ON ORBITS OF ORDER IDEALS OF MINUSCULE POSETS

DAVID B RUSH AND XIAOLIN SHI

Abstract. An action on order ideals of posets first analyzed in full generality by Fon-Der-Flaass is considered in the case of posets arising from minuscule representations of complex simple Lie algebras. For these minuscule posets, it is shown that the Fon-Der-Flaass action, together with the generating function that counts order ideals by their cardinality, exhibits the cyclic sieving phenomenon as defined by Reiner, Stanton, and White. The proof is uniform, and it is accomplished by investigation of a bijection due to Stembridge between order ideals of minuscule posets and fully commutative Weyl group elements arranged in Bruhat lattices, which allows for an equivariance between the Fon-Der-Flaass action and an arbitrary Coxeter element to be demonstrated.

If P is a minuscule poset, it is shown that the Fon-Der-Flaass action on order ideals of the Cartesian product $P \times [2]$ also exhibits the cyclic sieving phenomenon, only the proof is by appeal to the classification of minuscule posets and is not uniform.

1. Introduction

The Fon-Der-Flaass action on order ideals of a poset has been the subject of extensive study since it was introduced in its original form on hypergraphs by Duchet in 1974 [6]. The results that have been obtained to date, however, have yet to be assembled into a form sufficiently coherent to give an indication of the a priori nature of the operation. In this article, we attempt to clarify the picture by identifying a disparate collection of posets – called the minuscule posets – characterized by properties from representation theory for which the experience of the Fon-Der-Flaass action is somewhat uniform. We illustrate the commonality vis-a-vis the cyclic sieving phenomenon of Reiner-Stanton-White, which provides a unifying framework for organizing combinatorial data on orbits derived from cyclic actions.

If P is a poset, and $J(P)$ is the set of order ideals of P, partially ordered by inclusion, the Fon-Der-Flaass action Ψ maps an order ideal $I \in J(P)$ to the order ideal $\Psi(I)$ whose maximal elements are the minimal elements of $P \setminus I$. It should be clear that Ψ is invertible and thus generates a cyclic group $\langle \Psi \rangle$ acting on $J(P)$ for which the orbit structure is not immediately apparent.

In [9], Reiner, Stanton, and White observed many situations in which the orbit structure of the action of a cyclic group $\langle c \rangle$ on a finite set X may be predicted rather consistently by a polynomial $X(q) \in \mathbb{Z}[q]$. Following their lead, we say that the triple $(X, X(q), \langle c \rangle)$ exhibits the *cyclic sieving phenomenon* if, for any integer d, the number of elements x in X fixed by c^d is obtained by evaluating $X(q)$ at $q = \zeta^d$, where n is the order of c on X and ζ is any primitive n^{th} root of unity. In other words, the cyclic sieving phenomenon encapsulates an action's relevant enumerative attributes by expressing the number of orbits of each size as a particular specialization of an associated generating function. In the case when $X = J(P)$

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and c is the Fon-Der-Flaass action, the natural generating function to consider is the rank-generating function for $J(P)$, which we denote by $J(P;q)$, where the rank of an order ideal $I \in J(P)$ is given by the cardinality $|I|$ (so that $J(P;q) := \sum_{I \in J(P)} q^{|I|}$).

The minuscule posets are a class of posets arising in the representation theory of Lie algebras that enjoy some astonishing combinatorial properties, chief among them being that, if P is minuscule, the rank-generating function $J(P \times [m]; q)$ takes on a certain "nice" form for all positive integers m. (Here $[m]$ denotes the chain with m elements, and $P \times [m]$ denotes the Cartesian product.)

Let $\mathfrak{g}$ be a complex simple Lie algebra with Weyl group W and weight lattice Λ. There is a natural partial order on Λ called the *root order* in which one weight μ is considered to be smaller than another weight ω if the difference $\omega - \mu$ is a sum of positive roots. If $\lambda \in \Lambda$ is dominant and the only weights occuring in the irreducible highest weight representation V^λ are the weights in the W-orbit $W\lambda$, then λ is called *minuscule*, and the restriction of the root order to the set of weights $W\lambda$ (which is called the *weight poset*) has two alternate descriptions:

- Let W_J be the maximal parabolic subgroup of W stabilizing λ, and let W^J be the set of minimum-length coset representatives for the parabolic quotient W/W_J. Then there is a natural bijection

$$\begin{array}{rcl} W^J & \longrightarrow & W\lambda \\ w & \longmapsto & w_0 w \lambda. \end{array}$$

  (where w_0 denotes the longest element of W), and this map turns out to be an isomorphism of posets between the *strong Bruhat order* on W restricted to W^J and the root order on $W\lambda$.
- Let P be the poset of join irreducible elements of the root order on $W\lambda$. Then P is called the *minuscule poset for* λ; P is ranked, and there is an isomorphism of posets between the weight poset and $J(P)$.

If P is minuscule, it is a result of Proctor's ([8], Theorem 6) that P enjoys what Stanley calls the *Gaussian* property (cf. [12], Exercise 25): for all positive integers m,

$$J(P \times [m]; q) = \prod_{p \in P} \frac{1 - q^{m + r(p) + 1}}{1 - q^{r(p)+1}},$$

where $r(p)$ denotes the rank of the element p in P. This may be verified case-by-case, but in fact it follows uniformly from the *standard monomial theory* of Lakshmibai, Musili, and Seshadri, as is shown in [8].

Thus, for all positive integers m, we are led to consider the triple $(X, X(q), \langle \Psi \rangle)$, where $X = J(P \times [m])$, $X(q) = J(P \times [m]; q)$, and P is any minuscule poset. We are at last poised to state the first two of our main results, answering a question of Reiner's.

Theorem 1.1. *Let P be a minuscule poset. If $m = 1$, $(X, X(q), \langle \Psi \rangle)$ exhibits the cyclic sieving phenomenon.*

Theorem 1.2. *Let P be a minuscule poset. If $m = 2$, $(X, X(q), \langle \Psi \rangle)$ exhibits the cyclic sieving phenomenon.*

Interestingly enough, the analogous claim to Theorems 1.1 and 1.2 is false for $m = 3$; computations performed by Kevin Dilks [1] reveal that when $m = 3$ and P is the minuscule poset $P = [3] \times [3]$, the triple $(X, X(q), \langle\Psi\rangle)$ does not exhibit the cyclic sieving phenomenon. However, if P arises from a Lie algebra with root system of type D and no other Lie algebras (i.e., if P belongs to the third infinite family of minuscule posets; see the classification at the end of the introduction), the same triple exhibits the cyclic sieving phenomenon for all positive integers m, which we prove in section 10, and, in section 12, we include some speculation about the cases in which we suspect the cyclic sieving phenomenon to hold in general. The rest of this introduction is devoted to a discussion of Theorems 1.1 and 1.2 and a brief overview of our approach to their proofs.

It should be noted that several special cases of Theorem 1.1 already exist in the literature. When P arises from a Lie algebra with root system of type A, for instance, Theorem 1.1 reduces to a result of Stanley's in [13] coupled with Theorem 1.1(b) in [9], and it is recorded as Theorem 8.1 in [16]. The case when the root system is of type B turns out to be handled almost identically, and it is recorded as Corollary 8.4 in [16]. That being said, our theorem is a vast generalization of these results, and the novelty lies not so much in the statement of the theorem per se as in the method of proof, which relates Theorem 1.1 to a known cyclic sieving phenomenon for finite Coxeter groups, namely, Theorem 1.6 in [9], thus shedding more light onto the underlying structure of the orbits of order ideals of the previously studied minuscule posets and, for the first time, exposing the Fon-Der-Flaass action itself to new algebraic avenues of appraisal for which the prospects have only begun to be investigated.

If P is a finite poset, it is shown by Cameron and Fon-Der-Flaass in [5] that the Fon-Der-Flaass action Ψ may be expressed as a product of the involutive generators $\{t_p\}_{p\in P}$ for a larger group acting on the poset of order ideals $J(P)$. In [5], this group was denoted by $G(P)$ and left unnamed, but it is called the *toggle group* in [16] because for all $p \in P$ and $I \in J(P)$, $t_p(I)$ is obtained by *toggling I at p*, so that $t_p(I)$ is either the symmetric difference $I\Delta\{p\}$, if this forms an order ideal, or just I, otherwise.

On the other hand, results of Stembridge's in [15] provide a natural labelling of the elements of a minuscule poset P by the Coxeter generators $S = \{s_1, s_2, \ldots, s_n\}$ for the Weyl group W acting on the Bruhat lattice W^J. In particular, if P is a minuscule poset, there exists a labelling of P such that the linear extensions of the labelled poset (which is called a *minuscule heap*) index the reduced words for the fully commutative element of W representing the topmost coset w_0W_J. This labelling of the minuscule poset P is the one featured in Figure 2 (as well as in Figures 9, 10, 11, 14, 15 of the Appendix) and explained more thoroughly in Section 5, and it has the following important properties.

First, it realizes the poset isomorphism $J(P) \cong W^J$ explicitly. Given an order ideal $I \in J(P)$ and a linear extension $(x_1, x_2, \ldots, x_t)$ of the partial order restricted to the elements of I, if the corresponding sequence of labels is $(i_1, i_2, \ldots, i_t)$, define $\phi(I)$ to be $s_{i_t} \cdots s_{i_2} s_{i_1}$, considered as an element of W^J. Then the map $\phi : J(P) \to W^J$ is an order-preserving bijection.

Second, it indicates a correspondence between Coxeter elements in W and sequences of toggles in $G(P)$. To wit, the choice of a linear ordering on the Coxeter generators $S = (s_{i_1}, \ldots, s_{i_n})$ yields a choice of

[1] Computed using code in the computer algebra package `Maple`. The authors also thank Dilks for allowing them the use of his code for subsequent computations.

- an element $t_{(i_1,\dots,i_n)}$ in the toggle group that executes the following sequence of toggles: first toggle at all elements of P labelled by s_{i_n}, in any order; then toggle at all elements of P labelled by $s_{i_{n-1}}$, in any order;...; then toggle at all elements of P labelled by s_{i_2}, and, finally, toggle at all elements of P labelled by s_{i_1}, and
- a *Coxeter element* $c = s_{i_1}s_{i_2}\cdots s_{i_n}$ in the Weyl group, which acts on cosets W/W_J by left translation, i.e., $c(wW_J) = cwW_J$, and hence also acts on W^J.

The (original) theorems that reduce Theorem 1.1 to the aforementioned result of Reiner-Stanton-White are as follows.

Theorem 1.3. *For any minuscule poset P and any ordering of $S = (s_{i_1},\dots,s_{i_n})$, the actions Ψ and $t_{(i_1,\dots,i_n)}$ are conjugate in $G(P)$.*

Theorem 1.4. *For any minuscule poset P and any ordering of $S = (s_{i_1},\dots,s_{i_n})$, if $\phi : J(P) \to W^J$ is the isomorphism described above, then the following diagram is commutative.*

$$\begin{array}{rcl} J(P) & \xrightarrow{\phi} & W^J \\ t_{(i_1,\dots,i_n)} \downarrow & & c\downarrow \\ J(P) & \xrightarrow{\phi} & W^J. \end{array}$$

To see that these theorems suffice to demonstrate Theorem 1.1, we quote Theorem 1.6 from [9] and append the appropriate observations.

Theorem 1.5. *Let W be a finite Coxeter group; let S be the set of Coxeter generators, and let J be a subset of S. Let W^J be the set of minimum-length coset representatives, and let $W^J(q) = \sum_{w\in W^J} q^{l(w)}$, where $l(w)$ denotes the length of w. If $c \in W$ is a regular element in the sense of Springer's* [11], *then $(W^J, W^J(q), \langle c\rangle)$ exhibits the cyclic sieving phenomenon.*

Remark 1.6. If W^J is a distributive lattice, then the length function l doubles as a rank function, so $W^J(q)$ is the rank-generating function.

Remark 1.7. If c is a Coxeter element of W, then $c \in W$ is regular (cf. [11]).

The proofs of Theorem 1.3 and Theorem 1.4 are undertaken in section 6; sections 2, 3, 4, and 5 are all preliminary. Because offering a uniform proof of Theorem 1.4 requires a bit of a digression into the theory of fully commutative elements of Coxeter groups (for which the relevant background in elucidated in section 5) and because our rather more specific work delineating the desired equivariance for each individual minuscule family (which predates our uniform proof) is of some significance in its own right (especially for the minuscule posets associated to the classical root systems), we state the case-by-case bijections in the appendix, which also contains a complete description of every minuscule representation of a complex simple Lie algebra and the associated minuscule posets, and we suggest that the reader consult this "atlas" if she desires some concrete examples before delving too deeply into the more obscure aspects of the paper that may be of less immediate interest to someone working outside the field.

Unfortunately, we did not manage to adapt these techniques for the proof of Theorem 1.2, so here we opt for a somewhat less theoretical approach. We turn to work by Cameron and Fon-Der-Flaass, who devised a mechanism for encoding height 2 plane partitions (which may be thought of equivalently as order ideals of $P \times [2]$ for minuscule posets P arising from Lie algebras with root systems of type A) as sequences of dots and parentheses, which we refer to as bracket sequences. By performing elementary manipulations on these sequences, with the aid of generating functions, we are able to directly enumerate the number of orbits of each size under the Fon-Der-Flaass action and verify that the results we obtain are in concurrence with those predicted by assuming the cyclic sieving phenomenon holds. This is the focus of section 8 (section 7 is preliminary). In section 9, we apply the same techniques to demonstrate that the desired cyclic sieving phenomenon holds for symmetric height 2 plane partitions, which may be thought of equivalently as order ideals of $P \times [2]$ for minuscule posets P arising from Lie algebras with root systems of type B. Finally, in section 10, we present a proof that if P is a minuscule poset belonging to remaining infinite family, the order ideals of $P \times [m]$ obey the cyclic sieving phenomenon with respect to the Fon-Der-Flaass action for all positive integers m. Bracket sequences are applied as well, but, because m may be any positive integer, the bijection between bracket sequences and order ideals is more difficult to obtain. The claim of Theorem 1.2 for the two exceptional cases is checked by computer, using the software developed by Dilks, in section 11. While the proofs of these results which we assemble into Theorem 1.2 are purely combinatorial, it is our hope that representation-theoretic (or Bruhat-theoretic) simplifications are possible, and our outlook for future work on this problem is touched on in the concluding section (section 12).

We close the introduction with a description of the three infinite families and two exceptional cases of minuscule posets and the root systems associated to the Lie algebras from which they arise. As discussed, more detail is provided in the appendix. The following facts are well-known and may be found in, for instance, [3].

- For the root systems of the form A_{n-1}, there are $n-1$ possible minuscule weights, which lead to $n-1$ associated minuscule posets, namely all those posets of the form $[j] \times [n-j]$ such that $1 \leq j < n$. Posets of this form are considered to comprise the *first infinite family*. An example is depicted in Figure 9, part (a).
- For the root systems of the form B_n, there is 1 possible minuscule weight, which leads to 1 associated minuscule poset, namely $[n] \times [n]/S_2$. Posets of this form are considered to comprise the *second infinite family*. An example is depicted in Figure 10, part (a).
- For the root systems of the form C_n, there is 1 possible minuscule weight, which leads to 1 associated minuscule poset, namely $[2n-1]$. It should be clear that posets of this form already belong to the first infinite family. An example is depicted in Figure 11, part (a).
- For the root systems of the form D_n, there are 3 possible minuscule weights, which lead to 2 associated minuscule posets, namely $[n-1] \times [n-1]/S_2$ and $J^{n-3}([2] \times [2])$, because two of the minuscule weights both lead to the same minuscule poset. Posets of the latter form are considered to comprise the *third infinite family* (it should be clear that posets of the former form already belong to the second infinite family).
- For the root system E_6, there are 2 possible minuscule weights, which lead to 1 associated minuscule poset, namely $J^2([2] \times [3])$, because both minuscule weights lead to the same minuscule poset. This poset is considered to be the *first exceptional case*. It is depicted in Figure 14, part (b).
- For the root system E_7, there is 1 possible minuscule weight, which leads to 1 associated minuscule poset, namely $J^3([2] \times [3])$. This poset is considered to be the *second exceptional case*. It is depicted in Figure 15, part (b).

No other root systems admit minuscule weights.

2. The Fon-Der-Flaass Action

In this section, we introduce and analyze the Fon-Der-Flaass action. While we call it the Fon-Der-Flaass action because Fon-Der-Flaass devised it independently and was the first to make great strides in the development of the theory, it should be noted that the action actually appeared years earlier in Brouwer-Schrijver [4], so the credit for the invention should in fairness be considered theirs. Let $P = (X, <)$ be a partially ordered set, and let $J(P)$ be the set of order ideals of P, partially ordered by inclusion. Following the notation of [5], for all order ideals $I \in J(P)$, let $Z(I) = \{x \in I : y > x \Longrightarrow y \notin I\}$, and let $U(I) = \{x \notin I : y < x \Longrightarrow y \in I\}$. Then the *Fon-Der-Flaass action*, which we denote by Ψ, is formally defined as follows.

Definition 2.1. For all $I \in J(P)$, $\Psi(I)$ is the unique order ideal satisfying $Z(\Psi(I)) = U(I)$.

Remark 2.2. From Definition 2.1, it should be clear that Ψ permutes the order ideals of P.

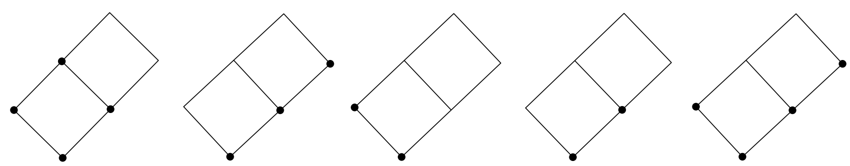

Figure 1. Fon-Der-Flaass Operation

Our understanding of the Fon-Der-Flaass action hinges upon our ability to decompose it into its constituent components. Recall from the introduction that for all $p \in P$ and $I \in J(P)$, we let $t_p : J(P) \to J(P)$ be the map defined by $t_p(I) = I \setminus \{p\}$ if $p \in Z(I)$, $t_p(I) = I \cup \{p\}$ if $p \in U(I)$, and $t_p(I) = I$ otherwise. The following theorem is equivalent to Lemma 1 in [5].

Theorem 2.3. *Let P be a poset. For all linear extensions $(p_1, p_2, \ldots, p_n)$ of P and order ideals $I \in J(P)$, $\Psi(I) = t_{p_1} t_{p_2} \cdots t_{p_n}(I)$.*

The group $G(P) := \langle t_p \rangle_{p \in P}$ is dubbed the *toggle group* by Striker and Williams [16], and it will be referred to as such in this work, as well. Note that for all x and y, the generators t_x and t_y commute unless x and y share a covering relation.

In the case that the poset P is ranked, there is a natural type of linear extension to consider in Theorem 2.3, namely the extensions that label the elements of P in order of increasing rank, which leads to further simplifications. For the purposes of this paper, we shall say that P is ranked if there is an integer-valued function r on X (called the *rank function*) such that $r(p) = 0$ for all minimal elements $p \in X$ and, for all $x, y \in X$, if x covers y, then $r(x) - r(y) = 1$. This condition is somewhat stronger than the standard, but it turns out that the posets we consider are all ranked in our sense of the word, so there is little to be lost by circumscribing full generality from the definition.

If P is a ranked poset, let the maximum value of r be R. For all $0 \leq i \leq R$, let $P_i = \{p \in P : r(p) = i\}$, and let $t_i = \prod_{p \in P_i} t_p$. We see that t_i is always well-defined because, for all i, t_x and t_y commute for all $x, y \in P_i$. By Theorem 2.3, $\Psi = t_0 t_1 \cdots t_R$. Note that t_i and t_j commute for all $|i - j| > 1$. We are now in a position to introduce the following theorem, also found in [5].

Theorem 2.4. *For all permutations σ of $\{0, 1, \ldots, R\}$, $\Psi_\sigma := t_{\sigma(0)} t_{\sigma(1)} \cdots t_{\sigma(R)}$ is conjugate to Ψ in $G(P)$.*

Corollary 2.5. *The action Ψ_σ has the same orbit structure as Ψ for all σ.*

Let $t_{\text{even}} = \prod_{i \text{ even}} t_i$, and let $t_{\text{odd}} = \prod_{i \text{ odd}} t_i$. It should be clear that t_{even} and t_{odd} are well-defined, and it follows from Theorem 2.4 that $t_{\text{even}} t_{\text{odd}}$ is conjugate to Ψ in $G(P)$. In words, this means that the action of toggling all the elements of odd rank, followed by toggling all the elements of even rank, is conjugate to the Fon-Der-Flaass action in the toggle group. As we shall see, this holds the key to demonstrating that the induced action of every Coxeter element of W on $J(P)$ under ϕ is conjugate to the Fon-Der-Flaass action as well. The expert reader may wish to skip directly to the proof of Theorem 1.3 in section 6 to witness this for herself.

3. Minuscule Posets

In this section, we introduce the primary objects of study for this paper – the minuscule posets. We begin with the requisite notation, following [14]. Let $\mathfrak{g}$ be a complex simple Lie algebra; let $\mathfrak{h}$ be a Cartan subalgebra; choose a set Φ^+ of positive roots α in $\mathfrak{h}^*$, and let $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be the set of simple roots. Let $(\cdot, \cdot)$ be the inner product on $\mathfrak{h}^*$, and, for each root α, let $\alpha^\vee = 2\alpha/(\alpha, \alpha)$ be the corresponding coroot. Finally, let $\Lambda = \{\lambda \in \mathfrak{h}^* : \alpha \in \Phi \to (\lambda, \alpha^\vee) \in \mathbb{Z}\}$ be the weight lattice.

For all $1 \leq i \leq n$, let s_i be the simple reflection corresponding to the simple root α_i, and let $W = \langle s_i \rangle_{1 \leq i \leq n}$ be the Weyl group of $\mathfrak{g}$. If s is conjugate to a simple reflection s_i in W, we refer to s as an (abstract) reflection.

Let V be a finite-dimensional representation of $\mathfrak{g}$. For each $\lambda \in \Lambda$, let

$$V_\lambda = \{v \in V : h \in \mathfrak{h} \Longrightarrow hv = \lambda(h)v\}$$

be the weight space corresponding to λ, and let Λ_V be the (finite) set of weights λ such that V_λ is nonzero. Recall that there is a standard partial order on Λ called the *root order* defined to be the transitive closure of the relations $\mu < \omega$ for all weights μ and ω such that $\omega - \mu$ is a simple root.

Definition 3.1. The *weight poset* Q_V of the representation V is the restriction of the root order on Λ to Λ_V.

If V is irreducible, it is known that Q_V has a unique maximal element, which is called the highest weight of V. This leads to the following definition.

Definition 3.2. Let V be a nontrivial, irreducible, finite-dimensional representation of $\mathfrak{g}$. V is a *minuscule representation* if the action of W on Λ_V is transitive. In this case, the highest weight of V is called the *minuscule weight.*

Theorem 3.3. *If V is minuscule, the weight poset Q_V is a distributive lattice.*

Remark 3.4. This result, due to Proctor (cf. [8], Propositions 3.2 and 4.1), was originally verified by exhaustive search, but, actually, it is a consequence of Theorem 5.8, for which a case-free proof was given using Bruhat-theoretic techniques by Stembridge in [15]. For now, however, we ask that the reader accept this result on faith.

Definition 3.5. If V is minuscule, let P_V be the poset of join irreducible elements of the weight poset Q_V, so that P_V is the unique poset satisfying $J(P_V) \cong Q_V$. Then P_V is the *minuscule poset* of V, and an arbitrary poset P is *minuscule* if and only if there exists a minuscule representation V for which $P_V = P$.

Remark 3.6. If V is a minuscule representation and λ is the highest weight of V, we refer to P_V as the minuscule poset for λ. A moment's thought reveals that this terminology is legitimate because, for all minuscule weights λ, the minuscule poset for λ is necessarily unique.

4. Bruhat Posets

In this section, we develop the framework through which we will obtain Theorems 1.3 and 1.4. We begin by discussing the Bruhat posets; then we illuminate the connection between these objects and the weight posets of minuscule representations, which we have already defined.

We continue with the notation of the previous section. Given a Weyl group W, we define a length function l on the elements of W as follows. For all $w \in W$, we let $l(w)$ be the minimum length of a word of the form $s_{i_1} s_{i_2} \dots s_{i_\ell}$ such that $w = s_{i_1} s_{i_2} \dots s_{i_\ell}$ and s_{i_j} is a simple reflection for all $1 \leq j \leq \ell$. This allows us to introduce a well-known partial order on W, known as the (strong) Bruhat order, for which l doubles as the rank function. The (strong) Bruhat order is defined to be the transitive closure of the relations $w <_B sw$ for all Weyl group elements w and (abstract) reflections s satisfying $l(w) < l(sw)$.

It turns out that what is of interest is not the Bruhat order on W per se, but rather the restrictions of the Bruhat order to parabolic quotients of W, for it is these orders that give rise to the Bruhat posets.

Definition 4.1. If J is a subset of $\{1, 2, \dots, n\}$, then $W_J := \langle s_i \rangle_{i \in J}$ is the *parabolic subgroup* of W generated by the corresponding simple reflections, and $W^J := W/W_J$ is the *parabolic quotient*.

It is well-known that each coset in W^J has a unique representative of minimum length, so the quotient W^J may be thought of equivalently as the subset of W comprising all the minimum-length coset representatives. This fact facilitates the definition of an analogous partial order on W^J.

Definition 4.2. The *Bruhat order* $<_B$ on the parabolic quotient W^J is the restriction of the Bruhat order on W to W^J. Posets of the form $(W^J, <_B)$ comprise the *Bruhat posets*.

We may also define the left Bruhat order on W to be the transitive closure of the relations $w <_L sw$ for all Weyl group elements w and *simple* reflections s satisfying $l(w) < l(sw)$. The analogous partial order on W^J is defined in precisely the same way: $(W^J, <_L)$ is the restriction of $(W, <_L)$ to the minimum-length coset representatives W^J. While the left Bruhat order is not, strictly speaking, necessary to illuminate the connection between the minuscule posets and the Bruhat posets, we introduce it here so that our work in this section may be compatible with the theory of fully commutative elements developed in section 5 and exploited in section 6.

We are now poised to state the following theorem, which appears as Proposition 4.1 in [8].

Theorem 4.3. *Let V be a minuscule representation with minuscule weight λ, and let $J = \{i : s_i\lambda = \lambda\}$. Then W_J is the stabilizer of λ in the Weyl group W, and the weight poset Q_V is isomorphic to the Bruhat poset $(W^J, <_B)$.*

Remark 4.4. There is a small subtlety in the proof Theorem 4.3 because the natural map $\varphi : W^J \to Q_V$ to consider, $w \mapsto w\lambda$, is order-reversing, rather than order-preserving. (In other words, for all $u, v \in W^J$, $u\lambda < v\lambda$ if and only if $v <_B u$.) However, composing φ with the order-reversing involution of Q_V given by $\omega \mapsto w_0\omega$, where w_0 is the unique longest element of W, yields a suitable isomorphism (as noted in the introduction). Alternatively, φ may be composed with the corresponding order-reversing involution of W^J given by $w \mapsto w_0w(w')^{-1}w_0$, where w' denotes the unique longest element of W^J. We omit the proofs of these claims, but the first involution, at least, may be found in the literature (cf., for instance, [14]), and anyway they are not altogether difficult. It should be noted, though, that, in Proctor's proof of Theorem 4.3, he circumvents this step by defining the partial order on the weights opposite to the root order. We avoid his approach here because it leads to unnecessary confusion over terms such as "highest weight," etc. (Why is the highest weight of the minuscule representation now the lowest weight?) The reader is encouraged to refer to [8] herself for more details.

Definition 4.5. The parabolic quotient W^J is *minuscule* if W_J is the stabilizer of a minuscule weight λ.

It is important to point out that the assumption that $\mathfrak{g}$ be simple implies that λ is fundamental (recall that the fundamental weights $\omega_1, \omega_2, \dots, \omega_n$ are defined by the condition $(\omega_i, \alpha_j^\vee) = \delta_{ij}$ for all $1 \leq i, j \leq n$, where δ_{ij} is the Kronecker delta). Hence if $\lambda = \omega_j$, $s_i\lambda = \lambda$ for all $i \neq j$. It follows that if W^J is minuscule, $J = \{1, 2, \dots, n\} \setminus \{j\}$, so W_J is a maximal parabolic subgroup of W, and, in general, a minuscule Bruhat poset is obtained precisely when the "missing" element of J is the index of a fundamental weight for which there exists a representation of $\mathfrak{g}$ in which that fundamental weight is minuscule.

We note that Bruhat posets W^J provide a natural setting for identifying instances of the cyclic sieving phenomenon because they come equipped with a group action, namely that of W, and a rank-generating function $W^J(q) := \sum_{w \in W^J} q^{l(w)}$, which is what motivated us to consider them in the first place. We now turn our attention to the labelling of the minuscule poset P_V and the construction of the isomorphism $\phi : J(P_V) \to W^J$, which lie behind the proofs of Theorems 1.3 and 1.4.

5. Fully Commutative Elements

We pause in this section to develop Stembridge's theory of fully commutative elements of Coxeter groups. At the end of this section, we quote a theorem of Stembridge's that the minuscule parabolic quotients are indeed distributive lattices, which was proven through this theory without reference to casework, and, in the next section, we shall see how this theory enables us to characterize the relationship between the action of the Weyl group on the elements of these lattices and the action of the toggle group on the order ideals of the corresponding minuscule posets.

Definition 5.1. Let W be a Weyl group, and let $S = \{s_1, s_2, \dots, s_n\}$ be the set of Coxeter generators. An element $w \in W$ is *fully commutative* if every reduced word for w can be obtained from every other by means of commuting braid relations only (i.e., via relations of the form $s_j s_{j'} = s_{j'} s_j$ for commuting Coxeter generators s_j and $s_{j'}$).

Given a fully commutative element w, we can define a labelled poset P_w that generates all the reduced words of w in the sense that the linear extensions of P_w, with labels in place of poset elements, are in bijection with the reduced words of w.

Definition 5.2. Let $s_{i_1} s_{i_2} \cdots s_{i_\ell}$ be a reduced word for w. Let $P_w = (\{1, 2, \ldots, \ell\}, <)$ be a partially ordered set, where the partial order on $\{1, 2, \ldots, \ell\}$ is defined to be the transitive closure of the relations $j > j'$ for all $j < j'$ in integers such that s_{i_j} and $s_{i_{j'}}$ do not commute. Then P_w is the *heap* of w, and, for all $1 \leq j \leq \ell$, s_{i_j} is the *label* of the heap element $j \in P_w$.

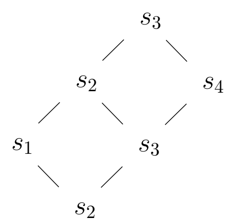

FIGURE 2. If W is the Weyl group arising from the root system A_4, then the element $w := s_3 s_2 s_4 s_1 s_3 s_2$ is fully commutative, and the heap P_w is as displayed above.

Let $\mathcal{L}(P_w) := \{\pi : \pi(1) \geq \pi(2) \geq \ldots \geq \pi(\ell)\}$ be the set of *reverse* linear extensions of P_w, and let $\mathcal{L}(P_w, w)$ be the set of labelled reverse linear extensions of P_w, i.e.,

$$\mathcal{L}(P_w, w) := \{s_{i_{\pi(1)}} s_{i_{\pi(2)}} \cdots s_{i_{\pi(\ell)}} : \pi \in \mathcal{L}(P_w)\}.$$

As alluded to above, the set $\mathcal{L}(P_w, w)$ is significant for the following reason.

Proposition 5.3. *$\mathcal{L}(P_w, w)$ is the set of reduced words for w in W.*

Proof. See Proposition 2.2 in [15]. □

Remark 5.4. Upon consulting Stembridge's paper, the reader may find the differences between our definitions and Stembridge's to be somewhat perplexing; for one thing, we define the partial order on P_w to be the reverse of Stembridge's order and consider reverse linear extensions rather than linear extensions, and, for another, Stembridge defines heaps for all words in W, whereas our definition is only correct for reduced words of fully commutative elements w. The implications for the theory are rather cosmetic, however, and we prefer our approach because it helps simplify some of the proofs.

It follows from Proposition 5.3 that, if w is fully commutative, the heaps of the reduced words for w are all equivalent, so we may refer to the heap of w unambiguously. This is also noted in [15]. The crucial claim is the next theorem.

Theorem 5.5. *Let $w \in W$ be fully commutative. Then $J(P_w) \cong \{x \in W : x \leq_L w\}$ is an isomorphism of posets.*

Proof. A proof is found in [15] (see Lemma 3.1), but because our definitions are different from Stembridge's, and because the map between the two posets will be of importance in its own right for our proof of Theorem 1.4, we provide our own adaptation of Stembridge's proof.

For all $1 \leq k \leq n$, let $C_k := \{j : s_{i_j} = s_k\}$ be the set of all heap elements labelled by s_k. We first note that each C_k is a totally ordered subset of P_w. The proof is by contradiction. Suppose that there exist incomparable elements $j, j' \in P_w$ such that $s_{i_j} = s_{i_{j'}} = s_k$. Then there exists a reverse linear extension of P_w in which j and j' occur consecutively, which implies that the corresponding reduced word of w contains two consecutive instances of s_k. This is of course impossible. Thus, we may write C_k in the form $\{k_1 < k_2 < \ldots < k_{\nu(k,w)}\}$, where $\nu(k, w)$ denotes the number of instances of s_k in a reduced word for w, and ν is well-defined because w is fully commutative.

We are now ready to define the bijection between $J(P_w)$ and $\{x \in W : x \leq_L w\}$. Given an order ideal $I \in J(P_w)$, let ρ be a linear extension of P_w such that $\rho(j) \in I$ for all $1 \leq j \leq |I|$ and $\rho(j) \notin I$ otherwise.

Definition 5.6.

$$\phi : J(P_w) \quad \longrightarrow \quad \{x \in W : x \leq_L w\}$$

is defined by

$$I \quad \longmapsto \quad s_{i_{\rho(|I|)}} \cdots s_{i_{\rho(2)}} s_{i_{\rho(1)}}.$$

Remark 5.7. The choice of the symbol ϕ to denote this map is deliberate, for when the heap P_w is minuscule (see Definition 6.1), ϕ is the map described in the introduction.

It is not immediately clear that ϕ is well-defined. However, if ρ and ρ' are both linear extensions of P_w such that $\rho(j), \rho'(j) \in I$ for all $1 \leq j \leq |I|$ and $\rho(j), \rho'(j) \notin I$ otherwise, then let $x = s_{i_{\rho(|I|)}} \cdots s_{i_{\rho(2)}} s_{i_{\rho(1)}}$ and $x' = s_{i_{\rho'(|I|)}} \cdots s_{i_{\rho'(2)}} s_{i_{\rho'(1)}}$. Since

$$(\rho(\ell), \ldots, \rho(|I| + 1), \rho(|I|), \ldots, \rho(2), \rho(1))$$

is a reverse linear extension of P_w,

$$s_{i_{\rho(\ell)}} \cdots s_{i_{\rho(|I|+1)}} s_{i_{\rho(|I|)}} \cdots s_{i_{\rho(2)}} s_{i_{\rho(1)}}$$

is a reduced word for w, so $s_{i_{\rho(\ell)}} \cdots s_{i_{\rho(|I|+1)}}$ is a reduced word for wx^{-1}. However,

$$(\rho(\ell), \ldots, \rho(|I| + 1), \rho'(|I|), \ldots, \rho'(2), \rho'(1))$$

is also a reverse linear extension of P_w. It follows that $s_{i_{\rho(\ell)}} \cdots s_{i_{\rho(|I|+1)}}$ is a reduced word for wx'^{-1}, so $x = x'$, as desired.

To see that ϕ is bijective, we define the inverse map $\phi^{-1} : \{x \in W : x \leq_L w\} \to J(P_w)$ by $x \mapsto \cup_{k=1}^{n} \{k_h : 1 \leq h \leq \nu(k, x)\}$. Because every reduced word for x is the final segment of a reduced word for w, it should be clear that $\phi^{-1}(x)$ is an order ideal of P_w for all $x \leq_L w$. It is a trivial matter to verify that $\phi^{-1}\phi$ is the identity on $J(P_w)$ and $\phi\phi^{-1}$ is the identity on $\{x \in W : x \leq_L w\}$, so this completes the proof. □

The following theorem rather persuasively demonstrates the relevance of the theory of fully commutative elements to our main results.

Theorem 5.8. *If W^J is minuscule, then the following three claims hold:*

(i) If $w \in W^J$, w is fully commutative;
(ii) $(W^J, <_L)$ is a distributive lattice;
(iii) $(W^J, <_B) = (W^J, <_L)$.

This theorem is a consequence of Theorems 6.1 and 7.1 in [15], where, as we've said, the proof is uniform. We will see how it enables us to apply our knowledge of fully commutative heaps to the minuscule setting in the next section.

6. The Main Results

In this section, we prove Theorems 1.3 and 1.4. We require the following definition and subsequent theorem.

Definition 6.1. If W^J is minuscule, and w_0^J is the longest element of W^J, then the heap $P_{w_0^J}$ is *minuscule*, and heaps of this form comprise the *minuscule heaps*.

Remark 6.2. Some of the minuscule heaps appear in Wildberger [17], but his construction differs from ours. In particular, he introduces a set of heaps that he calls *two-neighbourly*, and he shows that these are precisely the minuscule heaps arising from complex simple Lie algebras whose root systems are simply laced.

Theorem 6.3. *Let V be a minuscule representation of a complex simple Lie algebra $\mathfrak{g}$ with minuscule weight λ and Weyl group W. If $S = \{s_1, s_2, \ldots, s_n\}$ is the set of Coxeter generators and W_J is the maximal parabolic subgroup stabilizing λ, then the following claims hold:*

(i) If w_0^J is the longest element of W^J, then the poset $\{x \in W : x \leq_L w_0^J\}$ and the lattice $(W^J, <_L)$ are identical, and, furthermore, the minuscule heap $P_{w_0^J}$ and the minuscule poset P_V are isomorphic as posets.

(ii) The isomorphism $\phi : J(P_{w_0^J}) \to \{x \in W : x \leq_L w_0^J\} \cong (W^J, <_L) \cong (W^J, < B)$ defined in Definition 5.6 satisfies the following property: for all $1 \leq l \leq n$, the induced action of the Coxeter generator s_l on $J(P_{w_0^J})$ in the toggle group $G(P_{w_0^J})$ may be expressed in the form $\displaystyle\prod_{p \in P_{w_0^J} \text{ is labelled by } s_l} t_p$.

Example 6.4. In the case when the root system is A_4 and the minuscule weight is ω_2, Figure 3 shows the minuscule heap $P_{s_3s_2s_4s_1s_3s_2}$ (on the left) and the corresponding Bruhat poset $(W^J, <_B)$ (on the right). If I is the order ideal encircled by the solid line, then $\phi(I)$ is the coset representative encircled by the solid line, and $\prod_{p \in P_{s_3s_2s_4s_1s_3s_2} \text{ is labelled by } s_2} t_p(I)$ is the order ideal encircled by the dotted line. Furthermore, $\phi(\prod_{p \in P_{s_3s_2s_4s_1s_3s_2} \text{ is labelled by } s_2} t_p(I)) = s_2\phi(I)$ is the coset representative encircled by the dotted line, thus illustrating the statement (ii) in Theorem 6.3. (See the appendix for more details about this or other specific cases.)

Proof. (i) By Proposition 2.6 in [15], w_0^J is the unique maximal element of $(W^J, <_L)$. It follows that if $w \in W^J$, $w \leq_L w_0^J$. To see that the converse also holds, let x_0 be the longest element of W_J, and note that $w \in W^J$ if and only if wx_0 is reduced (i.e. if and only if the product of a reduced word for w and a reduced word for x_0 is necessarily a reduced word for wx_0). If $w \leq_L w_0^J$, then there exists a reduced word for w that is the final segment of a reduced word for w_0^J. Hence there exists a reduced word for w and a reduced word for x_0 such that their product is a reduced word for wx_0, and it follows from the fact that all reduced words for the same element are of the same length that wx_0 is reduced. We may conclude that $w \in W^J$, so, in general, $\{x \in W : x \leq_L w_0^J\} = (W^J, <_L)$. However, $J(P_{w_0^J}) \cong \{x \in W : x \leq_L w_0^J\}$, and $(W^J, <_L) = (W^J, <_B) \cong J(P_V)$ by Definition 3.5 and

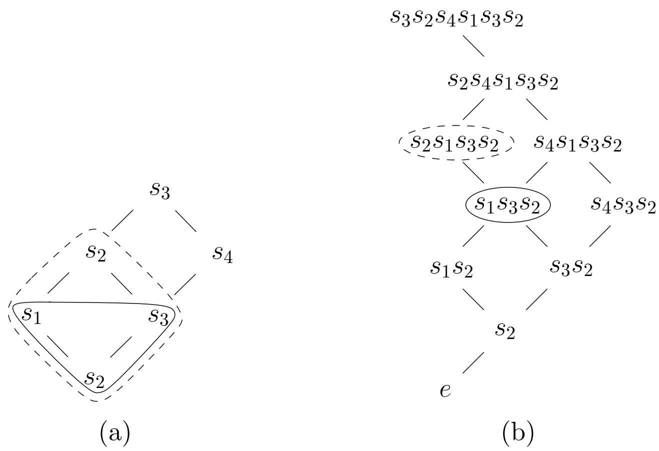

FIGURE 3. The map ϕ sends the indicated order ideals to the indicated coset representatives.

Theorems 4.3 and 5.8, so $J(P_{w_0^J}) \cong J(P_V)$, and it follows that $P_{w_0^J} \cong P_V$ is an isomorphism of posets, as desired.
(ii) Because $(W^J, <_L) = (W^J, <_B)$, it suffices to prove the claim with $(W^J, < L)$ in place of $(W^J, <_B)$. Following the notation in the proof of Theorem 5.5, for all $1 \leq l \leq n$, let C_l be the set of all heap elements labelled by s_l, and let t'_l be the toggle group element defined by $t'_l = \prod_{p \in C_l} t_p$. From section 5, we know that C_l is totally ordered, and, by definition of $P_{w_0^J}$, no two elements of C_l share a covering relation, so it follows that t'_l is well-defined for all l. Consider the following lemma:

Lemma 6.5. *For all order ideals $I \in J(P_{w_0^J})$, $t'_l(I)$ disagrees with I on at most one vertex of $P_{w_0^J}$.*

Proof. It suffices to show that if there exists one vertex on which the two disagree, then there cannot exist any other such vertices. If there exists a vertex on which the two disagree, then there must exist at least one vertex p_0 labelled by s_l such that $p_0 \in Z(I)$ or $p_0 \in U(I)$. If $p_0 \in Z(I)$, consider $\prod_{p \in C_l} t_p(I)$, where the toggles t_p are applied to I in order of increasing p. For all the elements p such that $p < p_0$, the toggle t_p has no effect because when t_p is applied, p is in the order ideal, but so is p_0, which is larger. For all the elements p such that $p > p_0$, the toggle t_p has no effect because when t_p is applied, p is not in the order ideal, but neither is p_0 (any longer), which is smaller. Similarly, if $p_0 \in U(I)$, consider $\prod_{p \in C_l} t_p(I)$, where the toggles t_p are applied to I in order of decreasing p. For all the elements p such that $p > p_0$, the toggle t_p has no effect because when t_p is applied, p is not in the order ideal, but neither is p_0, which is smaller. For all the elements p such that $p < p_0$, the toggle t_p has no effect because when t_p is applied, p is in the order ideal, but (now) so is p_0, which is larger. This completes the proof. $\square$

It follows immediately that in applying t'_l to an order ideal I, it suffices to determine whether or not there exists a p_0 labelled l in $Z(I)$ or $U(I)$, and, if there does, to apply t_{p_0} only, and, if there does not, to simply return I.

Let I be an order ideal, and let $w = \phi(I)$. There are two cases.

(i) If no elements of $P_{w_0^J}$ labelled by s_l belong to $Z(I)$ or $U(I)$, we claim that $s_l w W_J = wW_J$ (which is equivalent to saying that s_l maps the coset of w in W^J to itself). We divide into two subcases:

(1) $s_l w$ is not reduced. In this case, let $s_{i_{l(s_l w)}} \cdots s_{i_2} s_{i_1}$ be a reduced word for $s_l w$, and note that $s_l s_{i_{l(s_l w)}} \cdots s_{i_2} s_{i_1}$ is a reduced word for w, else $l(w) = l(s_l s_{i_{l(s_l w)}} \cdots s_{i_2} s_{i_1}) < l(s_{i_{l(s_l w)}} \cdots s_{i_2} s_{i_1}) = l(s_l w) < l(w)$, which is absurd. Let $i_{l(w)} = l$, and extend this word to a reduced word for w_0^J, $s_{i_\ell} \cdots s_{i_2} s_{i_1}$, where ℓ is the length of w_0^J. Without loss of generality, we may assume that the heap $P_{w_0^J}$ is built with reference to this particular reduced word (recall that the heaps of every reduced word for w_0^J are equivalent), in which case the vertex corresponding to $s_{i_{l(w)}}$ is maximal over the vertices in the order ideal $\phi^{-1}(w) = I$, which implies that there exists an element of $P_{w_0^J}$ labelled by s_l that belongs to $Z(I)$. Therefore, this case does not occur.

(2) $s_l w$ is reduced. In this case, $s_l w$ covers w in the left Bruhat order on W, so, by Corollary 2.5.2 in [2], it follows that $s_l w = w s_j$, where $j \in J$, or $s_l w \in W^J$. In the former case, since $s_j \in W_J$, $s_l w W_J = w s_j W_J = w W_J$, as desired, and it turns out that the latter case does not occur. To see this, assume that $s_l w \in W^J$, and let $s_{i_{l(w)}} \cdots s_2 s_1$ be a reduced word for w. Then $s_l s_{i_{l(w)}} \cdots s_2 s_1$ is a reduced word for $s_l w$. Let $i_{l(w)+1} = l$, and extend this word to a reduced word for w_0^J, $s_{i_\ell} \cdots s_2 s_1$, where ℓ is the length of w_0^J (this may be done because $s_l w \leq_L w_0^J$ follows from $s_l w \in W^J$). Without loss of generality, we may assume that the heap $P_{w_0^J}$ is built with reference to this particular reduced word, in which case the vertex corresponding to $s_{i_{l(w)+1}}$ is minimal over the vertices not in the order ideal $\phi^{-1}(w) = I$, which implies that there exists an element of $P_{w_0^J}$ labelled by s_l that belongs to $U(I)$. This contradiction completes the proof.

(ii) If there exists an element $p_0 \in P$ labelled by s_l such that $p_0 \in Z(I)$ or $p_0 \in U(I)$, we claim that $s_l w = t'_l(I)$. Again we divide into two subcases:

(1) $p_0 \in Z(I)$. In this case, let $s_{i_\ell} \cdots s_2 s_1$ be a reduced word for w_0^J, where ℓ is the length of w_0^J, and assume that the heap $P_{w_0^J}$ is built with reference to this particular reduced word. Since $p_0 \in Z(I)$, $I \setminus \{p_0\}$ is an order ideal of $P_{w_0^J}$. Let $(\rho(1), \rho(2), \ldots, \rho(|I|-1))$ be a linear extension of $I \setminus \{p_0\}$ (i.e. a linear extension of the poset with vertices in $I \setminus \{p_0\}$ and partial order given by the restriction of the partial order on $P_{w_0^J}$ to $I \setminus \{p_0\}$). Then $(\rho(1), \rho(2), \ldots, \rho(|I|-1), p_0)$ is a linear extension of I. Let $\rho(|I|) = p_0$, and extend this linear extension to a linear extension of $P_{w_0^J}$, $(\rho(1), \rho(2), \ldots, \rho(\ell))$. By definition of ϕ, $s_{i_{\rho(|I|)}} s_{i_{\rho(|I|-1)}} \cdots s_{i_{\rho(2)}} s_{i_{\rho(1)}}$ is a reduced word for w. Since p_0 is labelled by s_l, $s_{i_{p_0}} = s_l$, so it follows that $s_l w = s_{i_{\rho(|I|-1)}} \cdots s_{i_{\rho(2)}} s_{i_{\rho(1)}}$. However, $\prod_{p \in C_l} t_p(I) = I \setminus \{p_0\}$, and $\phi(I \setminus \{p_0\}) = s_{i_{\rho(|I|-1)}} \cdots s_{i_{\rho(2)}} s_{i_{\rho(1)}}$ as well, as desired.

(2) $p_0 \in U(I)$. In this case, let $s_{i_\ell} \cdots s_2 s_1$ be a reduced word for w_0^J, where ℓ is the length of w_0^J, and assume that the heap $P_{w_0^J}$ is built with reference to this particular reduced word. Since $p_0 \in U(I)$, $I \cup \{p_0\}$ is an order ideal of $P_{w_0^J}$. Let $(\rho(1), \rho(2), \ldots, \rho(|I|))$ be a linear extension of I; then $(\rho(1), \rho(2), \ldots, \rho(|I|), p_0)$ is a linear extension of $I \cup \{p_0\}$. Let $\rho(|I|+1) = p_0$, and extend this linear extension to a linear extension of $P_{w_0^J}$, $(\rho(1), \rho(2), \ldots, \rho(\ell))$. By definition of ϕ, $s_{i_{\rho(|I|)}} \cdots s_{i_{\rho(2)}} s_{i_{\rho(1)}}$ is a reduced word for w. Since p_0 is labelled by s_l, $s_{i_{p_0}} = s_l$, so it follows that $s_l w = s_{i_{\rho(|I|+1)}} s_{i_{\rho(|I|)}} \cdots s_{i_{\rho(2)}} s_{i_{\rho(1)}}$.

However, $\prod_{p \in C_l} t_p(I) = I \cup \{p_0\}$, and $\phi(I \cup \{p_0\}) = s_{i_{\rho(|I|+1)}} \cdots s_{i_{\rho(2)}} s_{i_{\rho(1)}}$ as well, as desired.

□

We proceed to the proofs of Theorems 1.3 and 1.4.

6.1. **Proof of Theorem 1.4.** Let P_V be a minuscule poset, and understand the label of each element of P_V to be the label of the corresponding element of $P_{w_0^J}$. From Theorem 6.3, it follows that, for all $1 \leq l \leq n$, the diagram below is commutative:

$$\begin{array}{rcl} J(P_V) & \xrightarrow{\phi} & W^J \\ t'l \downarrow & & s_l \downarrow \\ J(P_V) & \xrightarrow{\phi} & W^J. \end{array}$$

For any ordering of $S = (s_{i_1}, s_{i_2}, \ldots, s_{i_n})$, $c = s_{i_1} s_{i_2} \cdots s_{i_n}$ and $t_{(i_1,i_2,\ldots,i_n)} = t'_{i_1} t'_{i_2} \cdots t'_{i_n}$, so Theorem 1.4 follows immediately.

6.2. **Proof of Theorem 1.3.** Let P_V be a minuscule poset, and understand the labels of each element of P_V to be the label of the corresponding element of $P_{w_0^J}$. Theorem 6.3 embeds the Weyl group W as a subgroup of the toggle group $G(P_V)$, so, in light of Theorem 1.4, since the Coxeter elements are known to be pairwise conjugate in W, it suffices to exhibit a particular ordering $S = (s_{i_1}, s_{i_2}, \ldots, s_{i_n})$ such that $t_{(i_1,i_2,\ldots,i_n)} = t'_{i_1} t'_{i_2} \cdots t'_{i_n}$ is conjugate to Ψ in $G(P_V)$. However, in section 2, we saw that $t_{\text{even}} t_{\text{odd}}$ is conjugate to Ψ in $G(P_V)$. What we prove here is that there exists an ordering $(s_{i_1}, s_{i_2}, \ldots, s_{i_n})$ such that the toggle group elements $t'_{i_1} t'_{i_2} \cdots t'_{i_n}$ and $t_{\text{even}} t_{\text{odd}}$ are equal.

We start with two lemmas:

Lemma 6.6. *If P is a minuscule poset, then P is a ranked poset.*

Proof. As discussed in the introduction, minuscule posets are Gaussian. The fact that all Gaussian posets are ranked is recorded as Exercise 25(b) in [12]. □

Lemma 6.7. *If W is the Weyl group of a complex simple Lie algebra $\mathfrak{g}$, then the Dynkin diagram of the associated root system is acyclic and therefore bipartite.*

Proof. Assume for the sake of contradiction there the Dynkin diagram contains a cycle $(i_1, i_2, \ldots, i_k)$, where $k \geq 3$, so that s_{i_j} does not commute with $s_{i_{j+1}}$ for all $1 \leq i \leq k-1$, and s_{i_k} does not commute with s_{i_1}. Consider the infinite word $\cdots s_k \cdots s_2 s_1 \cdots s_k \cdots s_2 s_1 s_k \cdots s_2 s_1$. Since every final segment of this word is reduced, it follows that W is infinite, but this is not possible. □

Let r be the rank function for P_V. For all $1 \leq l \leq n$, we claim that the ranks of all the vertices labelled by s_l are of the same parity.

For the proof, the key observation is that each covering relation in the heap $P_{w_0^J}$ corresponds to an edge of the Dynkin diagram of the associated root system. (Recall that the partial order on $P_{w_0^J}$ is the transitive closure of the relations $j > j'$ for all $j < j'$ in integers such that s_{i_j} and $s_{i'_j}$ do not commute; cf. Definition 5.2.) Assume for the sake of contradiction that there exists an l such that $j, j' \in P_{w_0^J}$ are both labelled l but $r(j') - r(j)$ is odd. Without loss of generality, let $r(j') > r(j)$. Since C_l is totally ordered, it follows that $j' > j$, and there exists a set of vertices $\{j_1, j_2, \ldots, j_{2u}\}$ such that j' covers j_1, j_{2u} covers j, and j_i covers j_{i+1} for all $1 \leq i \leq 2u-1$. We may conclude that there exists a path of odd length in the Dynkin diagram from the vertex corresponding to s_l to itself. However, by Lemma 6.7, the graph of the Dynkin diagram is bipartite, so this is impossible.

Note that if $p, p' \in P_V$ and $r(p) \equiv r(p') \pmod 2$, then s_p and $s_{p'}$ commute in $G(P_V)$. Let S_{odd} be the set of all l such that s_l is a simple reflection and the rank of p is odd for all vertices $p \in P_{w_0^J}$ labelled by s_l. Similarly, let S_{even} be the set of all l' such that $s_{l'}$ is a simple reflection and the rank of p is even for all vertices $p \in P_{w_0^J}$ labelled by $s_{l'}$. It follows that $t_{\text{even}} t_{\text{odd}} = \prod_{l' \in S_{\text{even}}} t'_{l'} \prod_{l \in S_{\text{odd}}} t'_l$. This completes the proof.

7. Plane Partitions, Preliminaries

For the remainder of the paper, we shift the focus from the representation-theoretic aspects of minuscule posets to their combinatorial properties. From interpreting these properties judiciously, we are able to deduce a number of instances of the cyclic sieving phenomenon directly. We begin by recalling the Gaussian criterion from the introduction.

Definition 7.1. Let P be a ranked poset with rank function r. P is *Gaussian* if, for all positive integers m, the following equality holds:

$$J(P \times [m])(q) = \prod_{p \in P} \frac{1 - q^{m + r(p) + 1}}{1 - q^{r(p)+1}}.$$

Remark 7.2. It is a result of Proctor's that all minuscule posets are Gaussian (cf. [8], Theorem 6), and it is conjectured that all Gaussian posets are minuscule.

The two most important known families of Gaussian posets are the first two infinite families of minuscule posets. The order ideals of these posets may be identified with combinatorial objects that we refer to as plane partitions, which have been the subject of intensive study independent of the underlying posets. Thus, in establishing Theorem 1.2 for these cases, we are implicitly formulating new combinatorial identities that in some sense are already organized and explained.

Definition 7.3. A *plane partition* of an integer n is an array of integers $n_{i,j}$ of the form $n = \sum_{i,j \geq 1} n_{i,j}$, where the $n_{i,j}$ are non-negative and $n_{i,j} \geq n_{i,j+1}$, $n_{i,j} \geq n_{i+1,j}$.

Define a plane partition $n_{i,j}$ to be inside $m \times n \times k$ if it satisfies $0 \leq n_{i,j} \leq k$ for $1 \leq i \leq m$, $1 \leq j \leq n$, and $n_{i,j} = 0$ otherwise. If we think of such a plane partition as stacking $n_{i,j}$ cubes on the (i,j) square, then it is clear that any such plane partition corresponds to an order ideal of $[m] \times [n] \times [k]$. Conversely, any order ideal of $[m] \times [n] \times [k]$ corresponds to a plane partition inside $m \times n \times k$. In other words, there is a canonical bijection between the plane partitions inside $m \times n \times k$ and the order ideals of $[m] \times [n] \times [k]$.

Theorem 7.4. *(MacMahon) The generating function for plane partitions π inside $m \times n \times k$ is as follows:*

$$\sum_{\pi \subset m \times n \times k} q^{|\pi|} = \prod_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n \\ 1 \leq l \leq k}} \frac{[i+j+l-1]_q}{[i+j+l-2]_q}.$$

Remark 7.5. It follows that the MacMahon formula is also the rank-generating function for the poset of order ideals of $[m] \times [n] \times [k]$.

Definition 7.6. A *symmetric plane partition* of an integer n is a plane partition as defined in Definition 7.3, with the extra condition that $n_{i,j} = n_{j,i}$ for all i, j.

As expected, there is a canonical bijection between the symmetric plane partitions inside $n \times n \times k$ and the order ideals of the poset $([n] \times [n])/S_2 \times [k]$. The analogue to the MacMahon formula for symmetric plane partitions is called the Bender-Knuth formula.

Theorem 7.7. *The generating function for symmetric plane partitions π inside $n \times n \times k$ is as follows:*

$$\sum_{\substack{\pi \subset n \times n \times k \\ \pi \text{ symmetric}}} q^{|\pi|} = \prod_{i=1}^{n} \left(\frac{1 - q^{k+2i-1}}{1 - q^{2i-1}} \prod_{h=i+1}^{n} \frac{1 - q^{2(k+i+h-1)}}{1 - q^{2(i+h-1)}} \right).$$

Remark 7.8. By similar reasoning, the Bender-Knuth formula is the rank-generating function for the poset of order ideals of $([n] \times [n])/S_2 \times [k]$. The rank-generating functions for $J([m] \times [n] \times [k])$ and $J([n] \times [n])/S_2 \times [k]$ actually arise in a representation-theoretic setting in the sense that, up to a constant power of q, they are the q-analogues of the Weyl dimension formulas for the corresponding minuscule posets, which is shown in [14]. Thus, some papers treat these formulas as a given and use them to verify that the two families of posets in question are Gaussian, rather than the other way around.

8. Proof of Theorem 1.2 for the First Infinite Family

In this section, we will demonstrate that cyclic sieving phenomenon holds for all posets of the form $[m] \times [n] \times [2]$.

8.1. The Cameron Fon-Der-Flaass Bijection.

In [5], Cameron and Fon-Der-Flaass presented an equivariant bijection between order ideals of $[m] \times [n] \times [2]$ and what we call bracket sequences of length $m+n+1$ with respect to the Fon-Der-Flaass action and a natural action on bracket sequences, which they denoted by ψ. We will give only a brief outline of the bijection; for further details, see Theorem 6 in [5].

The bijection is as follows: an order ideal of $[m] \times [n] \times [2]$ is first mapped to a column of length $m + n + 1$. These columns are then placed side-by-side to form an infinite grid consisting of $m + n + 1$ rows. This done, each column is mapped to its first diagonal, and it is noted that a diagonal repeats itself if and only if its corresponding column repeats under an action conjugate to the Fon-Der-Flaass action. Finally, each diagonal is mapped to a balanced word W_t consisting of symbols among "$\bullet$", "(", ")", and ")(". Thus, demonstrating the cyclic sieving phenomenon for order ideals of $[m] \times [n] \times [2]$ under the Fon-Der-Flaass action is reduced to demonstrating the cyclic sieving phenomenon for such balanced words W_t under ψ. For convenience, in the rest of this section we will refer to the symbols "(", ")", ")(" as "brackets" and to W_t as a "bracket sequence." The action of ψ on W_t is summarized in the following proposition.

Proposition 8.1. *The operation ψ has the acts on W_t in the following way (in the below, the A_i are balanced sub-words, which are possibly empty):*

(1) If $W_t = \bullet A_1$ then $\psi(W_t) = A_1 \bullet$.

(2) *If* $W_t = (A_1)A_2$ *then* $\psi(W_t) = A_1(A_2)$.
(3) *If* $W_t = (A_1 \between \cdots \between A_i)A_{i+1}$, $i \geq 2$, *then* $\psi(W_t) = A_1(A_2 \between \cdots \between A_i \between A_{i+1})$.

Proof. See Theorem 6 in [5]. $\square$

The following theorem, found in [5], shows that ψ has order $m+n+1$.

Theorem 8.2. *If* $W_t = UV$, *where* U *and* V *are balanced sub-words, then* $\psi^{|U|}(W_t) = VU$.

Corollary 8.3. *Letting* $U = W_t$ *and* $V = \emptyset$ *in Theorem 8.2 yields* $\psi^{m+n+1}(W_t) = W_t$ *since* $|W_t| = m+n+1$.

8.2. **Substituting Roots of Unity into the MacMahon Formula.** Letting $k = 2$ in Theorem 7.4, the rank-generating function for $[m] \times [n] \times [2]$ becomes

$$X(m,n,q) = \begin{bmatrix} m+n+1 \\ n \end{bmatrix}_q \begin{bmatrix} m+n \\ n \end{bmatrix}_q \frac{[1]_q}{[n+1]_q}.$$

By symmetry of m and n, this is also equal to

$$\begin{bmatrix} m+n+1 \\ m \end{bmatrix}_q \begin{bmatrix} m+n \\ m \end{bmatrix}_q \frac{[1]_q}{[m+1]_q}.$$

Since the order of ψ is $m+n+1$, the cyclic sieving phenomenon predicts that for any $\ell \geq 0$,

$$X\left(m,n,q = (e^{\frac{2\pi i}{m+n+1}})^{\ell}\right) = \left|\{W_t \text{ balanced: } \psi^{\ell}(W_t) = W_t\}\right|.$$

For example, substituting $\ell = 1$ gives the number of elements whose order divides 1; substituting $\ell = 2$ gives the number of elements whose order divides 2, etc. It is easy to see that if the cyclic sieving phenomenon holds for some $\ell | m+n+1$, then it holds for all ℓ' such that $\ell = \gcd(\ell', m+n+1)$. Therefore it is suffices to verify the identities predicted by the cyclic sieving phenomenon for all $\ell | m+n+1$. The following lemma will prove useful in the subsequent computations.

Lemma 8.4. *Let* $n = n'd + r$, $k = k'd + s$, *where* $0 \leq r, s \leq d-1$. *Then*

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q=e^{\frac{2\pi i}{d}}} = \binom{n'}{k'} \begin{bmatrix} r \\ s \end{bmatrix}_{q=e^{\frac{2\pi i}{d}}}$$

Proof. If $r < s$, then it is easy to check that both sides of the equation evaluate to 0. Therefore it suffices to prove the claim when $r \geq s$. We can write

$$\begin{aligned}
\begin{bmatrix} n \\ k \end{bmatrix} &= \frac{[n'd+r]_q!}{[k'd+s]_q![(n'-k')d+(r-s)]_q!} \\
&= \left(\frac{[n'd+r]_q \cdots [n'd+1]_q}{[k'd+s]_q \cdots [k'd+1]_q[(n'-k')d+(r-s)]_q \cdots [(n'-k')d+1]_q}\right)\left(\frac{[n'd]_q!}{[k'd]_q![(n'-k')d]_q!}\right).
\end{aligned}$$

The first fraction, when $q = e^{\frac{2\pi i}{d}}$, is precisely $\begin{bmatrix} r \\ s \end{bmatrix}_{q=e^{\frac{2\pi i}{d}}}$. The second fraction, when $q = e^{\frac{2\pi i}{d}}$, evaluates to $\binom{n'}{k'}$. Thus the lemma is proved. $\square$

For $\ell < m+n+1$, if we let $d = \frac{m+n+1}{\ell}$, then $q = \left(e^{\frac{2\pi i}{m+n+1}}\right)^{\ell}$ is a d^{th} root of unity. Expanding the MacMahon formula, we have the following:

$$\begin{bmatrix} m+n+1 \\ n \end{bmatrix}_q \begin{bmatrix} m+n \\ n \end{bmatrix}_q \frac{[1]_q}{[n+1]_q} = \left(\frac{[m+n+1]_q \cdots [m+2]_q}{[n]_q \cdots [1]_q}\right)\left(\frac{[m+n]_q \cdots [m+1]_q}{[n]_q \cdots [1]_q}\right)\frac{[1]_q}{[n+1]_q}.$$

If we let q be a d^{th} root of unity where $d > n+1$, then in the above expression $[m+n+1]_q = 0$, and all of the other terms are nonzero. Hence the expression as a whole evaluates to 0. Also, suppose $d \nmid n$ and $d \nmid n+1$; then, since $d | m+n+1$, by Lemma 8.4, it follows that

$$\begin{bmatrix} m+n+1 \\ n \end{bmatrix}_{q=e^{\frac{2\pi i}{d}}} = \binom{\ell}{\ell'} \begin{bmatrix} r \\ r' \end{bmatrix}_{q=e^{\frac{2\pi i}{d}}},$$

where $m+n+1 = \ell d + r$, $b = \ell' d + r'$, $0 \le r, r' \le d-1$. Since $d | m+n+1$ and $d \nmid n$, it follows that $r = 0$ and $r' > 0$. Therefore, the expression evaluates to 0.

The only cases left are when $d|n$ or $d|n+1$. If $d|n+1$, then since $d|m+n+1$, we get that $d|m$. By symmetry between m and n, it suffices to analyze the case when $d|n$. Let $m+n+1 = \ell d$, $n = \ell' d$. Applying Lemma 8.4, we have

$$\begin{bmatrix} m+n+1 \\ n \end{bmatrix}_{q=e^{\frac{2\pi i}{d}}} = \binom{\ell}{\ell'}.$$

Consider the term

$$\begin{bmatrix} m+n \\ n \end{bmatrix}_q = \frac{[m+n]_q \cdots [m+1]_q}{[n]_q \cdots [1]_q}$$

Since $d|n$, $d|m+1$, and $d|m+n+1$, the number of terms in $m+1\ldots,m+n$ divisible by d is the same as the number of terms in $1,\ldots,n$ divisible by d. The rest of the terms cancel each other out when we let $q = e^{\frac{2\pi i}{d}}$. Hence the expression evaluates to

$$\frac{(\ell-\ell')(\ell-\ell'+1)\cdots(\ell-1)}{\ell'!} = \binom{\ell-1}{\ell'}.$$

Finally, since $d \ge 2$, it follows that $\frac{[1]_q}{[n+1]_q} = 1$. Therefore, when $d|n$, the expression evaluates to

$$\binom{\ell-1}{\ell'}\binom{\ell}{\ell'},$$

where $\ell = \frac{m+n+1}{d}$, $\ell' = \frac{n}{d}$. We may conclude that there are $\binom{\ell-1}{\ell'}\binom{\ell}{\ell'}$ elements whose orbits are of size dividing ℓ.

Remark 8.5. The above argument only works for $d \ge 2$. In the case that $d = 1$, what is predicted by the cyclic sieving phenomenon is equivalent to the MacMahon formula.

8.3. **Bracket Positions.** If we count "(", ")", ")(", and "•" as having length one each, all words W_t from the Cameron Fon-Der-Flass bijection have length $m+n+1$. Treating the brackets "(", ")", and ")(" as identical for now, their positions can be considered as a subset of $[m+n+1] = \{1,\ldots,m+n+1\}$. The following proposition follows directly from Proposition 8.1.

Proposition 8.6. *Every time we apply ψ to W_t, the entries in the bracket position subset decrease by 1, where the entries are considered modulo $m+n+1$.*

Example 8.7. Consider the sequence shown below:

$$\begin{array}{rll}
 & (\bullet\bullet \between \bullet \between \bullet\bullet) & \{1,4,6,9\} \\
\xrightarrow{\psi} & \bullet\,\bullet\,(\bullet \between \bullet\bullet \between) & \{3,5,8,9\} \\
\xrightarrow{\psi} & \bullet(\bullet \between \bullet\bullet \between)\bullet & \{2,4,7,8\} \\
\xrightarrow{\psi} & (\bullet \between \bullet\bullet \between)\,\bullet\,\bullet & \{1,3,6,7\} \\
\xrightarrow{\psi} & \bullet(\bullet\bullet \between\between \bullet\bullet) & \{2,5,6,9\} \\
 & \vdots & \vdots
\end{array}$$

The entries in the subset decrease by 1 $\pmod 9$ every time we apply ψ.

Suppose there are ℓ brackets in total, and suppose furthermore that there exists $d|m+n+1$ such that $\psi^{\frac{m+n+1}{d}}(W_t) = (W_t)$. By Proposition 8.6, it follows that the position subset is a disjoint union of subsets among the following:

$$\left\{0, \frac{m+n+1}{d}, \cdots, \frac{(d-1)(m+n+1)}{d}\right\},$$
$$\left\{1, 1+\frac{m+n+1}{d}, \cdots, 1+\frac{(d-1)(m+n+1)}{d}\right\}, \ldots,$$
$$\left\{\frac{m+n+1}{d}-1, \frac{2(m+n+1)}{d}-1, \cdots, (m+n+1)-1\right\}.$$

(Since the position subset is invariant under shifts by $\frac{m+n+1}{d} \pmod{m+n+1}$, if the position subset contains one element of a subset listed above, it must contain all elements of that subset.)

Observe that each of the $\frac{m+n+1}{d}$ subsets above has cardinality d. It follows that $d|\ell$. Since we need to choose $\frac{\ell}{d}$ subsets from above, there are a total of $\binom{\frac{m+n+1}{d}}{\frac{\ell}{d}}$ ways to choose the bracket positions.

8.4. **Underlying Bracket Sequence.** Once we fix the positions of the brackets, we consider the types of the brackets. We start with the following definition.

Definition 8.8. The *underlying bracket sequence* $U(W_t)$ of a word W_t is the bracket sequence obtained by deleting all the $\bullet$'s from W_t.

Example 8.9. If $W_t = ((\bullet) \between \bullet)$, then its underlying bracket sequence is $U(W_t) = (() \between)$.

Suppose $\psi^{\frac{m+n+1}{d}}(W_t) = (W_t)$; then the underlying bracket sequence remains the same after we apply ψ to W_t $\frac{m+n+1}{d}$ times. Thus, it should be easy to see from Proposition 8.1 that the following proposition holds.

Proposition 8.10. *If $W_t = \bullet A_1$, then $U(\psi(W_t)) = U(W_t)$. Otherwise, $U(\psi(W_t)) = \psi(U(W_t))$.*

Example 8.11. If we start with $((\bullet)\between\bullet)$, then we have the following sequence:

$$\begin{array}{rll} & ((\bullet)\between\bullet) & (()\between) \\ \xrightarrow{\psi} & (\bullet)(\bullet\between) & ()(\between) \\ \xrightarrow{\psi} & \bullet((\bullet\between)) & ((\between)) \\ \xrightarrow{\psi} & ((\bullet\between))\bullet & ((\between)) \\ \xrightarrow{\psi} & (\bullet\between)(\bullet) & (\between)() \\ & \vdots & \vdots \end{array}$$

When $W_t = \bullet A_1$, then after applying ψ, the underlying bracket sequence doesn't change. Otherwise, the underlying bracket sequence changes from $U(W_t)$ to $\psi(U(W_t))$.

In the previous subsection, we chose $\frac{\ell}{d}$ subsets from

$$\left\{0, \frac{m+n+1}{d}, \cdots, \frac{(d-1)(m+n+1)}{d}\right\},$$
$$\left\{1, 1+\frac{m+n+1}{d}, \cdots, 1+\frac{(d-1)(m+n+1)}{d}\right\}, \ldots,$$
$$\left\{\frac{m+n+1}{d}-1, \frac{2(m+n+1)}{d}-1, \cdots, (m+n+1)-1\right\}$$

(where ℓ is the number of brackets). Since each of the chosen subsets contains exactly one position with index $\leq \frac{m+n+1}{d}$, applying ψ $\frac{\ell}{d}$ times to the underlying bracket sequence returns the original sequence. In other words,

$$\psi^{\frac{\ell}{d}}(U(W_t)) = U(W_t).$$

Hence the underlying bracket sequence must have order dividing $\frac{\ell}{d}$.

From now on, we will use the term "bracket sequence" to mean a balanced word consisting of only brackets and no $\bullet$'s.

Definition 8.12. For $1 \leq i \leq m$, a bracket sequence is said to have *type* $(i, m-i)$ if there are i "(", i ")", and $m-i$ "$\between$" in the sequence.

By Definition 8.12, the length of a bracket sequence of type $(i, m-i)$ is $\ell = i+i+m-i = m+i$.

Definition 8.13. For any $d \geq 1$ such that d divides $m+n+1$, let $N_d(i, m-i)$ be the number of bracket sequences of type $(i, m-i)$ whose order divides $\frac{\ell}{d} = \frac{m+i}{d}$.

Definition 8.13, coupled with the previous result about bracket positions, yields the following theorem.

Theorem 8.14. *The number of words W_t whose order divides $\frac{m+n+1}{d}$ is*

$$\sum_{i=1}^{m} \binom{\frac{m+n+1}{d}}{\frac{m+i}{d}} N_d(i, m-i),$$

where, for convenience, if $d \nmid m+i$, we let $\binom{\frac{m+n+1}{d}}{\frac{m+i}{d}} = 0$.

We proceed to obtain a closed formula for $N_d(i, m-i)$.

8.5. **Analyzing Bracket Sequences.** For convenience, if S is a balanced word, we denote the length of S by $|S|$.

Definition 8.15. A balanced word S is called *reducible* if there exist nonempty balanced sub-words S_1 and S_2 such that $S = S_1S_2$. If S is not reducible, then S is called *irreducible.*

It is easy to see from Definition 8.15 that irreducible words are either of the form (A) or $(A_1 \between A_2 \between \cdots \between A_i)$, with $i \geq 2$, where $A, A_1, \ldots, A_i$ are balanced words.

Definition 8.16. If W is a balanced word such that $W = A_1A_2$, where A_1 is irreducible, then A_1 is called the *first irreducible sub-word* of W. (A_2 may be empty.)

We present two lemmas about the length of irreducible sub-words in balanced words W when we apply ψ.

Lemma 8.17. *Let $W = UV$, where W, U, and V are balanced words. For all $1 \leq i \leq |U|-1$, the length of the first irreducible sub-word in $\psi^i(W)$ is either greater than $|V|$ or less than $|U|$.*

Proof. Let γ denote the function that maps each balanced word to the length of its first irreducible sub-word. We induct on $|U|$. The base case when $U = ()$ is trivial. (Consider the sequence $()V \to (V) \to V()$, and note that (V) is an irreducible word of length $|V|+2 > |V|$.)

Suppose the claim is true for all $W = UV$, where $|U| < k$. We will show that the claim is true when $|U| = k$. The proof will be divided into three cases.

(i) U is reducible. We can write U as U_1U_2, where U_1, U_2 are non-empty balanced words. We can apply the inductive process two times in between $U_1U_2V \to U_2VU_1 \to VU_1U_2$ to see that between each listed configuration either $\gamma > |V|$ or $\gamma < |U|$. This follows from the fact that $|U_1|, |U_2| < |U|$ and $|U_2V|, |VU_1| > |V|$. Furthermore, for the configuration U_2VU_1, $\gamma \leq |U_2| < |U|$, which implies that the claim is true in this case.

(ii) $U = (U')$, where U' is a balanced word. Applying ψ once to $W = (U')V$ maps $(U')V$ to $U'(V)$. Now, to invoke the inductive hypothesis, let $V' = (V)$. By induction, for all configurations occurring between of $U'V'$ and $V'U'$, either $\gamma < |U'| < |U|$ or $\gamma > |V'| > |V|$. Furthermore, for the initial configuration $U'V'$, $\gamma \leq |U'| < |U|$, and for the final configuration $V'U'$, $\gamma = |V'| > |V|$. Once the configuration $V'U' = (V)U'$ is reached, applying ψ once more gives $V(U') = VU$, so the claim is true in this case.

(iii) $U = (U_1 \between U_2 \between \cdots \between U_i)$. Applying ψ once to UV maps UV to $U_1(U_2 \between \cdots \between U_i \between V)$. To invoke the inductive hypothesis, let $U' = U_1$ and $V' = (U_2 \between \cdots \between U_i \between V)$. By induction on $U'V'$, for all configurations occurring between $U'V'$ and $V'U'$, either $\gamma < |U'| < |U|$ or $\gamma > |V'| > |V|$. Furthermore, for the initial configurations $U'V'$, $\gamma \leq |U'| < |U|$, and for the final configuration $V'U'$, $\gamma = |V'| > |V|$. Since $V'U' = (U_2 \between \cdots \between U_i \between V)U_1$, applying ψ once more gives $\psi(V'U') = U_2(U_3 \between \cdots \between V \between U_1)$, and reprising the argument above shows that for all configurations reached by successive applications of ψ between UV and VU, the length of the first irreducible sub-word is always less than $|U|$ or greater than $|V|$.

Therefore, the inductive hypothesis holds for $|U| = k$, and, by induction, the claim is proved for all balanced words U, as desired. $\square$

Lemma 8.18. *Let $W = UV$, where W, U, and V are balanced words. Furthermore, suppose $|V| \geq |U|$. Then for all $1 \leq i \leq |U| - 1$, the following two claims hold:*

- *If V is irreducible, then the longest irreducible sub-word in $\psi^i(W)$ has length at most $|V|$.*
- *If U is irreducible (here V is not necessarily irreducible), then the longest irreducible sub-word in $\psi^i(W)$ has length greater than $|V|$.*

Proof. Let Γ denote the function that maps each balanced word to the length of its longest irreducible sub-word. To prove this lemma, we induct on the length of U for both claims simultaneously. Again, the base is trivial because the sequence to consider is $()V \to (V) \to V()$, and (V) is an irreducible word of length greater than $|V|$.

For the inductive step, suppose both claims are true for $|U| < k$. Let $W = UV$ such that $|U| = k$ and $|V| \geq |U|$, and suppose first that $|V|$ is irreducible but U is reducible. If $U = U_1 U_2 \cdots U_i$, where $U_1, \ldots, U_i$ are all irreducible, $i \geq 2$, then we may apply the inductive hypothesis corresponding to the second claim to see that for all configurations occurring between UV and VU, namely

$$U_1 \cdots U_i V \to U_2 \cdots U_i V U_1 \to U_3 \cdots U_i V U_1 U_2 \to \cdots \to V U_1 \cdots U_i,$$

it is always true that $\Gamma \geq |V|$. This follows from the observation that, for any configuration between two listed configurations, $\Gamma > |V|$, and that, for each of the shown configurations, $\Gamma \geq |V|$. Therefore, the first claim holds for $|U| = k$ when V is irreducible and U is reducible.

We proceed to show that if U is irreducible (but V is not necessarily irreducible), then the second claim is true. This will imply that the first claim is true when U and V are both irreducible. We may write U as $(U_1 \between U_2 \between \cdots \between U_i)$, $i \geq 1$ (if $i = 1$, then $U = (U_1)$). Applying ψ once to $W = UV = (U_1 \between U_2 \between \cdots \between U_i)V$ maps UV to

$$U_1(U_2 \between \cdots \between U_i \between V).$$

Since $(U_2 \between \cdots \between U_i \between V)$ is an irreducible word of length greater than $|V|$ and $|U_1| < |U|$, applying the inductive hypothesis corresponding to the first claim to $U = U_1$, $V = (U_2 \between \cdots \between U_i \between V)$ shows that, for all configurations between $U_1(U_2 \between \cdots \between U_i \between V)$ and $(U_2 \between \cdots \between U_i \between V)U_1$, $\Gamma \geq |(U_2 \between \cdots \between U_i \between V)| > |V|$. Since this argument may be reprised to show that for all configurations between $(U_1 \between U_2 \between \cdots \between U_i)V$ and $V(U_1 \between U_2 \between \cdots \between U_i)$, $\Gamma > |V|$, the second claim holds for $|U| = k$ when U is irreducible but V is not necessarily irreducible.

It follows by induction that both claims are true for all balanced words U. Thus the lemma is proved. $\square$

Now, given $d \geq 2$, the following theorem characterizes all orbits whose lengths divide $\frac{\ell}{d}$, where ℓ is the length of the bracket sequence.

Theorem 8.19. *Any orbits whose length divides $\frac{\ell}{d}$ contains at least one of the following elements:*

- *$A \cdots A$, where A is a balanced word appearing d times.*
- *$(A_1 \between A_2 \between \cdots \between A_k \between A_1 \between \cdots \between A_k \between \cdots \between A_1 \between \cdots \between A_{k-1})A_k$, where $A_1, \ldots, A_k$ are balanced words appearing d times each.*

Proof. Let S be an element whose orbit length divides $\frac{\ell}{d}$. We may write $S = S_1S_2\cdots S_n$, where $S_1,\ldots,S_n$ are irreducible balanced words. If $|S_i| + |S_{i+1}| + \cdots + |S_j| = \frac{\ell}{d}$ for some $1 \le i \le j$ (indices are taken modulo n), then if we let $A = S_iS_{i+1}\cdots S_j$, $A\cdots A$ is in the orbit, where A appears d times. Therefore we may assume for the remainder of this proof that $|S_i|+|S_{i+1}|+\cdots+|S_j| \ne \frac{\ell}{d}$ for all $1 \le i \le j$. We will now divide into three cases based on the value of $\max\{|S_1|,\ldots,|S_n|\}$.

(i) $\max\{|S_1|,\ldots,|S_n|\} < \frac{\ell}{d}$. By Theorem 8.2, we can apply ψ an appropriate number of times to make the largest S_i appear at the front. Therefore without loss of generality we may assume $|S_1| = \max\{|S_1|,\ldots,|S_n|\}$. Since $|S_j| < \frac{\ell}{d}$ for all $1 \le j \le n$, there exists $1 \le i \le n-1$ such that $|S_1| + \cdots + |S_{i-1}| < \frac{\ell}{d}$ and $|S_1| + \cdots + |S_i| > \frac{\ell}{d}$. Applying ψ $|S_1| + \cdots + |S_{i-1}|$ times maps $S_1S_2\cdots S_n$ to $S_iS_{i+1}\cdots S_nS_1\cdots S_{i-1}$.

Since the orbit has length dividing $\frac{\ell}{d}$, the original word must appear as one of the configurations between $S_iS_{i+1}\cdots S_nS_1\cdots S_{i-1}$ and $S_{i+1}\cdots S_nS_1\cdots S_{i-1}S_i$. However, invoking Lemma 8.17 for $U = S_i$, $V = S_i\cdots S_nS_1\cdots S_{i-1}$ tells us that for all such configurations the first irreducible sub-word has length $< |S_i| \le |S_1|$ or $> |S_i\cdots S_nS_1\cdots S_{i-1}| \ge |S_1|$. Therefore, no such configuration may have first irreducible sub-word of length S_1, which is the the length of the first irreducible sub-word of the original word. Hence this case does not occur.

(ii) $\frac{\ell}{d} < \max\{|S_1|,\ldots,|S_n|\} \le \frac{(d-1)\ell}{d}$. As above, without loss of generality, we may assume that $|S_1|$ is the maximum. We write the word as S_1X, where $X = S_2\cdots S_n$, and, since S_1 is irreducible, we may write S_1 as $(A_1 \between A_2 \between \cdots \between A_i)$ (or (A), but this does not affect the analysis). Observe that the orbit contains the following elements:

$$(A_1 \between A_2 \between \cdots \between A_i)X, (A_2 \between \cdots \between A_i \between X)A_1, (A_3 \between \cdots \between X \between A_1)A_2, \cdots .$$

We may apply ψ and re-label if necessary so that for the new word $(A_1' \between A_2' \between \cdots \between A_i')X'$, $(A_1',\ldots A_i', X)$ is a cyclic permutation of $(A_1,\ldots,A_i,X)$, and $|X'| = \max\{|A_1|,\ldots,|A_i|,|X|\}$. It follows that $|X'| \ge |X| \ge \frac{\ell}{d}$ and $|X'| \le \frac{(d-1)\ell}{d}$ because if X' is one of the A_i, then it is smaller than S_1, which has length $\le \frac{(d-1)\ell}{d}$, and if X' is the old X, then since $|S_1| > \frac{\ell}{d}$, $|X'| = |X| < \frac{(d-1)\ell}{d}$. Therefore, we have $|(A_1' \between A_2' \between \cdots \between A_i')| \ge \frac{\ell}{d}$ and $|X'| \ge \frac{\ell}{d}$. Applying ψ to the word $(A_1' \between A_2' \between \cdots \between A_i')X'$ once, we obtain

$$A_1'(A_2' \between \cdots \between A_i' \between X').$$

We now show that the original word cannot occur as one of the configurations between $A_1'(A_2' \between \cdots \between A_i' \between X')$ and $(A_2' \between \cdots \between A_i' \between X')A_1'$.

Invoking Lemma 8.17 with $U = A_1'$, $V = (A_2' \between \cdots \between A_i' \between X')$, the length of the first irreducible sub-word for every such configuration is either $< |A_1'| \le |(A_1' \between A_2' \between \cdots \between A_i')|$ or $> |(A_2' \between \cdots \between A_i' \between X')| \ge |(A_1' \between A_2' \between \cdots \between A_i')|$ (since $X' \ge A_1'$). It follows that no such configuration can be identical to $(A_1' \between A_2' \between \cdots \between A_i')X'$. By successive applications of this argument, we see that the only possible configurations that could potentially match the original word are the following:

$$(A_2' \between \cdots \between A_i \between X)A_1, (A_3' \between \cdots \between A_i \between X \between A_1)A_2, \cdots .$$

Since applying ψ $\frac{\ell}{d}$ times returns the original word, the word $(A_1' \between A_2' \between \cdots \between A_i')X'$ is of the form

$$(A_1' \between A_2' \between \cdots \between A_k' \between A_1' \between \cdots \between A_k' \between \cdots \between A_1' \between \cdots \between A_{k-1}')A_k'.$$

(There are d $A_1'A_2' \cdots A_k'$.) However, this implies that $X' = A_k'$, which contradicts the assumptions $|A_k'| < \frac{\ell}{d}$ and $|X'| \geq \frac{\ell}{d}$, so this case does not occur.

(iii) $\max\{|S_1|, \ldots, |S_n|\} > \frac{(d-1)\ell}{d}$. Without loss of generality, let $|S_1| = \max\{|S_1|, \ldots, |S_n|\}$. Let $X = S_2 \cdots S_n$. Since S_1 is irreducible, we may write $S_1 X$ as either $(A)X$ or $(A_1 \between A_2 \between \cdots \between A_i)X$ (the analysis is the same either way). There are two sub-cases to consider:

(1) $\max\{|A_1|, \ldots, |A_i|, |X|\} \leq \frac{(d-1)\ell}{d}$. In this case, we may apply ψ an appropriate number of times to obtain the word $(A_1' \between A_2' \between \cdots \between A_i')X'$, where $(A_1', \ldots A_i', X)$ is a cyclic permutation of $(A_1, \ldots, A_i, X)$ and $|X'| = \max\{|A_1|, \ldots, |A_i|, |X|\}$. Therefore, it follows from the same reasoning we used in case (ii) that one of the elements in the orbit must be of the form

$$(A_1' \between A_2' \between \cdots \between A_k' \between A_1' \between \cdots \between A_k' \between \cdots \between A_1' \between \cdots \between A_{k-1}')A_k',$$

where $A_1', \ldots A_k'$ appears d times each.

(2) $\max\{|A_1|, \ldots, |A_i|, |X|\} > \frac{(d-1)\ell}{d}$. In this case, suppose $|A_j| = \max\{|A_1|, \ldots, |A_i|, |X|\}$, where $1 \leq j \leq i$ (X can't be the longest among $\{A_1, \ldots, A_i, X\}$ because the assumption $|S_1| > \frac{(d-1)\ell}{d}$ implies that $|X| < \frac{\ell}{d}$). We may apply ψ an appropriate number of times to obtain the word

$$(A_{j+1} \between \cdots \between A_i \between X \between A_1 \between \cdots \between A_{j-1})A_j.$$

Note that $|A_j| < |(A_1 \between \cdots \between A_i)|$, so we have successfully reduced the length of the maximal irreducible sub-word. Therefore, we may continue this process until the word we obtain belongs to one of the previous cases, which we have handled already. This completes the proof.

$\square$

8.6. **The Generating Functions f and g.** In this section, we define two bivariate generating functions $f(x, y)$ and $g(x, y)$ that will help us demonstrate the cyclic sieving phenomenon. Recall that by Definition 8.13, $N_1(i, m-i)$ is the total number of balanced words of type $(i, m-i)$. Analogously, for $m \geq 1$, $1 \leq i \leq m$, let $U_1(i, m-i)$ be the number of irreducible balanced words of type (i, j).

Definition 8.20. Let $f(x, y)$ and $g(x, y)$ be defined as follows:

$$f(x, y) = 1 + \sum_{1 \leq m, 1 \leq i \leq m} N_1(i, m-i) x^i y^{m-i}.$$

$$g(x, y) = 1 + \sum_{1 \leq m, 1 \leq i \leq m} U_1(i, m-i) x^i y^{m-i}.$$

Remark 8.21. The coefficient $[x^i y^j] f(x, y)$ gives the total number of balanced words of type (i, j), where by convention the number of balanced words of type $(0, 0)$ is 1, representing the empty word, and the coefficient $[x^i y^j] g(x, y)$ gives the total number of irreducible balanced words of type (i, j).

We proceed to compute f and g explicitly. We start by identifying two relationships between the functions rooted in their definitions.

Proposition 8.22. $f = \frac{1}{2-g}$.

Proof. Every non-empty balanced word S can be decomposed as $S = S_1 S_2 \cdots S_i$, where $S_1, \cdots, S_i$ are nonempty irreducible balanced words. It follows immediately that

$$f = 1 + (g-1) + (g-1)^2 + \cdots = \frac{1}{1-(g-1)} = \frac{1}{2-g},$$

as desired. ☐

Proposition 8.23. $g = 1 + \frac{xf}{1-yf}$.

Proof. Every irreducible balanced word of type (i, j), $i \geq 1$, is of the form

$$(A), (A \between B), (A \between B \between C), \ldots$$

Hence the number of irreducible balanced word of type (i, j) is equal to

$$[x^{i-1}y^j]f + [x^{i-1}y^{j-1}]f^2 + [x^{i-1}y^{j-2}]f^3 + \cdots = [x^i y^j](xf + xyf + xy^2 f^3 + \cdots)$$

It follows that

$$g = 1 + xf + xyf^2 + xy^2 f^3 + \cdots = 1 + xf(1 + yf + y^2 f^2 + \cdots),$$

which simplifies to

$$g - 1 = \frac{xf}{1-yf},$$

as desired. ☐

Since $f = \frac{1}{2-g}$, we have $g = 2 - \frac{1}{f}$. Substituting this into the second relation yields

$$\frac{f-1}{f} = \frac{xf}{1-yf},$$

which simplifies to

$$(x+y)f^2 - (1+y)f + 1 = 0.$$

Solving for f as a solution of the quadratic equation, we obtain

$$f = \frac{(1+y) \pm \sqrt{(1+y)^2 - 4(x+y)}}{2(x+y)}$$

It is easy to see that the sole valid solution is

$$f = \frac{(1+y) - \sqrt{(1-y)^2 - 4x}}{2(x+y)}$$

because the alternative does not have a power series expansion at $x = y = 0$. Substituting this solution into $g = 2 - \frac{1}{f}$ yields

$$g = \frac{3 - y - \sqrt{(1-y)^2 - 4x}}{2}$$

Remark 8.24. If we let $y = 0$, f becomes the generating function of the Catalan numbers because the number of balanced words of type $(i, 0)$ is C_i by definition.

8.7. **Cyclic Sieving Phenomenon for d dividing m.** In this section, we first show that it suffices to demonstrate the cyclic sieving phenomenon for d dividing m, and then we reduce the cyclic sieving phenomenon in that case to verification of a combinatorial identity.

Recall that, by Theorem 8.19, every orbit contains an element of the form $A \cdots A$, where A appears d times, or $(A_1 \between A_2 \between \cdots \between A_k \between A_1 \between \cdots \between A_k \between \cdots \between A_1 \between \cdots \between A_{k-1})A_k$, where there are d $A_1 \cdots A_k$. In the former case, if A is of type (a, b), then $m = d(a+b) \equiv 0 \pmod{d}$, so d must divide m. In the latter case, if $A_1 \cdots A_k$ is of type (a, b), then the word must be of type $(1, di-2) + d(a, b) = (da+1, db+di-2)$, which implies that $m = d(a+b+i) - 1 \equiv -1 \pmod{d}$. Since d divides $m+n+1$, it follows that d divides n. Therefore, it suffices by symmetry to restrict our attention to the case in which d divides m.

Let $m = dr$. Given an orbit whose length divides $\frac{\ell}{d}$, there must exist an element of the form $A \cdots A$ in the orbit, where A appears d times. The type of the bracket sequence in the orbit must be (da, db), where (a, b) is the type of A. Therefore, the only types for which there can exist a bracket sequence of order $\frac{\ell}{d}$ are $(dr, 0), (dr-d, d), \ldots, (d, dr-d)$.

Suppose for the type $(di, d(r-i))$, $1 \leq i \leq r$, there are a_i words of that type whose orbits have lengths dividing $\frac{\ell}{d}$ $(a_i = N_d(di, d(r-i)))$. Then by Proposition 8.14, the total number of words (with $\bullet$'s) that have orbit size dividing $\frac{m+n+1}{d} = r + \frac{n+1}{d}$ is

$$\sum_{i=1}^{r} a_i \binom{r + \frac{n+1}{d}}{r+i} = a_r \binom{r + \frac{n+1}{d}}{2r} + a_{r-1} \binom{r + \frac{n+1}{d}}{2r-1} + \cdots + a_1 \binom{r + \frac{n+1}{d}}{r+1}.$$

If cyclic sieving occurs, we must have

$$\sum_{i=1}^{r} a_i \binom{r + \frac{n+1}{d}}{r+i} = \binom{r + \frac{n+1}{d}}{r} \binom{r + \frac{n+1}{d} - 1}{r}.$$

Let $X = \binom{r+\frac{n+1}{d}}{r}$. Since we have

$$\binom{r + \frac{n+1}{d}}{r+i} = X \cdot \frac{\binom{\frac{n+1}{d}}{i}}{\binom{r+i}{i}},$$

if we let $c = \frac{n+1}{d}$, it remains to show

$$\sum_{i=1}^{r} \binom{c}{i} \frac{a_i}{\binom{r+i}{i}} = \binom{c + (r-1)}{r}.$$

But by the binomial identity

$$\binom{c + (r-1)}{r} = \sum_{i=1}^{r} \binom{r-1}{i-1} \binom{c}{i},$$

since $c = \frac{n+1}{d}$ can vary, it is equivalent to show

$$\frac{a_i}{\binom{r+i}{i}} = \binom{r-1}{i-1},$$

or

$$a_i = \binom{r+i}{i} \binom{r-1}{i-1},$$

for all $1 \leq i \leq r$. That is the task of the next section.

8.8. **Computing a_i.** We will now compute a_i, the number of elements of type $(di, d(r-i))$ whose orbits have length dividing $\frac{\ell}{d}$. Since $d|m$, by Theorem 8.19, the orbit must contain an element of the form $A \cdots A$, where A appears d times.

Suppose $A = A_1 A_2 \cdots A_i$, where $A_1, A_2, \ldots, A_i$ are irreducible, and consider the sequence

$$A_1 \cdots A_i \underbrace{A \cdots A}_{d-1\ A\text{'s}} \xrightarrow{\psi^{|A_1|}} A_2 \cdots A_i \underbrace{A \cdots A}_{d-1\ A\text{'s}} A_1 \xrightarrow{\psi^{|A_2|}} A_3 \cdots A_i \underbrace{A \cdots A}_{d-1\ A\text{'s}} A_1 A_2 \xrightarrow{\psi^{|A_3|}} \cdots \xrightarrow{\psi^{|A_i|}} \underbrace{A \cdots A}_{d-1\ A\text{'s}} A_1 \cdots A_i.$$

For all configurations occurring between any two listed configurations, it follows from Lemma 8.18 that the longest irreducible sub-word has length $> \frac{\ell}{d}$. Therefore, no such configuration can be a representative element of the form $B \cdots B$, where B is a balanced word appearing d times. We claim that the length of the orbit divided by the number of representative elements of the form $B \cdots B$ contained in the orbit is $\frac{\ell}{id}$. To see this, we write $A_1 \cdots A_i = (A_1 \cdots A_{i/s}) \times s$, where $s \geq 1$ and $W \times s$ denotes the concatenation of s copies of W, in such a way that the order of the i/s-tuple $(A_1, \cdots, A_{i/s})$ under cyclic permutation is i/s. Then the orbit of the whole word has length $\frac{\ell}{sd}$, there are i/s representative elements in the orbit, so the length of the orbit divided by the number of representative elements is $\frac{\ell/(sd)}{i/s} = \frac{\ell}{id}$, as desired.

It should now be easy to see that since the word is of type $(di, d(r-i))$, A is of type $(i, r-i)$, and $\ell = d(r+i)$, the number a_i is the coefficient of the term $x^i y^{r-i}$ in the generating function $\left((g-1) + \frac{(g-1)^2}{2} + \frac{(g-1)^3}{3} + \cdots\right)(r+i)$.

To compute the coefficient of $x^i y^j$ in $\left((g-1) + \frac{(g-1)^2}{2} + \frac{(g-1)^3}{3} + \cdots\right)$, we make use of the Lagrange inversion formula. Recall the following relations from Proposition 8.22 and Proposition 8.23:

$$\begin{aligned} g - 1 &= \frac{xf}{1 - yf} \\ f &= \frac{1}{2 - g} \end{aligned}$$

Substituting the second relation into the first relation yields

$$g - 1 = \frac{x}{2 - g - y}.$$

Let $g' = g - 1$. The relation becomes

$$g' = \frac{x}{1 - y - g'}$$

or

$$x = g'(1 - y - g').$$

Let $s(x) = x(1 - y - x)$; then $s(x) = g'^{-1}(x)$. The Lagrange inversion formula tells us that

$$[x^n] g'^m = \frac{m}{n} [x^{n-m}] \left(\frac{x}{x(1 - y - x)} \right)^n.$$

Therefore,

$$[x^n] \frac{g'^m}{m} = \frac{1}{n} [x^{n-m}] \left(\frac{1}{1 - x - y} \right)^n.$$

Since

$$(1-(x+y))^{-n} = \sum_{i=0}^{\infty} \binom{n+i-1}{i} (x+y)^i,$$

the coefficient for x^{n-m} in $(1-(x+y))^{-n}$ is equal to

$$\sum_{i=0}^{\infty} \binom{n+i-1}{i} \binom{i}{n-m} y^{i-(n-m)} x^{n-m}.$$

Hence the coefficient $[x^n]\left(\sum_{m=1}^{\infty} \frac{g'^m}{m}\right)$ is equal to

$$\sum_{m=1}^{\infty} \frac{1}{n} \sum_{i=0}^{\infty} \binom{n+i-1}{n-1} \binom{i}{n-m} y^{i-(n-m)} = \sum_{m=1}^{\infty} \sum_{i=0}^{\infty} \frac{\binom{n+i}{n}\binom{i}{n-m}}{n+i} y^{i-(n-m)}.$$

To extract the coefficient of y^r for each m, we let $i-(n-m) = r$, i.e. $i = n-m+r$. The coefficient of y^r in the above sum is equal to

$$\sum_{m=1}^{\infty} \frac{\binom{2n+r-m}{n}\binom{n-m+r}{n-m}}{2n+r-m} = \sum_{m=1}^{\infty} \frac{\binom{2n-m+r}{n}\binom{n-m+r}{r}}{2n+r-m}. \tag{8.1}$$

We will now use hyper-geometric identities as described in [10] to show that this sum is equal to $\frac{\binom{2n+r}{n}\binom{n+r-1}{n-1}}{2n+r}$.

Let $C_m = \frac{\binom{2n-m+r}{n}\binom{n-m+r}{r}}{2n+r-m}$. Then

$$\begin{aligned}
\frac{C_{m+1}}{C_m} &= \frac{(m-n)}{m-(2n+r-1)} \\
&= \frac{(m-n)(m+1)}{(m-(2n+r-1))(m+1)}
\end{aligned}$$

Thus, it suffices to compute

$${}_2F_1\begin{pmatrix} -n, & 1 \\ & -(2n+r-1) \end{pmatrix}.$$

From Section 2 in [10], it is given that

$${}_2F_1\begin{pmatrix} -n, & -b \\ & c \end{pmatrix} = \frac{(c+b)_n}{(c)_n},$$

where the $(\alpha)_n$ are the shifted factorials defined by

$$(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1), \qquad (\alpha)_0 = 1.$$

Therefore,

$${}_2F_1\begin{pmatrix} -n, & 1 \\ & -(2n+r-1) \end{pmatrix} = \frac{(-(2n+r-1)-1)_n}{(-(2n+r-1))_n} = \frac{2n+r}{n+r}.$$

Hence the right-hand side in Equation 8.1 is equal to $C_0 \cdot \left(\frac{2n+r}{n+r} - 1\right)$, which is equal to

$$\frac{\binom{2n+r}{n}\binom{n+r}{n}}{2n+r} \cdot \frac{n}{n+r} = \frac{\binom{2n+r}{n}\binom{n+r-1}{n-1}}{2n+r}.$$

This is the coefficient of $[x^n y^r]$ in $(g-1) + \frac{(g-1)^2}{2} + \cdots$.

Since we demonstrated that a_i is the coefficient of $[x^i y^{r-i}]$ in $\left((g-1) + \frac{(g-1)^2}{2} + \cdots\right)(r+i)$, letting $n = i$ and $r = r-i$ in our formula above gives

$$a_i = \frac{\binom{r+i}{i}\binom{r-1}{i-1}}{r+i} \cdot (r+i) = \binom{r+i}{i}\binom{r-1}{i-1}.$$

Indeed, this is exactly what we wanted to show. Therefore, the cyclic sieving phenomenon holds, as desired.

Remark 8.25. It has been verified via Kevin Dilks's Maple code that cyclic sieving does not occur in the poset $[3] \times [3] \times [3]$. Furthermore, the order of the Fon-Der-Flaass action is 33 for the poset $[4] \times [4] \times [4]$, so, in particular, it is not true in general that the order of the Fon-Der-Flaass action is $m + n + k - 1$ for the poset $[m] \times [n] \times [k]$ (although it is conjectured by Cameron and Fon-Der-Flaass that if $m + n + k - 1$ is prime, then the order of the Fon-Der-Flaass action is divisible by $m + n + k - 1$ – and they have proved this for all posets in which k exceeds $(m-1)(n-1)$).

9. Proof of Theorem 1.2 for the Second Infinite Family

In this section, we demonstrate that the cyclic sieving phenomenon holds for $m = 2$ for minuscule posets that belong to the second infinite family, i.e. posets of the form $([n+1] \times [n+1])/S_2$. The fact that the triple $(J(([n+1] \times [n+1])/S_2 \times [2]), J(([n+1] \times [n+1])/S_2 \times [2]; q), \Psi)$ exhibits the cyclic sieving phenomenon is obtained as a consequence of the fact that $(J(([n+1] \times [n+1]) \times [2]), J(([n+1] \times [n+1]) \times [2]; q), \Psi)$ cyclic sieving phenomenon. From the Bender-Knuth formula (Theorem 7.7), we see that the rank-generating function for $J(([n+1] \times [n+1])/S_2 \times [2])$ is $\begin{bmatrix} 2n+3 \\ n+1 \end{bmatrix}_q$. It follows by Lemma 8.4 that if q is a $(2n+3)^{\text{th}}$ root of unity, substituting q into the Bender-Knuth expression yields 0 unless $q = 1$. It therefore suffices to show that all orbits of order ideals of $([n+1] \times [n+1])/S_2 \times [2]$ are free of length $2n+3$.

Assume for the sake of contradiction that there exists an orbit of length $\frac{2n+3}{d}$, where $d > 1$. Since the Fon-Der-Flaass action on plane partitions viewed as order ideals of $[n+1] \times [n+1] \times [2]$ restricts to the Fon-Der-Flaass action on symmetric plane partitions viewed as order ideals of $([n+1] \times [n+1])/S_2 \times [2]$, this orbit must also be an orbit of order ideals of $[n+1] \times [n+1] \times 2$ under the Fon-Der-Flaass action. However, by Section 8.2, we see that $d | n+1$. Then $\gcd(n+1, 2n+3) = 1$ contradicts the assumption that $d > 1$, so all the orbits are free of length $2n+3$. This confirms that the cyclic sieving phenomenon holds, as desired.

Remark 9.1. It has been verified via Dilks's Maple code that the cyclic sieving does not occur in the poset $([6] \times [6])/S_2 \times [4]$. Every poset of the form $([n+1] \times [n+1])/S_2 \times [3]$ (that we tested!) was found to obey the cyclic sieving phenomenon, however, and it is tempting to conjecture that the cyclic sieving phenomenon holds for all such posets, but we do not attempt to provide a proof.

10. Proof of Theorem 1.2 for the Third Infinite Family

In this section, we demonstrate that the cyclic sieving phenomenon holds for all positive integers m for minuscule posets that belong to the third infinite family. Recall from the introduction that the third infinite family consists of posets of the form $J^{n-3}([2] \times [2])$. We may describe these posets equivalently as the posets of the form $[r] \oplus ([1]+[1]) \oplus [r]$, for it is easily checked that $[r] \oplus ([1]+[1]) \oplus [r] = J^{n-2}([2] \times [2])$ (cf. [12], Chapter 3, for a description of the relevant notation). As Cameron and Fon-Der-Flaass did for the case $[m] \times [n] \times [2]$, we present a bijection from order ideals of $([r] \oplus ([1]+[1]) \oplus [r]) \times [m]$ to bracket sequences that is equivariant with respect to the Fon-Der-Flaass action on order ideals and a variant of ψ on bracket sequences that we denote by ψ'. We were motivated to make this construction by a new bijection from order ideals of $[m] \times [n] \times [2]$ to bracket sequences found in [16], which the authors of that preprint devised after seeing our work in section 8.

Remark 10.1. The choice of the symbol ψ' to denote the variant of ψ is deliberate, for the action of ψ' on bracket sequences meaningfully resembles the action of ψ given by the Cameron-Fon-Der-Flaass defined in section 8.

We draw the poset $([r] \oplus ([1]+[1]) \oplus [r]) \times [m]$ as two $(r+1) \times m$ rectangles, one northeast of the other, such that if $p < p'$ is a covering relation in the poset, then p' is either northeast or northwest of p. Call a path starting from the west of the west-most vertex and ending to the east of the east-most vertex a "boundary path" if it consists of directed segments running northeast or southeast. We view each order ideal of $([r] \oplus ([1]+[1]) \oplus [r]) \times [m]$ as determining two "boundary paths," one of which separates the elements in the order ideal from the elements not in the order ideal in the northeast rectangle, and the other of which separates the elements in the order ideal from the elements not in the order ideal in the southwest rectangle. We call the former path the "upper boundary path" and the latter path the "lower boundary path." To ensure that, no matter the order ideal, the boundary paths are always well-defined, we impose the following condition:

- The two boundary paths coincide unless it is necessary for them to separate.

For example, the boundary paths for the order ideal are shown below:

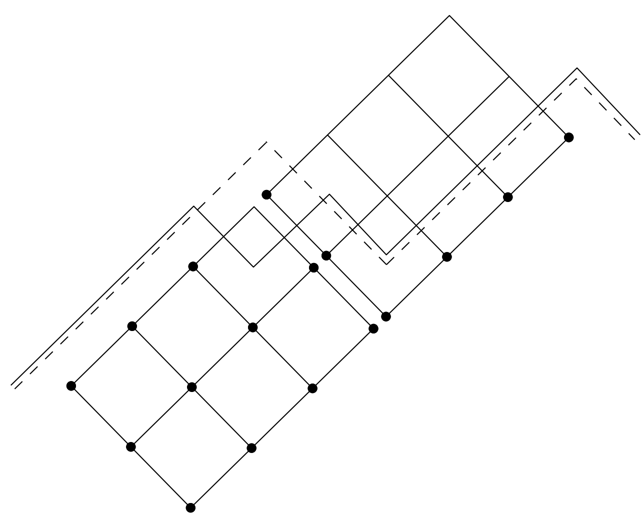

FIGURE 4. Boundary paths for an order ideal in $([3] \oplus ([1]+[1]) \oplus [3]) \times [3]$.

It should be clear that this condition implies the following:

Lemma 10.2. *The two boundary paths coincide outside the area immediately bordering the "double-line." (What we call the double-line is determined by the near-coincidence of the northeast edge of the southwest rectangle and the southwest edge of the northeast rectangle and is indicated in the above diagram.)*

We represent each boundary path by a binary sequence of length $m+2r+1$. The procedure is simple: the segments of the boundary path pointing toward the Northeast correspond to 1's and the segments pointing toward the Southeast correspond to 0's. The order of the binary sequence is given by the order of the segments in the boundary path.

Given upper and lower boundary paths expressed as binary sequences, we convert the data into a bracket sequence as follows.

- If there are $2k+1$ $\begin{pmatrix}1\\1\end{pmatrix}$, we replace the first k instances by "(", the last k instances by ")", and the middle instance by "$)\!\!($". If there are $2k$ $\begin{pmatrix}1\\1\end{pmatrix}$, then we replace the first k instances by "(", and the last k instances by ")".
- We replace every instance of $\begin{pmatrix}0\\1\end{pmatrix}$ by "(".
- We replace every instance of $\begin{pmatrix}1\\0\end{pmatrix}$ by ")".
- We replace every instance of $\begin{pmatrix}0\\0\end{pmatrix}$ by "$\bullet$".

The next theorem delineates some of the basic properties of bracket sequences.

Theorem 10.3. *The bracket sequences satisfy the following properties:*

(i) The bracket sequences are balanced in the sense of section 8.
(ii) Each bracket sequence contains either m $\bullet$'s or $(m-1)$ $\bullet$'s.
(iii) If a bracket sequence contains exactly $(m-1)$ $\bullet$'s, there must be exactly one instance of $\begin{pmatrix}0\\1\end{pmatrix}$ and exactly one instance of $\begin{pmatrix}1\\0\end{pmatrix}$ in the binary sequences from which it arises. Furthermore, these instances must be preceded by exactly r instances of $\begin{pmatrix}1\\1\end{pmatrix}$ and followed by exactly r instances of $\begin{pmatrix}1\\1\end{pmatrix}$.

Proof. We first prove *(ii)* and *(iii)*.

To prove *(ii)*, note that, if the two boundary paths coincide, then there are m $\bullet$'s. Therefore, there can be no instances of $\begin{pmatrix}0\\1\end{pmatrix}$ or $\begin{pmatrix}1\\0\end{pmatrix}$ in the binary sequences. On the other hand, if the two paths are different, then they must fork immediately below the "double line" and rejoin immediately above the "double line," so the subsequences where they differ must be of the form either

$$\begin{aligned} U &: 100\cdots 00 \\ L &: 000\cdots 01 \end{aligned}$$

or

$$\begin{aligned} U &: 00\cdots 01 \\ L &: 10\cdots 00. \end{aligned}$$

Either way, we obtain exactly one instance of $\begin{pmatrix}1\\0\end{pmatrix}$ and exactly one instance of $\begin{pmatrix}0\\1\end{pmatrix}$, which in turn implies there are exactly $m-1$ $\bullet$'s. Hence *(ii)* and *(iii)* are proved.

It should be clear that *(i)* follows immediately. □

We will now define a map on order ideals of the poset conjugate in the toggle group to the Fon-Der-Flaass action.

Definition 10.4. *Promotion* Let ψ' be the map on order ideals of $([r]\oplus([1]+[1])\oplus[r])\times[m]$ that toggles all the elements of the poset in succession, north-south column by north-south column, from west to east.

Remark 10.5. When we say ψ' toggles the elements from west to east, we mean that if a is west of a', then a is toggled before a' under ψ'.

Lemma 10.6. *If the two boundary paths of an order ideal coincide, then after applying ψ', the boundary paths of the new order ideal coincide. Furthermore, ψ' acts on the binary sequences by left rotation.*

Proof. Treating the two boundary paths as one path, we may represent the order ideal as a $2r+1$-tuple:

$$(\underbrace{a_1,\ldots,a_1}_{n_1},\underbrace{a_2,\ldots,a_2}_{n_2},\cdots,\underbrace{a_{i-1},\ldots,a_{i-1}}_{n_{i-1}},\underbrace{0,\ldots,0}_{n_i}),$$

where $\sum_{j=1}^{i} n_j = 2r+1$. An example is shown in the diagram below:

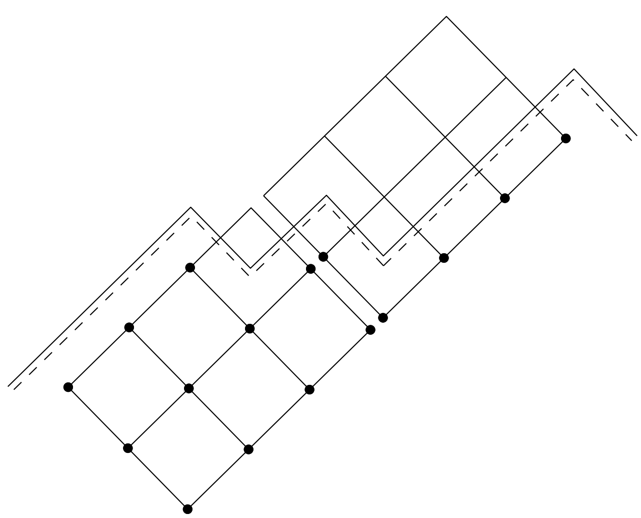

FIGURE 5. Two boundary paths coincide, represented by the tuple $(3,3,3,2,1,1,1)$.

If $a_1 = m$, then the identical binary sequences are:

$$\underbrace{1\cdots1}_{n_1}\underbrace{0\cdots0}_{a_1-a_2}\cdots\underbrace{1\cdots1}_{n_{i-1}}\underbrace{0\cdots0}_{a_{i-1}}\underbrace{1\cdots1}_{n_i}.$$

Applying ψ' deletes the southwest-most elements of the order ideal and shifts what is left of the "stair-case" southwest by one element. The resulting binary sequences are:

$$\underbrace{1\cdots1}_{n_1-1}\underbrace{0\cdots0}_{a_1-a_2}\cdots\underbrace{1\cdots1}_{n_{i-1}}\underbrace{0\cdots0}_{a_{i-1}}\underbrace{1\cdots1}_{n_i+1},$$

which are cyclic rotations of the original sequences.

If $a_1 < m$, then the identical binary sequences are:

$$\underbrace{0\cdots0}_{m-a_1}\underbrace{1\cdots1}_{n_1}\underbrace{0\cdots0}_{a_1-a_2}\cdots\underbrace{1\cdots1}_{n_{i-1}}\underbrace{0\cdots0}_{a_{i-1}}\underbrace{1\cdots1}_{n_i}.$$

Applying ψ' shifts the stair-case northwest by one element and brings all the southeast-most elements of the poset into the order ideal. The resulting binary sequences are:

$$\underbrace{0\cdots0}_{m-a_1-1}\underbrace{1\cdots1}_{n_1}\underbrace{0\cdots0}_{a_1-a_2}\cdots\underbrace{1\cdots1}_{n_{i-1}}\underbrace{0\cdots0}_{a_{i-1}}\underbrace{1\cdots1}_{n_i}0,$$

which are again cyclic rotations of the original sequences, as desired. $\square$

Lemma 10.7. *If the upper and lower binary sequences of an order ideal do not coincide, ψ' shifts the positions of $\begin{pmatrix}0\\0\end{pmatrix}$ to the left by 1 $\pmod{2r+m+1}$. Furthermore, if the binary sequences start with an instance of $\begin{pmatrix}1\\1\end{pmatrix}$, then ψ' reverses the relative order in which $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ appear in the sequence. Otherwise, if the binary sequences start with an instance of $\begin{pmatrix}0\\0\end{pmatrix}$, then the relative order of $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ after ψ' is applied remains unchanged.*

Proof. We first consider the case in which the upper boundary path is above the lower boundary path (i.e. $\begin{pmatrix}1\\0\end{pmatrix}$ occurs before $\begin{pmatrix}0\\1\end{pmatrix}$). For any column outside the area immediately bordering the "double line," the upper path and the lower path are coincident. As with the previous lemma, we may show that ψ' shifts the positions of $\begin{pmatrix}0\\0\end{pmatrix}$ occurring in the segments where the paths coincide to the left by one position. Consider the columns (L_1, L_2, U_1, U_2) as shown in the diagram below.

Since the upper path lies above the lower path, $L_2 < U_1$. If we start and end at the positions indicated in the diagram above, our binary sequences sequences are

$$\begin{aligned} U &: 1\overbrace{0\cdots0}^{L_1-U_1}1\overbrace{0\cdots0}^{U_1-L_2}\overbrace{0\cdots0}^{L_2-U_2} \\ L &: 1\underbrace{0\cdots0}_{L_1-U_1}\underbrace{0\cdots0}_{U_1-L_2}1\underbrace{0\cdots0}_{L_2-U_2} \end{aligned}$$

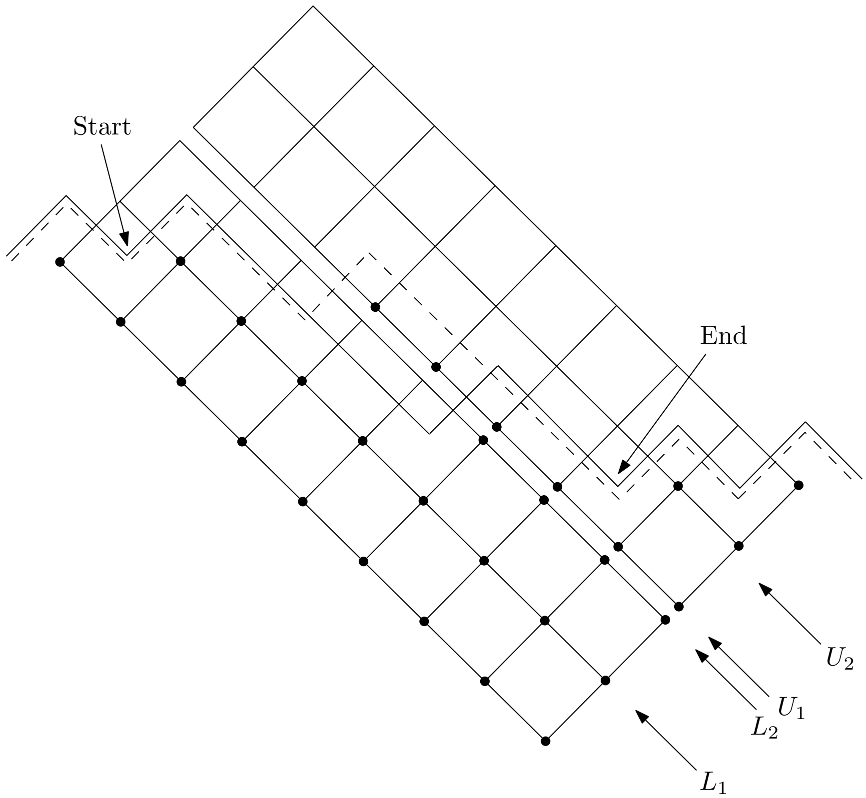

FIGURE 6. The upper path lies above the lower path, with $(L_1, L_2, U_1, U_2) = (8, 4, 6, 2)$.

There are two cases to consider. If the full binary sequences start with an instance of $\binom{1}{1}$, then, under ψ', $L_1 \mapsto U_1$, $L_2 \mapsto L_2 + 1$, and $U_1 \mapsto U_2$. The resulting binary sequences for this segment are:

$$\begin{array}{lcl} U' & : & \overbrace{0\cdots0}^{L_1-U_1}1\;\overbrace{0\cdots0}^{U_1-L_2-1}\;\overbrace{0\cdots0}^{L_2-U_2+1}\;1 \\ L' & : & \underbrace{0\cdots0}_{L_1-U_1}1\;\underbrace{0\cdots0}_{U_1-L_2-1}\;1\;\underbrace{0\cdots0}_{L_2-U_2+1} \end{array}$$

It should be clear that, in this case, ψ' shifted the positions of $\binom{0}{0}$ to the left by one position and reversed the relative order of $\binom{0}{1}$ and $\binom{1}{0}$.

On the other hand, suppose the full binary sequences start with an instance of $\binom{0}{0}$; then, under ψ', $L_1 \mapsto L_1 + 1$, $L_2 \mapsto L_2 + 1$, $U_1 \mapsto U_1 + 1$, and $U_2 \mapsto U_2 + 1$. The resulting path

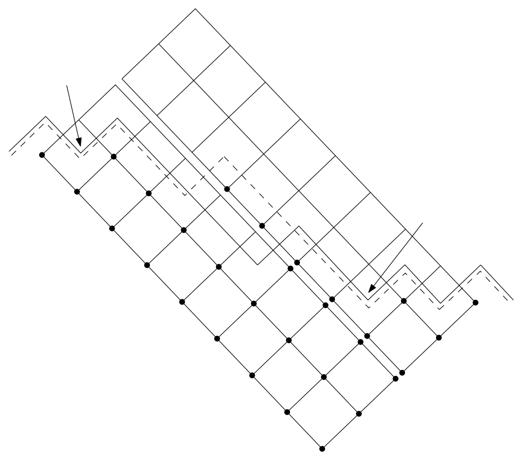

FIGURE 7. Example of promotion when the full binary sequences start with an instance of $(1,1)^t$.

sequences for this segment are:

$$\begin{array}{rcl} U' & : & \overbrace{0\cdots0}^{L_1-U_1}\,1\,\overbrace{0\cdots0}^{U_1-L_2}\overbrace{0\cdots0}^{L_2-U_2}\,1 \\ L' & : & \underbrace{0\cdots0}_{L_1-U_1}\underbrace{0\cdots0}_{U_1-L_2}\,1\,\underbrace{0\cdots0}_{L_2-U_2}\,1, \end{array} \tag{10.1}$$

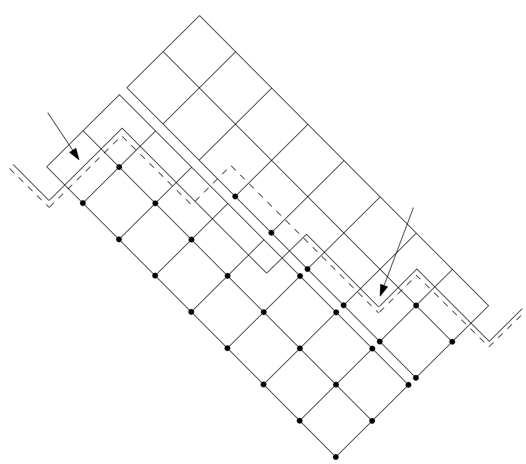

FIGURE 8. Example of promotion when the full binary sequences start with an instance of $(0,0)^t$.

and it is again clear that ψ' shifted the positions of $\begin{pmatrix}0\\0\end{pmatrix}$ to the left by one position, but this time it left the relative order of $\begin{pmatrix}0\\1\end{pmatrix}$ and $\begin{pmatrix}1\\0\end{pmatrix}$ unchanged. Thus the lemma is proved for the cases in which the upper path lies above the lower path.

However, analogous reasoning suffices to prove the lemma for the cases in which the lower path lies above the upper path, so this completes the proof.

□

Combining the two lemmas above, we arrive at the following theorem.

Theorem 10.8. *Each time we apply ψ' to an order ideal, the corresponding bracket sequence changes as follows:*

(i) The positions of the $\bullet$'s, when treated as a subset of $[m+2r+1]$, shift to the left by $1 \pmod{m+2r+1}$.
(ii) The bracket sequence obtained by deleting the leftmost $r-1$ instances of "(" and the rightmost $r-1$ instances of ")", as well as all $\bullet$'s, changes in accordance with the Cameron-Fon-Der-Flaass rule.

Remark 10.9. The "underlying" bracket sequence from *(ii)* is always among ()(), (()), and $(\between)$.

It should be clear that the map from order ideals to balanced bracket sequences satisfying the requisite conditions (i.e., to balanced bracket sequences containing at least r instances of "(" and at least r instances of ")") is invertible. It follows that this map is bijective. Therefore, for every positive integer $d|m+2r+1$, the number of order ideals whose orbits under ψ' have lengths dividing $\frac{m+2r+1}{d}$ is equal to the number of bracket sequences whose orbits under ψ' have lengths dividing $\frac{m+2r+1}{d}$.

Finally, we are ready to demonstrate the cyclic sieving. Suppose we wish to compute the number of bracket sequences whose orbits under ψ' have lengths dividing $\frac{m+2r+1}{d}$ for some positive integers d dividing $m+2r+1$. We first consider the positions of the $\bullet$'s, and we note that, given such a bracket sequence, after applying ψ' $\frac{m+2r+1}{d}$ times, the positions of the $\bullet$'s must be the same as in the original bracket sequence. There are two cases:

(i) There are m $\bullet$'s. This implies $d|m$. By reasoning analogous to that employed in the analyses of $\bullet$ positions in the $[m]\times[n]\times[2]$ case, we see that there are $\binom{\frac{m+2r+1}{d}}{\frac{m}{d}}$ ways to choose the positions of the $\bullet$'s. Since the two boundary paths coincide, it follows from the Remark 10.9 that the underlying bracket sequence is $(\between)$, which is left unchanged after ψ' is applied. Therefore, the number of bracket sequences arising in this case is $\binom{\frac{m+2r+1}{d}}{\frac{m}{d}}$.

(ii) There are $m-1$ $\bullet$'s. This implies $d|m-1$. There are $\binom{\frac{m+2r+1}{d}}{\frac{m-1}{d}}$ ways to choose the positions of the $\bullet$'s. It follows from Remark 10.9 that the underlying bracket sequence is either ()() or (()). Since there are $\frac{m-1}{d}$ $\bullet$'s in the first $\frac{m+2r+1}{d}$ positions; each time we apply ψ' the positions of $\bullet$'s shift to the left by one position, and the bracket sequence only changes when the left most character is not a $\bullet$, the bracket sequence changes

$$\frac{m+2r+1}{d} - \frac{m-1}{d} = \frac{2r+2}{d}$$

times over the course of $\frac{m+2r+1}{d}$ applications of ψ'. Since $\frac{m+2r+1}{d}$ applications of ψ' returns the original bracket sequence, it follows that $\frac{2r+2}{d}$ must be even, which in turn implies that $\frac{r+1}{d}$ is an integer. Because there are two ways to choose the underlying bracket sequence, the total number of bracket sequences arising in this case is

$$2\binom{\frac{m+2r+1}{d}}{\frac{m-1}{d}}.$$

It remains to show this matches the results given by substituting the appropriate roots of unity into the rank-generating function for $J(([r]\oplus([1]+[1])\oplus[r])\times[m])$. Since $[r]\oplus([1]+[1])\oplus[r]$ is Gaussian,

$$J(([r]\oplus([1]+[1])\oplus[r])\times[m];q)=\begin{bmatrix}m+2r+1\\m\end{bmatrix}_q\left(\frac{1-q^{m+r+1}}{1-q^{r+1}}\right).$$

We wish to compute the number of elements whose order divides $\frac{m+2r+1}{d}$, so we let q be a primitive d^{th} root of unity and divide into four cases:

(i) $d|m$. We first demonstrate that $1-q^{r+1}\neq 0$. Suppose $1-q^{r+1}=0$; then d divides $r+1$. Also, since d divides m and d divides $m+2r+1$, d divides $2r+1$. This implies that $d=1$, which contradicts the assumption that $d\geq 2$. Therefore, $1-q^{m+r+1}=1-q^{r+1}\neq 0$. Substituting q into $\begin{bmatrix}m+2r+1\\m\end{bmatrix}_q$, the expression evaluates to $\binom{\frac{m+2r+1}{d}}{\frac{m}{d}}$.

(ii) $d|m-1$ and $d|r+1$. The rank-generating function is

$$\begin{bmatrix}m+2r+1\\m-1\end{bmatrix}_q\cdot\frac{(1-q^{2r+2})(1-q^{m+r+1})}{(1-q^m)(1-q^{r+1})}.$$

As in case *(i)*, it is easily shown that $1-q^{m+r+1}=1-q^m\neq 0$. Furthermore, since d divides $r+1$, $q^{r+1}=1$. It follows that $\frac{1-q^{2r+2}}{1-q^{r+1}}=1+q^{r+1}=2$. Substituting q into the rank-generating function, the expression evaluates to

$$2\cdot\binom{\frac{m+2r+1}{d}}{\frac{m-1}{d}},$$

as desired.

(iii) $d|m-1$ and $d\nmid r+1$. Since d divides both $m-1$ and $m+2r+1$, d divides $2r+2$. As in case *(ii)*, it is easily shown that $1-q^m\neq 0$. Furthermore, $1-q^{r+1}\neq 0$, but $1-q^{2r+2}=0$, so the whole expression evaluates to 0.

(iv) $d\nmid m$, $d\nmid m-1$. Since d does not divide $m-1$, it follows from Lemma 8.4 that $\begin{bmatrix}m+2r+1\\m-1\end{bmatrix}_q=0$. We claim that $1-q^{r+1}\neq 0$. Suppose $1-q^{r+1}=0$; then $d|r+1$. Since d also divides $m+2r+1$, d divides $m-1$, which contradicts the assumption $d\nmid m-1$. Hence $1-q^{r+1}\neq 0$. Substituting q into the rank-generating function, the expression evaluates to 0.

This completes our proof that the triple

$$(J((r\oplus(1+1)\oplus r)\times[m]),J((r\oplus(1+1)\oplus r)\times[m];q),\Psi)$$

exhibits the cyclic sieving phenomenon for all positive integers m.

11. Proof of Theorem 1.2 for the Exceptional Cases

We verified via Dilks's code that the cyclic sieving phenomenon holds for the triple $(J(J^2([2] \times [3]) \times [m]), J(J^2([2] \times [3]) \times [m]; q), \Psi)$ when $m \leq 4$ and for the triple $(J(J^3([2] \times [3]) \times [m]), J(J^3([2] \times [3]) \times [m]; q), \Psi)$ when $m \leq 3$. For reference, we provide the data on the orbit structures corresponding to both exceptional posets for the cases $m = 1$ and $m = 2$. This data is not required to establish Theorem 1.1 because the proof of Theorem 6.3 is uniform, but it is required to show that Theorem 1.2 holds in the exceptional cases as well as in the infinite families.

	$P = J^2([2] \times [3])$	$P = J^3([2] \times [3])$
$m = 1$	$2 \times 12 + 1 \times 3$	$3 \times 18 + 1 \times 2$
$m = 2$	27×13	77×19

Table 1. For each entry, the table indicates the number of Fon-Der-Flaass action orbits of each size that occur in the indicated poset of order ideals. For instance, the poset $J(J^3([2] \times [3]) \times [1])$ is composed of 3 orbits of order 18 and 1 orbit of order 2.

It is tempting to propose the following conjecture.

Conjecture 11.1. *The triples* $(J(J^2([2]\times[3])\times[m]), J(J^2([2]\times[3])\times[m]; q), \Psi)$ *and* $(J(J^3([2]\times[3]) \times [m]), J(J^3([2] \times [3]) \times [m]; q), \Psi)$ *exhibit the cyclic sieving phenomenon for all positive integers* m*.*

12. Concluding Remarks

In this article, we demonstrated that, for all minuscule posets P, the triple $(J(P \times [m]), J(P \times [m]; q), \Psi)$ exhibits the cyclic sieving phenomenon for positive integers $m \leq 2$ if P belongs to the first or second infinite family and for all m if P belongs to the third infinite family, and we checked that the triple exhibits the cyclic sieving phenomenon for $m \leq 4$ if P is the first exceptional case and for $m \leq 3$ if P is the second exceptional case. It remains an open problem to determine a priori, given a minuscule poset P and a positive integer m, whether the triple exhibits the cyclic sieving phenomenon, but, as discussed, we conjecture that the cyclic sieving phenomenon holds for all m for the exceptional cases and for $m = 3$ for members of the second infinite family. That leaves only the first infinite family for which even hints about the behavior of its order ideals under the Fon-Der-Flaass action for large m have proved elusive.

It is our hope that the combinatorial techniques developed in sections 7-11, which provide more robust methods for representing the Fon-Der-Flaass action as an operation on bracket sequences as well as new tools for analyzing those bracket sequences, will be of some help should future researchers seek to tackle concretely the cases not covered in this paper. Furthermore, we would one day like to see a uniform resolution of this problem that draws upon the more advanced algebraic techinques of sections 2-6. That, in particular, may seem like a tall order, but it should be noted that Stembridge proved that an instance of the $q = -1$ phenomenon (a special case of the cyclic sieving phenomenon for actions of order 2) holds uniformly for all Cartesian products $P \times [m]$ in [14], so we have good reason to be optimistic, even though the situation in the case of general cyclic sieving is considerably more complicated.

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14. Appendix

In this section, for every minuscule poset P_V arising from a minuscule representation V of a complex simple Lie algebra $\mathfrak{g}$, we present an explicit description of the isomorphism $\phi : J(P_V) \to W^J$ satisfying the properties of Theorem 6.3. These results are not necessary for our uniform proof of Theorem 6.3, but they could constitute a case-by-case proof of Theorem 6.3 if we justified the assertions in Theorems A.2, B.2, C.2, D.2, D.4, and E.1, and we include them in order to state specifically, for every P_V, which elements of P_V are labelled by which Coxeter generators, where the labels of P_V are always understood to be the labels of the corresponding minuscule heap $P_{w_0^J}$. We adopt the notation of Stembridge's in his appendix to [14].

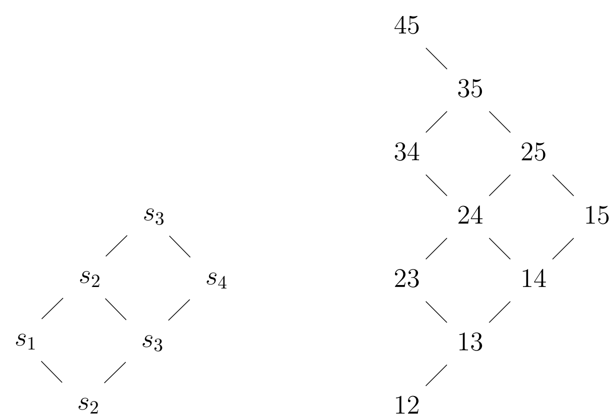
Appendix A. The Case A_{n-1}

For positive integers $n \geq 2$, we consider the Lie algebras $\mathfrak{sl}(n)$, for which the associated root systems are of the form A_{n-1}. Let $\alpha_j = \epsilon_{j+1} - \epsilon_j$ for all $1 \leq j \leq n-1$. The possible minuscule weights are $\omega_1, \omega_2, \ldots, \omega_{n-1}$. It is possible to consider all these cases simultaneously. If V is a representation in which ω_k is minuscule, $P_V = [k] \times [n-k]$, and $J = \{1, 2, \ldots, n-1\} \setminus \{k\}$. The Weyl group W is the group of all $n \times n$ permutation matrices, so W is the group of permutations of the n basis vectors $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$, and, as a shorthand, we regard W as the group of permutations of the n letters $1, 2, \ldots, n$ by setting $w(i) = i'$ if $w(\epsilon_i) = \epsilon_{i'}$ for all $1 \leq i \leq n$. We see then that the Coxeter generator s_j swaps the letters j and $j+1$ for all $1 \leq j \leq n-1$.

Note that the elements of W^J considered as minimum-length coset representatives are precisely the permutations w satsifying $w(1) < w(2) < \ldots < w(k)$ and $w(k+1) < w(k+2) < \ldots < w(n)$. We proceed to define the desired bijection. Let $P_V = \{(i,j) \in \mathbb{Z}^2 : 1 \leq i \leq k \text{ and } 1 \leq j \leq n-k\}$, where the partial order is defined to be the transitive closure of the relations $(i,j) < (i+1,j)$ for all $1 \leq i \leq k-1, 1 \leq j \leq n-k$ and $(i,j) < (i,j+1)$ for all $1 \leq i \leq k, 1 \leq j \leq n-k-1$. Given an order ideal $I \in J(P_V)$, we define I_i for all $1 \leq i \leq k$ as follows: if $(i,1) \notin I$, then $I_i = 0$; otherwise, I_i is the largest positive integer such that $(i, I_i) \in I$.

Definition A.1. Let $\phi : J(P_V) \to W^J$ be the map defined by setting $\phi(I)$ to be the unique minimum-length coset representative satisfying $w(i) = i + I_{k+1-i}$ for all $1 \leq i \leq k$. (See Figure 9.)

Theorem A.2. *The induced action of s_l on $J(P_V)$ may be expressed as* $\prod_{(i,j):i+l=j+k} t_{(i,j)}$*; in other words, the elements of P_V labelled by the simple reflection s_l are precisely those (i,j) for which $i+l=j+k$.*



(a) (b)

FIGURE 9. In the case when the root system is A_4 and the minuscule weight is ω_2, the map ϕ sends order ideals of the minuscule heap P_{45123} (at left) to elements of the Bruhat poset $(W^J, <_B)$, where $J = \{1,3,4\}$ (at right).

APPENDIX B. THE CASE B_n

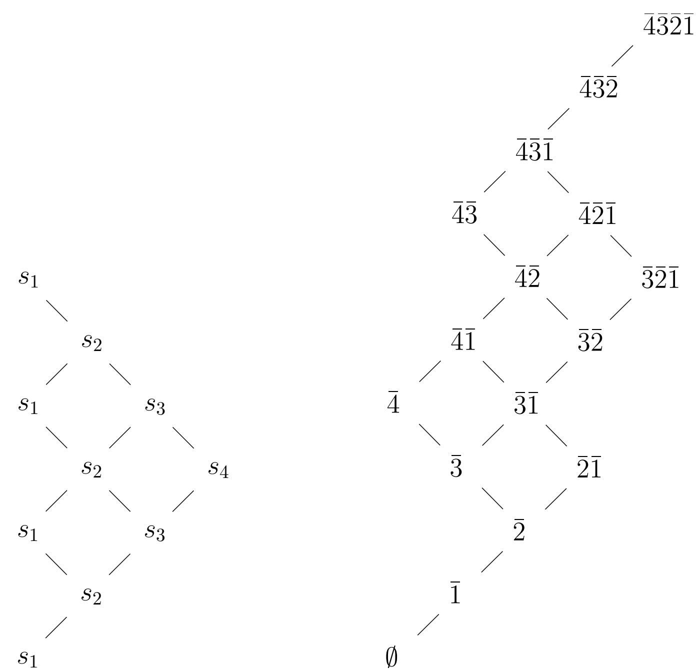
For positive integers $n \geq 2$, we consider the Lie algebras $\mathfrak{so}(2n+1)$, for which the associated root systems are of the form B_n. Let $\alpha_1 = \epsilon_1$, and let $\alpha_j = \epsilon_j - \epsilon_{j-1}$ for all $2 \leq j \leq n$. The only possible minuscule weight is ω_1. If V is a representation in which ω_1 is minuscule, $P_V = [n] \times [n]/S_2$, and $J = \{1, 2, \ldots, n\} \setminus \{1\}$. The Weyl group W is the group of all $n \times n$ signed permutation matrices – a signed permutation matrix is like a permutation matrix except that the nonzero entries may be 1 or -1 – so W is the group of signed permutations of the n basis vectors $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$, and, as a shorthand, we regard W as a subgroup of the group of permutations of the $2n$ letters $1, 2, \ldots, n, \overline{1}, \overline{2}, \ldots, \overline{n}$ by setting $w(i) = i'$ and

$w(\bar{i}) = \overline{i'}$ if $w(\epsilon_i) = \epsilon_{i'}$ and $w(i) = \overline{i'}$ and $w(\bar{i}) = i'$ if $w(\epsilon_i) = -\epsilon_{i'}$, for all $1 \le i \le n$. We see then that the Coxeter generator s_1 swaps the letters 1 and $\overline{1}$, and the Coxeter generator s_j swaps the letters $j-1$ and j and the letters $\overline{j-1}$ and $\overline{j}$ for all $2 \le j \le n$.

Note that the elements of W^J considered as minimum-length coset representatives are precisely the permutations w satisfying $w(1) < w(2) < \ldots < w(n)$, where in comparing two letters among $1, 2, \ldots, n, \overline{1}, \overline{2}, \ldots, \overline{n}$ we treat $\bar{i}$ as a stand-in for $-i$. We proceed to define the desired bijection. Let $P_V = \{(i,j) \in \mathbb{Z}^2 : 1 \le i \le j \le n\}$, where the partial order is defined to be the transitive closure of the relations $(i,j) < (i+1,j)$ for all $2 \le i+1 \le j \le n$ and $(i,j) < (i,j+1)$ for all $1 \le i \le j \le n-1$. Given an order ideal $I \in J(P_V)$, we define I_i for all $1 \le i \le n$ as follows: if $(i,i) \notin I$, then $I_i = 0$; otherwise, I_i is the largest positive integer such that $(i, I_i + i - 1) \in J$. We also define $M(I)$ to be 0 if $I_1 = 0$ and to be the largest positive integer such that $I_{M(I)} > 0$ otherwise.

Definition B.1. Let $\phi : J(P_V) \to W^J$ be the map defined by setting $\phi(I)$ to be the unique minimum-length coset representative satisfying $w(i) = \overline{I(i)}$ for all $1 \le i \le M(I)$ and $w(i) > 0$ otherwise, where we consider $\overline{\overline{j}}$ to be j for all $1 \le j \le n$. (See Figure 10.)

Theorem B.2. *The induced action of s_l on $J(P_V)$ may be expressed as* $\displaystyle\prod_{(i,j):i+l=j+1} t_{(i,j)}$*; in other words, the elements of P_V labelled by the simple reflection s_l are precisely those (i,j) for which $i+l=j+1$.*



(a) (b)

FIGURE 10. In the case when the root system is B_4 and the minuscule weight is ω_1, the map ϕ sends order ideals of the minuscule heap $P_{\bar{4}\bar{3}\bar{2}\bar{1}}$ (at left) to elements of the Bruhat poset $(W^J, <_B)$, where $J = \{2,3,4\}$ (at right).

Appendix C. The Case C_n

For positive integers $n \geq 2$, we consider the Lie algebras $\mathfrak{sp}(2n)$, for which the associated root systems are of the form C_n. Let $\alpha_1 = 2\epsilon_1$, and let $\alpha_j = \epsilon_j - \epsilon_{j-1}$ for all $2 \leq j \leq n$. The only possible minuscule weight is ω_n. If V is a representation in which ω_n is minuscule, $P_V = [2n-1]$, and $J = \{1, 2, \ldots, n\} - \{n\}$. The Weyl group W is identical to that for the root systems of the form B_n, i.e. it remains the group of all $n \times n$ signed permutation matrices, and we retain the shorthand by which we regard W as a subgroup of the group of permutations of the $2n$ letters $1, 2, \ldots, n, \overline{1}, \overline{2}, \ldots, \overline{n}$ by setting $w(i) = i'$ and $w(\overline{i}) = \overline{i'}$ if $w(\epsilon_i) = \epsilon_{i'}$ and $w(i) = \overline{i'}$ and $w(\overline{i}) = i'$ if $w(\epsilon_i) = -\epsilon_{i'}$, for all $1 \leq i \leq n$.

Note that the elements of W^J considered as minimum-length coset representatives are precisely the permutations w satisfying $0 < w(1) < w(2) < \ldots < w(n-1)$. We proceed to define the desired bijection. Let $P_V = \{i \in \mathbb{Z} : -n+1 \leq i \leq n-1\}$, where the partial order is defined to be the transitive closure of the relations $i < i+1$ for all $-n+1 \leq i \leq n-2$. Note that P_V is in fact totally ordered. Given an order ideal $I \in J(P_V)$, we define $M(I)$ to be $-n$ if I is empty and to be the largest element of I otherwise.

Definition C.1. Let $\phi : J(P_V) \to W^J$ be the map defined by setting $\phi(I)$ to be the unique minimum-length coset representative satisfying $w(n) = -M(I)$ if $M(I) < 0$ and $w(n) = \overline{M(I)+1}$ otherwise. (See Figure 11.)

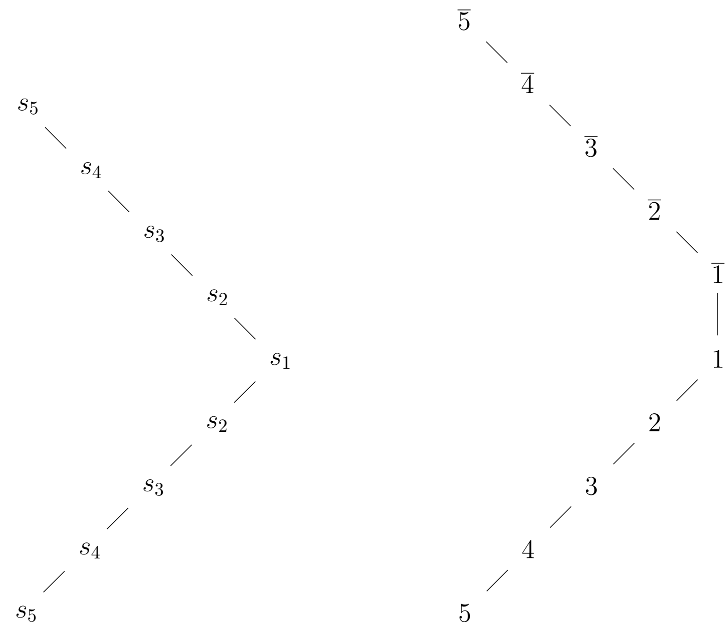
Theorem C.2. *The induced action of s_l on $J(P_V)$ may be expressed as $\prod_{i:|i|=l-1} t_i$; in other words, the elements of P_V labelled by the simple reflection s_l are precisely those i for which $|i| = l-1$.*

Appendix D. The Case D_n

For positive integers $n \geq 4$, we consider the Lie algebras $\mathfrak{so}(2n)$, for which the associated root systems are of the form D_n. Let $\alpha_1 = \epsilon_1 + \epsilon_2$, and let $\alpha_j = \epsilon_j - \epsilon_{j-1}$ for all $2 \leq j \leq n$. The only possible minuscule weights are ω_1, ω_2, and ω_n. However, the minuscule representation with minuscule weight ω_2 is isomorphic to the minuscule representation with minuscule weight ω_1. Therefore, we need only consider the cases in which the minuscule weight is ω_1 or ω_n.

If V is a representation in which ω_1 is minuscule, $P_V = ([n-1] \times [n-1])/S_2$, and $J = \{1, 2, \ldots, n\} \setminus \{1\}$. The Weyl group W is the group of all even $n \times n$ signed permutation matrices – an even signed permutation matrix is like a signed permutation matrix except that the number of negative entries must be even – so W is the group of even signed permutations of the n basis vectors $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$, and, as a shorthand, we regard W as a subgroup of the group of permutations of the $2n$ letters $1, 2, \ldots, n, \overline{1}, \overline{2}, \ldots, \overline{n}$ by setting $w(i) = i'$ and $w(\overline{i}) = \overline{i'}$ if $w(\epsilon_i) = \epsilon_{i'}$ and $w(i) = \overline{i'}$ and $w(\overline{i}) = i'$ if $w(\epsilon_i) = -\epsilon_{i'}$, for all $1 \leq i \leq n$. We see then that the Coxeter generator s_1 swaps the letters 1 and $\overline{2}$ and the letters 2 and $\overline{1}$, and the Coxeter generator s_j swaps the letters $j-1$ and j and the letters $\overline{j-1}$ and $\overline{j}$ for all $2 \leq j \leq n$.

Note that the elements of W^J considered as minimum-length coset representatives are precisely the permutations w satisfying $w(1) < w(2) < \ldots < w(n)$. We proceed to define the desired bijection. Let $P_V = \{(i, j) \in \mathbb{Z}^2 : 1 \leq i \leq j \leq n-1\}$, where the partial order is



(a) (b)

FIGURE 11. In the case when the root system is C_5 and the minuscule weight is ω_5, the map ϕ sends order ideals of the minuscule heap $P_{1234\overline{5}}$ (at left) to elements of the Bruhat poset $(W^J, <_B)$, where $J = \{1, 2, 3, 4\}$ (at right).

defined to be the transitive closure of the relations $(i, j) < (i+1, j)$ for all $2 \leq i+1 \leq j \leq n-1$ and $(i, j) < (i, j+1)$ for all $1 \leq i \leq j \leq n-2$. Given an order ideal $I \in J(P_V)$, we define I_i for all $1 \leq i \leq n-1$ as follows: if $(i, i) \notin I$, then $I_i = 0$; otherwise, I_i is the largest positive integer such that $(i, I_i + i - 1) \in I$. We also define $M(I)$ to be 0 if $I_1 = 0$ and to be the largest positive integer such that $I_{M(I)} > 0$ otherwise.

Definition D.1. Let $\phi : J(P_V) \to W^J$ be the map defined by setting $\phi(I)$ to be the unique minimum-length coset representative satisfying $w(i) = \overline{I_i + 1}$ for all $1 \leq i \leq M(I)$, $w(M(I) + 1) = (-1)^{M(I)}$, and $w(i) > 0$ otherwise, i.e., $w(M(I) + 1)$ is defined to be 1 if $M(I)$ is even and $\overline{1}$ if $M(I)$ is odd. (See Figure 12.)

Theorem D.2. *The induced action of s_1 on $J(P_V)$ may be expressed as* $\prod_{(i,i): i \text{ is odd}} t_{(i,i)}$*; the induced action of s_2 on $J(P_V)$ may be expressed as* $\prod_{(i,i): i \text{ is even}} t_{(i,i)}$*, and the induced action of s_l, for all $3 \leq l \leq n$, may be expressed as* $\prod_{(i,j): i+l=j+2} t_{(i,j)}$*. In other words, the elements of P_V labelled by the simple reflection s_1 are precisely those (i, i) for which i is odd; the elements of P_V labelled by the simple reflection s_2 are precisely those (i, i) for which i is even, and the elements of P_V labelled by the simple reflection s_l, for all $3 \leq l \leq n$, are precisely those (i, j) for which $i + l = j + 2$.*

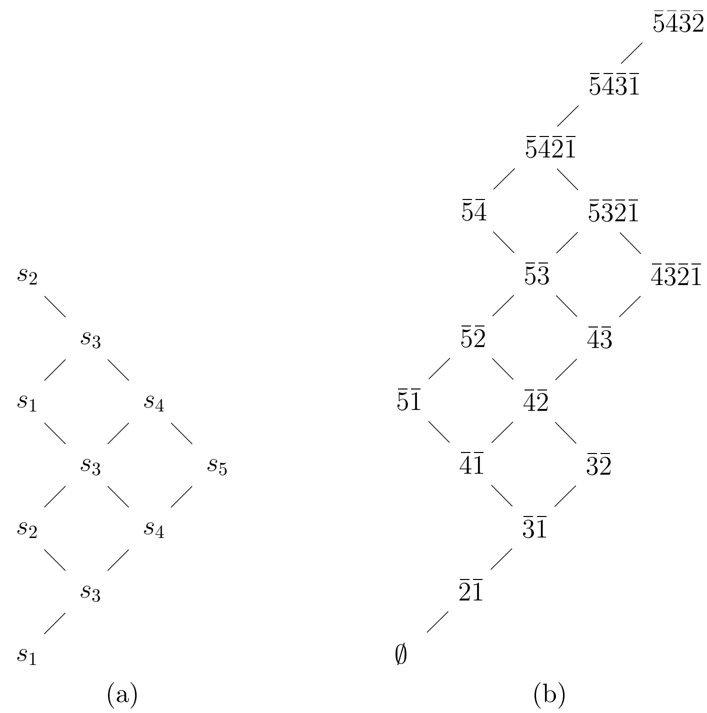


FIGURE 12. In the case when the root system is D_5 and the minuscule weight is ω_1, the map ϕ sends order ideals of the minuscule heap $P_{\bar{5}\bar{4}\bar{3}\bar{2}1}$ (at left) to elements of the Bruhat poset $(W^J, <_B)$, where $J = \{2,3,4,5\}$ (at right).

If V is a representation in which ω_n is minuscule, $P_V = J^{n-3}([2] \times [2]) = (n-2) \oplus (1+1) \oplus (n-2)$, and $J = \{1,2,\ldots,n\} \setminus \{n\}$. This time the elements of W^J considered as minimum-length coset representatives are precisely the permutations w satisfying $\overline{2} \leq w(1) \leq 2 \leq w(2) < w(3) \ldots < w(n-1)$. We proceed to define the desired bijection. Let $P_V = \{(i,j) \in \mathbb{Z} : 1 \leq |i| \leq n-2$ and $j = 0$ or $i = 0$ and $|j| = 1\}$, where the partial order is defined to be the transitive closure of the relations $(i,j) < (i',j')$ for all $(i,j),(i',j')$ satisfying $i' - i = 1$. Given an order ideal $I \in J(P_V)$, we define $M(I)$ to be $-n+1$ if I is empty and to be the largest integer for which there exists a j satisfying $(M(I),j) \in I$ otherwise. We also define $N(I)$ to be 1 if $(0,1) \in I$ but $(0,-1) \notin I$, -1 if $(0,-1) \in I$ but $(0,1) \notin I$, and 0 otherwise.

Definition D.3. Let $\phi : J(P_V) \to W^J$ be the map defined by setting $\phi(I)$ to be the unique minimum-length coset representative satisfying $w(n) = -M(I)+1$ if $M(I) < 0$, $w(n) = \overline{M(I)+2}$ if $M(I) > 0$, and $w(n) = N(I)$ if $M(I) = 0$. (See Figure 13.)

Theorem D.4. *The induced action of s_1 on $J(P_V)$ may be expressed as $t_{(0,-1)}$; the induced action of s_2 may be expressed as $t_{(0,1)}$, and the induced action of s_l, for all $3 \leq l \leq n$, may be expressed as $\prod_{i:|i|=l-2} t_{(i,0)}$; in other words, the element of P_V labelled by the simple reflection s_1 is precisely $(0,-1)$; the element of P_V labelled by the simple reflection s_2 is precisely $(0,1)$, and the elements of P_V labelled by the simple reflection s_l, for all $3 \leq l \leq n$, are precisely those $(i,0)$ for which $|i| = l-2$.*

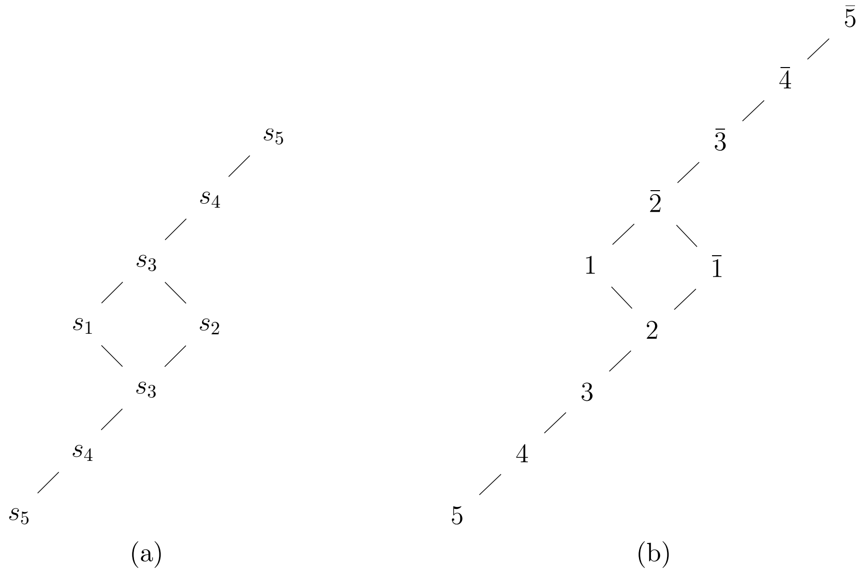


FIGURE 13. In the case when the root system is D_5 and the minuscule weight is ω_5, the map ϕ sends order ideals of the minuscule heap $P_{1234\bar{5}}$ (at left) to elements of the Bruhat poset $(W^J, <_B)$, where $J = \{1, 2, 3, 4\}$ (at right).

Appendix E. The Exceptional Cases

For the Lie algebras E_6 and E_7, for which the root systems are E_6 and E_7, respectively, we let the diagrams of the minuscule heaps speak for themselves. For the case E_6, let the Coxeter-Dynkin diagram for the Weyl group be as shown in part (a) of Figure 14. The only possible minuscule weights are ω_1 and ω_6, but the minuscule representation with minuscule weight ω_6 is isomorphic to the minuscule representation with minuscule weight ω_1. Therefore, we need only consider the case in which the minuscule weight is ω_1. Part (b) of Figure 14 illustrates the minuscule heap $P_{w_0^J}$.

For the case E_7, let the Coxeter-Dynkin diagram for the Weyl group be as shown in part (a) of Figure 15. The only possible minuscule weight is ω_7. Part (b) of Figure 15 illustrates the minuscule heap $P_{w_0^J}$.

The following theorem may be easily verified by hand or by computer.

Theorem E.1. *The statement of Theorem 6.3 holds for the cases E_6 and E_7, where the minuscule heaps named in the theorem are as they appear in Figures 14 and 15, respectively.*

References

[1] G. E. Andrews, Plane partitions (II): the equivalence of the Bender-Knuth and MacMahon conjectures, *Pacific J. Math.* **72** (1977), 283-291.

[2] A. Björner, F. Brenti, *Combinatorics of Coxeter Groups*, New York: Springer, 2005.

[3] N. Bourbaki, *Lie Groups and Lie Algebras: chapters 4-6*, Berlin: Springer, 2002.

[4] A. Brouwer and A. Schrijver, On the period of an operator, defined on antichains, *Math. Centrum Report* ZW **24** (1974).

[5] P. J. Cameron, D. G. Fon-Der-Flaass, Orbits of antichains revisited, *European J. Combin.* **16** (1995), 545-554.

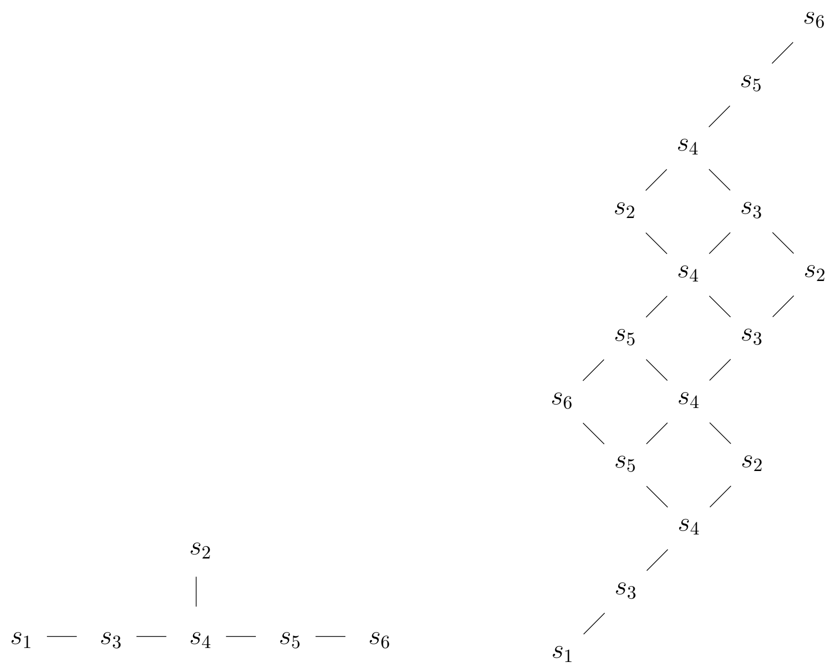


FIGURE 14. The labelling of the Coxeter-Dynkin diagram for the Weyl group of E_6 indicated at left corresponds to the labelling of the minuscule heap for E_6 displayed at right.

[6] P. Duchet, Sur les hypergraphes invariants, *Discrete Math.* **8** (1974), 269-280.
[7] D. G. Fon-Der-Flaass, Orbits of antichains in ranked posets, *European J. Combin.* **14** (1993), 17-22.
[8] R. A. Proctor, Bruhat lattices, plane partition generating functions, and minuscule representations, *European J. Combin.* **5** (1984), 331-350.
[9] V. Reiner, D. Stanton, D. White, The cyclic sieving phenomenon, *J. Combin. Theory Ser. A* **108** (2004), 17-50.
[10] R. Roy, Binomial identities and hypergeometric series, *Amer. Math. Monthly* **94** (1987), 36-46.
[11] T. A. Springer, Regular elements of finite reflection groups, *Invent. Math.* **25** (1974), 159-198.
[12] R. P. Stanley, *Enumerative Combinatorics: volume I*, New York: Cambridge University Press, 1997.
[13] R. P. Stanley, Promotion and evacuation, *Electron. J. Combin.* **16** (2009), R9.
[14] J. R. Stembridge, On minuscule representations, plane partitions and involutions in complex Lie groups, *Duke Math. J.* **73** (1994), 469-490.
[15] J. R. Stembridge, On the fully commutative elements of Coxeter groups, *J. Algebraic Combin.* **5** (1996), 353-385.
[16] J. Striker, N. Williams, Promotion and rowmotion, preprint arXiv:1108.1172v1 [math.CO].
[17] N. J. Wildberger, Minuscule posets from neighbourly graph sequences, *European J. Combin.* **24** (2003), 741-757.

DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MA
E-mail address: `dbr@mit.edu`

DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MA
E-mail address: `dannyshi@mit.edu`

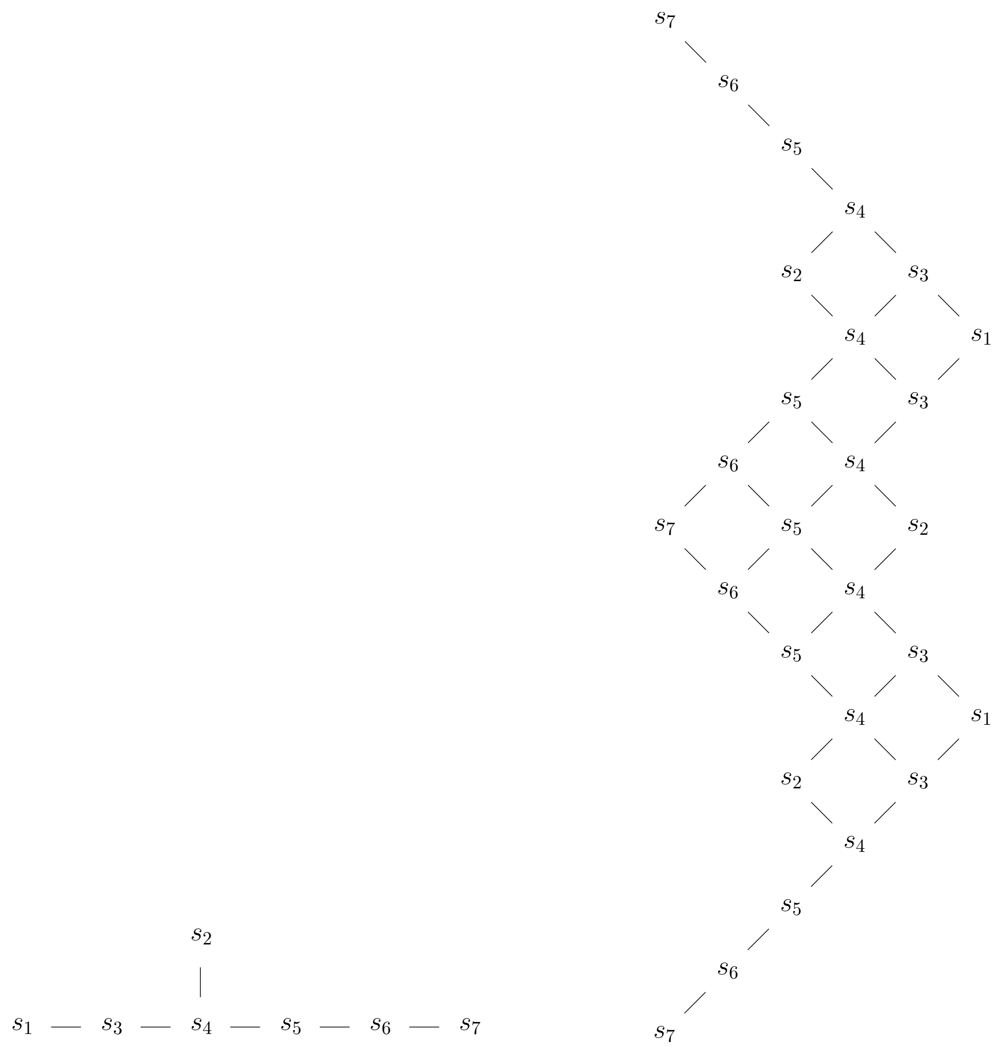


FIGURE 15. The labelling of the Coxeter-Dynkin diagram for the Weyl group of E_7 indicated at left corresponds to the labelling of the minuscule heap for E_7 displayed at right.