^5 \end{pmatrix}} S(-5)^4 \xleftarrow{\begin{pmatrix} -y & 0 & 0 \\ x & -y^3 & 0 \\ 0 & x^3 & -y \\ 0 & 0 & x \end{pmatrix}} \begin{matrix} S(-6) \\ \oplus \\ S(-8) \\ \oplus \\ S(-6) \end{matrix} \leftarrow 0$$

The columns of the rightmost matrix form **syzygies** of the generators of I.

Our work

Minimal free resolution of smallest stable ideals

A **block** is a subsequence of the base-p expansion of d of the form $(p-1, p-1, \ldots, a)$, where $a < p-1$.

Theorem (C-D-G-G-S, 2023+)

Let $S = k[x, y]$, where $\mathrm{char}(k) = p$. Let I be the smallest stable ideal generated in degree d. Based entirely on the **blocking** *of the base-p expansion of d, we can find the following:*

1. *The number of generators of I.*
2. *The number of distinct degrees of syzygies of the minimal generators of I.*
3. *The distinct degrees of syzygies of the minimal generators of I.*
4. *The multiplicity of each degree of syzygy.*

Example

Let $p = 5$ and $d = 994$. The MFR of the smallest stable ideal in this degree is:

$$I \longleftarrow S(-994)^{600} \longleftarrow \begin{array}{c} S(-994-6)^{28} \\ \bigoplus \\ S(-994-1)^{570} \\ \bigoplus \\ S(-994-256)^{1} \end{array} \longleftarrow 0$$

Example

Let $p = 5$ $\qquad d = 994$

$$(4,3,4,2,1)$$

Number of generators:
$(4+1)\cdot(3+1)\cdot(4+1)\cdot(2+1)\cdot(1+1) = 600$

Example

Let $p = 5$ $\qquad d = 994$

$(4,3,4,2,1)$

Number of generators:
$(4+1)\cdot(3+1)\cdot(4+1)\cdot(2+1)\cdot(1+1) = 600$

$$I \longleftarrow S(-994)^{600} \longleftarrow \_\_ \longleftarrow 0$$

Example

Let $p = 5$ $\qquad$ $d = 994$

$$(4,3,4,2,1)$$

Example

Let $p = 5$ $\qquad d = 994$

$$(4,3,4,2,1)$$

$$\downarrow$$

$$\left(\underset{5^0\ 5^1}{\boxed{4,3}} \ , \ \underset{5^2\ 5^3}{\boxed{4,2}} \ , \ \underset{5^4}{\boxed{1}} \right)$$

Number of distinct syzygy degrees: 3

Example

Let $p = 5$ $\quad d = 994$

$$\begin{array}{c}(4,3,4,2,1)\\ \downarrow\\ \left(\ \underset{5^0\ 5^1}{\boxed{4,3}}\ ,\ \underset{5^2\ 5^3}{\boxed{4,2}}\ ,\ \underset{5^4}{\boxed{1}}\ \right)\end{array}$$

Number of distinct syzygy degrees: 3

$$I \longleftarrow S(-994)^{600} \longleftarrow \begin{array}{c} S(-994 - \_\_)^{\_\_} \\ \bigoplus \\ S(-994 - \_\_)^{\_\_} \\ \bigoplus \\ S(-994 - \_\_)^{\_\_} \end{array} \longleftarrow 0$$

Example

Let $p = 5$ $\qquad d = 994$

$$(4,3,4,2,1)$$

$$\downarrow$$

$$\Big(\underset{5^0\ 5^1}{\boxed{4,3}} \;,\; \underset{5^2\ 5^3}{\boxed{4,2}} \;,\; \underset{5^4}{\boxed{1}} \Big)$$

Degrees of syzygies:

$5^2 - (4 + 3 \cdot 5) = 6$

1

256 $\qquad x^{625}y^{369} \leftrightarrow x^{369}y^{625}$

Example

Let $p = 5$ $d = 994$

$$(4,3,4,2,1)$$

$$\downarrow$$

$$\Big(\underset{5^0\ 5^1}{\boxed{4,3}}\ ,\ \underset{5^2\ 5^3}{\boxed{4,2}}\ ,\ \underset{5^4}{\boxed{1}} \Big)$$

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Let $p = 5$ $\qquad$ $d = 994$

$$(4,3,4,2,1)$$

$$\downarrow$$

$$\Big(\underset{5^0\ 5^1}{\boxed{4,3}} \;,\; \underset{5^2\ 5^3}{\boxed{4,2}} \;,\; \underset{5^4}{\boxed{1}} \Big)$$

Number of syzygies of each degree:

1: $(4 \cdot 5^0 + 3 \cdot 5^1) \cdot (4 + 1) \cdot (2 + 1) \cdot (1 + 1) = 570$

6: 28

256: 1

Example

Let $p = 5$ $\quad d = 994$

$$\begin{array}{c}(4{,}3{,}4{,}2{,}1)\\ \downarrow\end{array}$$

$$\Big(\underset{5^0\ 5^1}{\boxed{4,3}}\ ,\ \underset{5^2\ 5^3}{\boxed{4,2}}\ ,\ \underset{5^4}{\boxed{1}}\Big)$$

Number of syzygies of each degree:

1: $(4 \cdot 5^0 + 3 \cdot 5^1) \cdot (4+1) \cdot (2+1) \cdot (1+1) = 570$

6: 28

256: 1

$$I \longleftarrow S(-994)^{600} \longleftarrow \begin{array}{c} S(-994-6)^{28} \\ \bigoplus \\ S(-994-1)^{570} \\ \bigoplus \\ S(-994-256)^{1} \end{array} \longleftarrow 0$$

MFR of other stable ideals

Given a degree $d = (d_i)$ and carry pattern $c = (c_i)$, define the sequence $d^c = (d_i^c)$ by

$$d_i^c := d_i + pc_{i+1} - c_i$$

A **block** is a subsequence of d^c satisfying certain conditions.

Theorem (C-D-G-G-S, 2023+)

Let $S = k[x, y]$, where $\mathrm{char}(k) = p$. Let I be the smallest stable ideal generated in degree d. Based entirely on the **blocking** *of d^c, we can find the following:*

1. *The number of generators of I.*
2. *The number of distinct degrees of syzygies of the minimal generators of I.*
3. *The distinct degrees of syzygies of the minimal generators of I.*
4. *The multiplicity of each degree of syzygy.*

Example

Let $p = 5$, $d = 994 = (4, 3, 4, 2, 1)$, $c = (0, 1, 0, 1)$.
Then:

$$d^c = (4, 8, 3, 7, 0)$$

Example

Let $p = 5$, $d = 994 = (4, 3, 4, 2, 1)$, $c = (0, 1, 0, 1)$.
Then:

$$d^c = (4, 8, 3, 7, 0)$$

A block is a string of numbers at least $p - 1$ followed by a number less than $p - 1$:

$$d^c = (\boxed{4, 8, 3}, \boxed{7, 0})$$

We can read off the minimal free resolution from the blocking of d^c.

MFR of other stable ideals

Let $p = 5$, $d = 994 = (4, 3, 4, 2, 1)$, $c = (0, 1, 0, 1)$.

$$d^c = (\boxed{4, 8, 3}, \boxed{7, 0})$$

The minimal free resolution of $I = I_c$ is

$$I \longleftarrow S(-994)^{960} \longleftarrow \begin{matrix} S(-994-1)^{952} \\ \bigoplus \\ S(-994-6)^{7} \end{matrix} \longleftarrow 0$$

Thank you!

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