Topology of three complexes from matroids

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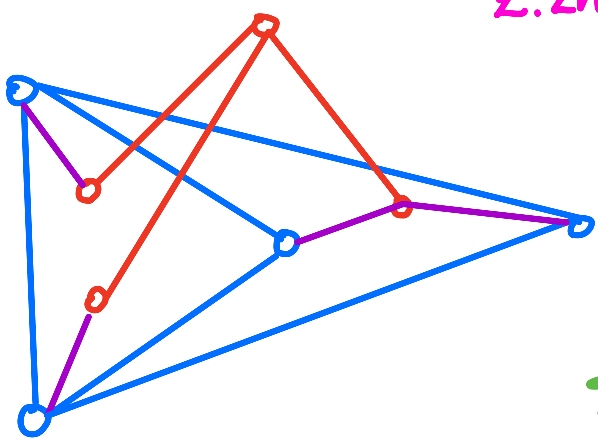


Brown Combinatorics Seminar
April 19, 2022

1. Review matroids M
   - independent sets $\mathcal{I}(M)$
   - flats $\mathcal{F}(M)$

2. Shellability

3. Independent set complex $\mathcal{I}(M)$,
   Bergman complex $\underline{\Delta}_M$

4. Augmented Bergman complex Δ_M

5. Two kinds of **shellings** of Δ_M
   and corollaries

1. Review matroids M

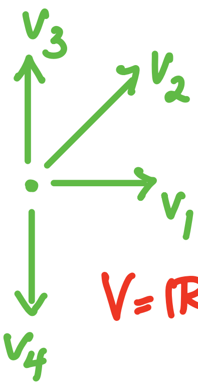
A matroid M of rank r on groundset $E = \{1, 2, \dots, n\}$ abstracts vectors $v_1, v_2, \dots, v_n$ spanning an r-dimensional vector space V over some field k

EXAMPLE

$n=4$

$k=\mathbb{R}$

$r=2$



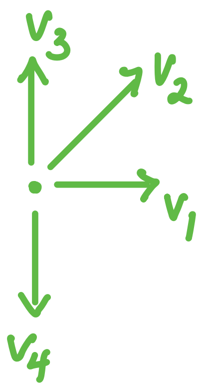
$V = \mathbb{R}^2$

$$\longleftrightarrow \quad \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \atop \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

an $r \times n$ full rank matrix having v_i as its columns

The matroid M associated to $v_1, v_2, \dots, v_n$ forgets their coordinates, but records the subscripts of (linearly) independent sets

$$\mathcal{I}(M) \underset{\text{DEF'N}}{:=} \left\{ I \subseteq \{1,2,\dots,n\} : \begin{array}{l} \{v_i\}_{i \in I} \text{ are} \\ \text{linearly independent} \end{array} \right\}$$



$$\rightsquigarrow \mathcal{I}(M) = \left\{ \begin{array}{lll} \emptyset, & 1, & 12, \\ & 2, & 13, \\ & 3, & 14, \\ & 4 & 23, \\ & & 24 \end{array} \right\}$$

Note: $34 \notin \mathcal{I}(M)$ since $\{v_3, v_4\}$ are dependent

$ijk \notin \mathcal{I}(M) \ \forall i,j,k$

$\mathcal{I}(M)$ always satisfies independent set axioms:

(I0) $\emptyset \in \mathcal{I}(M)$

(I1) $I \subseteq J$ and $J \in \mathcal{I}(M) \Rightarrow I \in \mathcal{I}(M)$

(I2) $I, J \in \mathcal{I}(M)$ and $\#I < \#J$
(Exchange axiom) $\Rightarrow \exists j \in J \setminus I$ with $I \cup \{j\} \in \mathcal{I}(M)$

and this is our first definition of a matroid M:
a collection $\mathcal{I}(M)$ of subsets of $E = \{1, 2, \ldots, n\}$
satisfying axioms (I0), (I1), (I2).

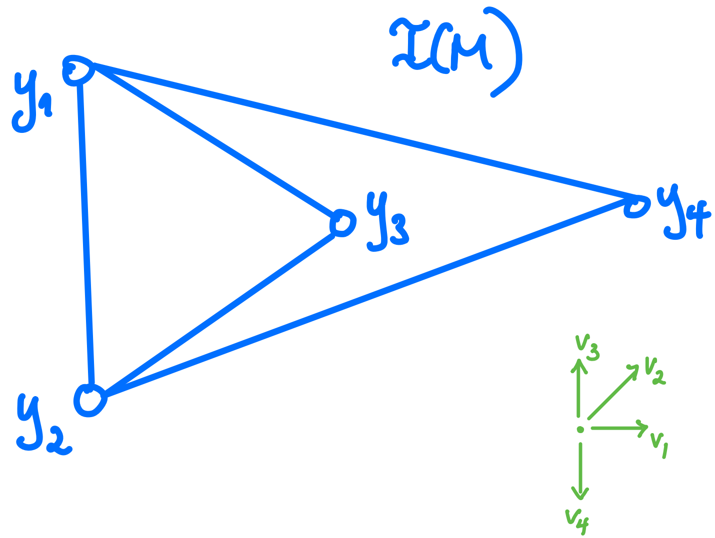
(I0) $\emptyset \in \mathcal{I}(M)$

(I1) $I \subseteq J$ and $J \in \mathcal{I}(M) \Rightarrow I \in \mathcal{I}(M)$

(I2) $I, J \in \mathcal{I}(M)$ and $\#I < \#J$
$\Rightarrow \exists\, j \in J \setminus I$ with $I \cup \{j\} \in \mathcal{I}(M)$

Axioms (I0), (I1) say $\mathcal{I}(M)$ is an abstract simplicial complex on vertices $\{y_1, y_2, \ldots, y_n\}$

Axiom (I2) implies all inclusion-maximal independent sets, called the bases $\mathcal{B}(M)$, have same cardinality $r =:$ rank $r(M)$.

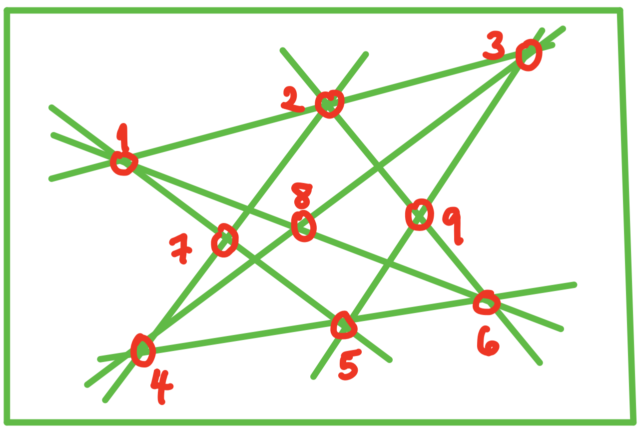


$\Rightarrow$ $\mathcal{I}(M)$ is a pure simplicial complex of dimension $r(M) - 1$.

Not all matroids M are representable by vectors $v_1, v_2, \ldots, v_n$

EXAMPLE The non-Pappus matroid M on $E = \{1, 2, \ldots, 9\}$ of rank 3 has

$$\mathcal{I}(M) = \left\{ \begin{array}{l} \text{all } I \subset \{1,2,\ldots,9\} \text{ with } \#I \leq 3, \\ \text{except the collinear triples shown} \end{array} \right\}$$



$789 \in \mathcal{I}(M)$ violates Pappus's Theorem

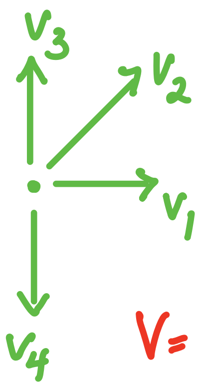
But $\mathcal{I}(M)$ satisfies axioms (I0), (I1), (I2).

An alternate axiomatization of M uses the flats $\mathcal{F}(M)$ which are (when M is represented by $v_1, v_2, \ldots, v_n$ in V)

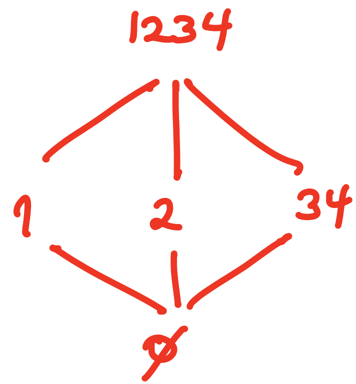
$$\mathcal{F}(M) := \left\{ F \subseteq \{1,2,\ldots,n\} : \{v_i\}_{i\in F} = W \cap \{v_1, v_2, \ldots, v_n\} \text{ for some subspace } W \text{ of } V \right\}$$

EXAMPLE

flats $\mathcal{F}(M) = \{ \emptyset, 1, 2, 34, 1234 \}$



$V = \mathbb{R}^2$



the poset $\mathcal{F}(M)$ ordered via inclusion

We could have defined a matroid M on $E = \{1, 2, \_\_, n\}$ as a collection $\mathcal{F}(M)$ of subsets $F \subseteq E$, satisfying

flat axioms:

(F0) $E = \{1, 2, \_, n\} \in \mathcal{F}(M)$

(F1) $F, G \in \mathcal{F}(M) \Rightarrow F \cap G \in \mathcal{F}(M)$

(F2) $F \in \mathcal{F}(M)$ and $i \in E \setminus F \Rightarrow$
$\exists !\ G \in \mathcal{F}(M)$ covering F with $i \in G$.

(F0), (F1) $\Rightarrow$ the poset $\mathcal{F}(M)$ is a lattice, with $F \wedge G = F \cap G$.

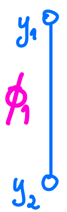
(F2) $\Rightarrow$ $\mathcal{F}(M)$ is actually a geometric lattice.

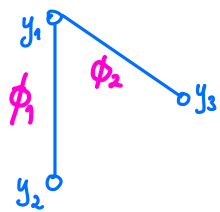
atomic + upper semimodular

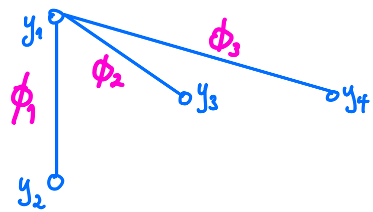
2. Shellability

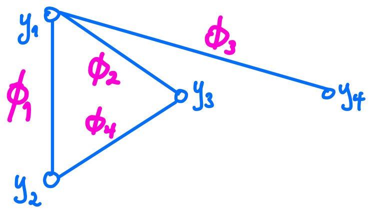
DEF'N: A pure $(r-1)$-dimensional simplicial complex Δ is shellable if we can order its facets $\phi_1, \phi_2, \ldots, \phi_t$ in a shelling order:

$\forall j \geq 2$, ϕ_j intersects the subcomplex generated by $\phi_1, \phi_2, \ldots, \phi_{j-1}$ in a pure $(r-2)$-dim'l subcomplex

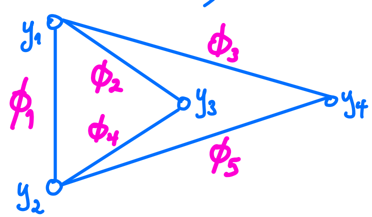








$\mathcal{I}(M)$



Shelling determines the homotopy type of Δ

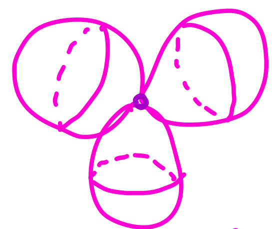
DEF'N: Call ϕ_j a homology facet in the shelling $\phi_1, \phi_2, \ldots, \phi_t$ if ϕ_j intersects the subcomplex gen'd by $\phi_1, \phi_2, \ldots, \phi_{j-1}$ in its entire boundary $\mathrm{Bd}\,\phi_j$

PROPOSITION: When Δ is pure d-dimensional and shellable,

then $\|\Delta\| \approx \underbrace{S^d \vee S^d \vee \cdots \vee S^d}_{\beta\text{-fold 1-point wedge of } d\text{-spheres } S^d}$

geometric realization of Δ

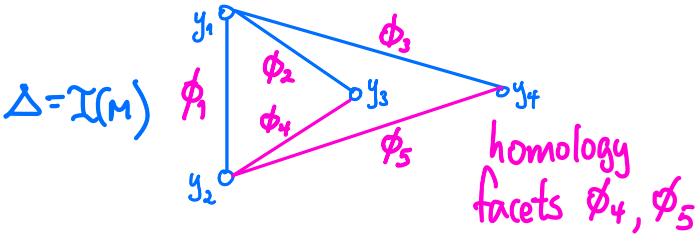
homotopy equivalent

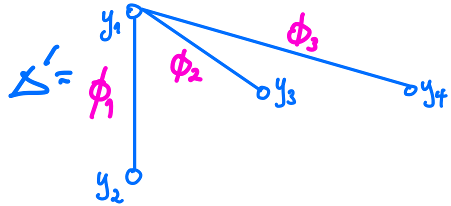


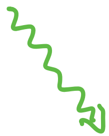
where $\beta :=$ # of homology facets ϕ_j in any shelling order

In fact, whenever Δ is shellable,

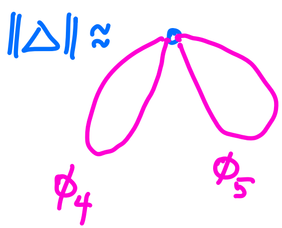
then $\Delta' := \Delta - \left\{\begin{matrix}\text{all homology}\\ \text{facets } \phi_j\end{matrix}\right\}$ is contractible:



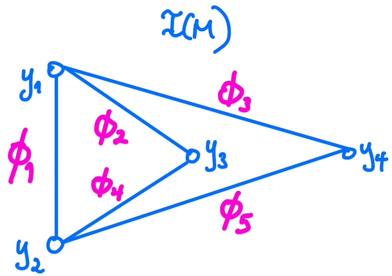




contract Δ' to a point



3. THEOREM (Provan-Billera 1980) For a matroid M, the independent set complex $\mathcal{I}(M)$ is shellable, via lexicographic order on the bases $\mathcal{B}(M)$.



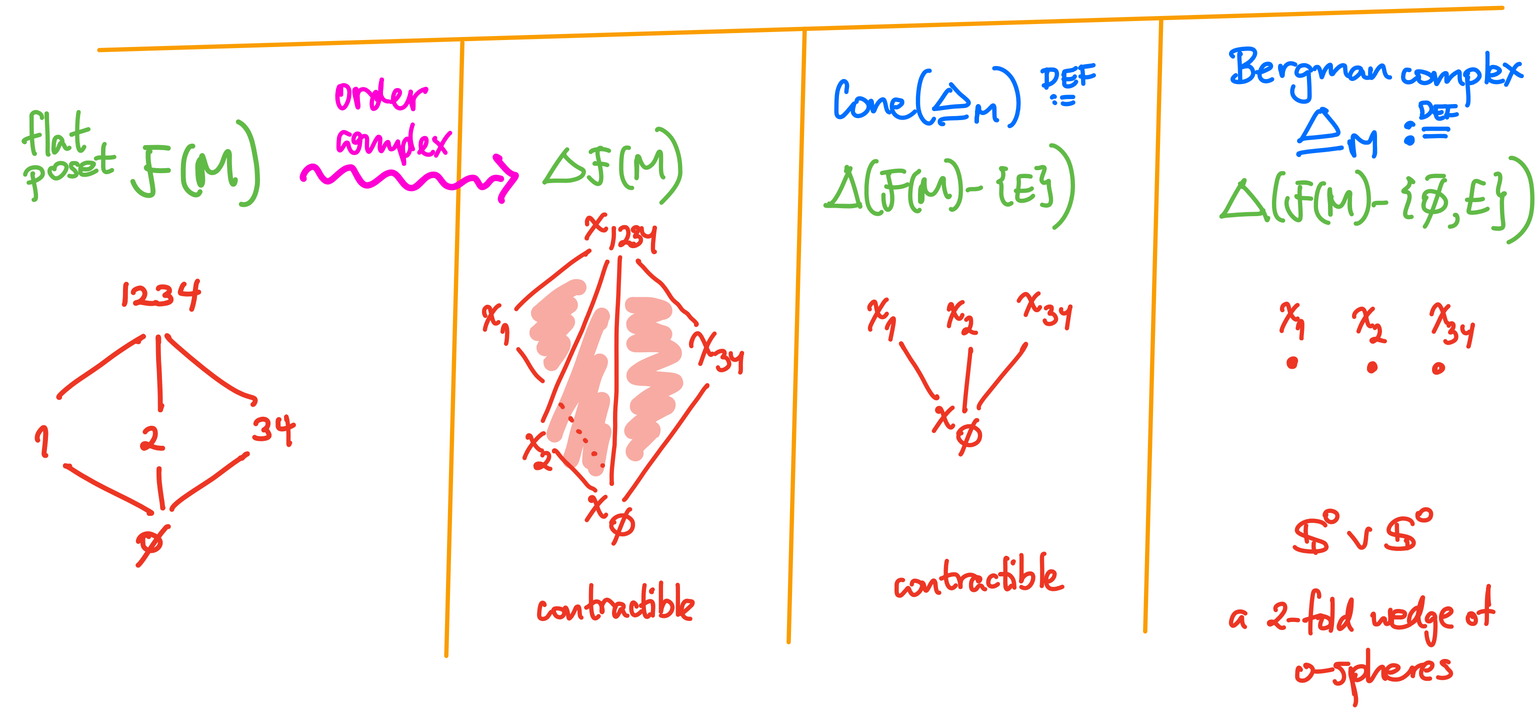
$$\underset{12}{\phi_1} <_{lex} \underset{13}{\phi_2} <_{lex} \underset{14}{\phi_3} <_{lex} \underset{23}{\phi_4} <_{lex} \underset{24}{\phi_5}$$

Furthermore, the number of homology facets is

$\beta = T_M(0,1)$ = Tutte polynomial $T_M(x,y)$ evaluated at $x=0, y=1$
= # bases $B \in \mathcal{B}(M)$ of internal activity zero

COROLLARY: $\|\mathcal{I}(M)\| \simeq \underbrace{S^{r(M)-1} \vee \cdots \vee S^{r(M)-1}}_{T_M(0,1)\text{-fold wedge}}$

The flats $\mathcal{F}(M)$ as a poset P gives us another simplicial complex, the order complex $\Delta P :=$ simplicial complex with vertex set $\{x_p\}_{p\in P}$ and simplices/faces the totally ordered subsets $\{x_{p_1}, x_{p_2}, \dots, x_{p_\ell}\}$ if $p_1 < p_2 < \dots < p_\ell$ in P



ASIDE: Why call Δ_M the Bergman complex ?

THEOREM (Ardila-Klivans 2003) The subspace $V \subseteq k^n$ cut out by linear forms whose coefficients are the rows of (represented) matroid $M = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}$

has a natural simplicial subdivision for its

Bergman fan / tropical variety $\mathrm{Trop}(V) \subset \mathbb{R}^n$

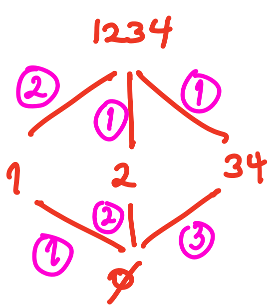
with $\mathrm{link}_{\mathrm{Trop}(V)}(\underline{0}) \cong \Delta_M$

link at the origin in the simplicial fan

THEOREM (Garsia 1980) For a matroid M, all three of
$$\left\{\begin{array}{l}\Delta\mathcal{F}(M)\\ \Delta(\mathcal{F}(M)-\{E\})\\ \Delta(\mathcal{F}(M)-\{\emptyset,E\}) =: \underline{\Delta}_M\end{array}\right.$$
are shellable, via lexicographic order on the edge-label sequences on maximal chains $\emptyset \subset F_1 \subset F_2 \subset F_3 \subset \cdots \subset F_{r(M)-1} \subset E$ in $\mathcal{F}(M)$

edge-labels: $(\min F_1,\ \min(F_2 \setminus F_1),\ \min(F_3 \setminus F_2), \ldots,\ \min(E \setminus F_{r(M)-1}))$

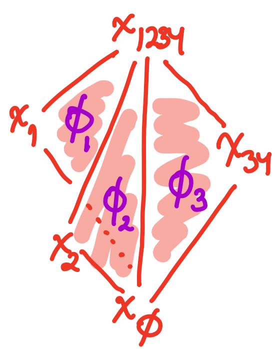
$\mathcal{F}(M)$



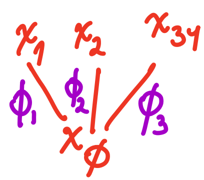
$(1,2) <_{lex} (2,1) <_{lex} (3,1)$

$\phi_1 \qquad \phi_2 \qquad \phi_3$

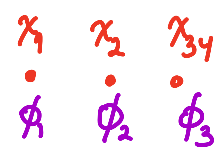
$\Delta\mathcal{F}(M)$



$\Delta(\mathcal{F}(M)-\{E\})$



$\Delta(\mathcal{F}(M)-\{\emptyset,E\})$



Furthermore, the number of homology facets is

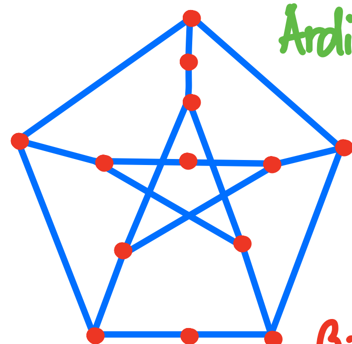
$\beta = T_M(1,0)$ = Tutte polynomial $T_M(x,y)$ evaluated at $x=1, y=0$

= # bases $B \in \mathcal{B}(M)$ of external activity zero

COROLLARY: $\|\underline{\Delta}(M)\| \approx \underbrace{S^{r(M)-2} \vee \cdots \vee S^{r(M)-2}}_{T_M(1,0)\text{-fold wedge}}$

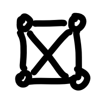
Bergman complex

$\underline{\Delta}_M := \Delta(\mathcal{F}(M) - \{\bar{\emptyset}, E\})$



$\beta = 6$

A nice Ardila-Klivans example: $\underline{\Delta}_M$ for graphic matroid $M = M(K_4)$



4. Augmented Bergman complex Δ_M

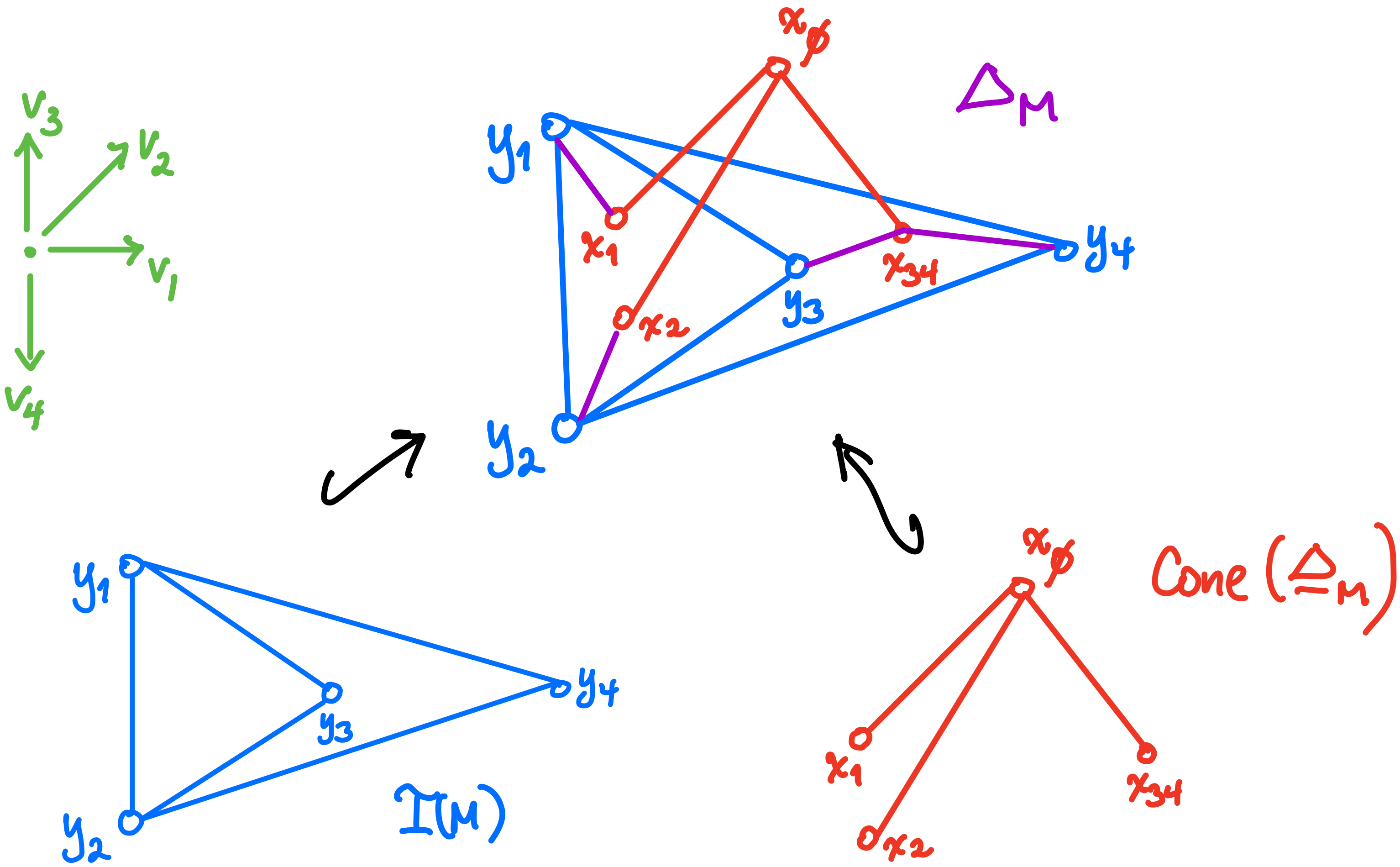
In a monumental pair of 2020 papers,
Braden-Huh-Matherne-Proudfoot-Wang introduced a hybrid.

DEF'N: The augmented Bergman complex Δ_M
has vertex set $\{y_1, y_2, \ldots, y_n\} \cup \{x_F\}_{\emptyset \subseteq F \subsetneq E}$
(proper flats $F \in \mathcal{F}(M)$)

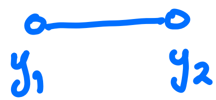
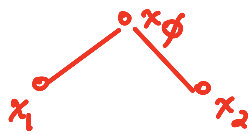
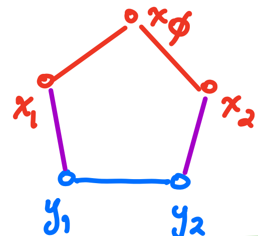
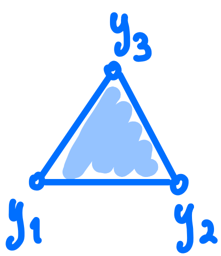
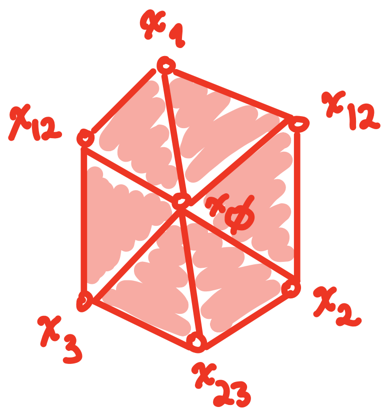
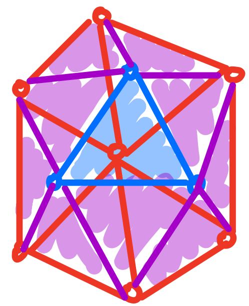
with simplices/faces $\{y_i\}_{i \in I} \cup \{x_{F_1}, x_{F_2}, \ldots, x_{F_\ell}\}$

when
- $I \in \mathcal{I}(M)$ is independent
- $F_1, F_2, \ldots, F_\ell$ are proper flats
- $I \subseteq F_1 \subset F_2 \subset \cdots \subset F_\ell \quad (\neq E)$

Δ_M is pure of dimension $r(M)-1$, containing both $\mathcal{I}(M)$ and $\mathrm{Cone}(\underline{\Delta}_M)$ as subcomplexes:



SPECIAL CASE : Boolean matroid M of rank n

	$\mathcal{I}(M)$ $= (n-1)$-simplex $2^{\{1,2,\dots,n\}}$	$\mathrm{Cone}(\Delta_M)$ = barycentric subdivision of $(n-1)$-simplex	Δ_M = boundary of stellohedron
$n=2$			
$n=3$			

Why did BHMPW introduce Δ_M ?

Its Stanley-Reisner ring has an amazing Artinian quotient by certain linear forms

$CH(M)$ = augmented Chow ring of M

$\cup$

$IH(M)$ = intersection cohomology of M (an $H(M)$-submodule)

$\cup$

$H(M)$ = graded Möbius algebra of M (a subalgebra)

instrumental in their proof of

- Dowling-Wilson's Top Heavy Conj. (1974) for M
- nonnegativity of Kazhdan-Lusztig polynomial for M

They used this weaker property of Δ_M than shellabity:

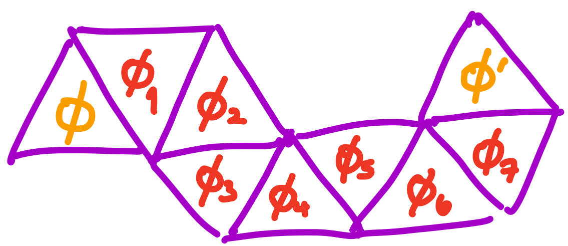
PROPOSITION (BHMPW) 2020: For any matroid M,

Δ_M is gallery-connected:

any two facets ϕ, ϕ' are connected by a gallery of facets

$$\phi = \phi_0, \phi_1, \phi_2, \ldots, \phi_{t-1}, \phi_t = \phi'$$

with each $\phi_i \cap \phi_{i+1}$ of dimension $r(M)-2$ (= codimension 1)



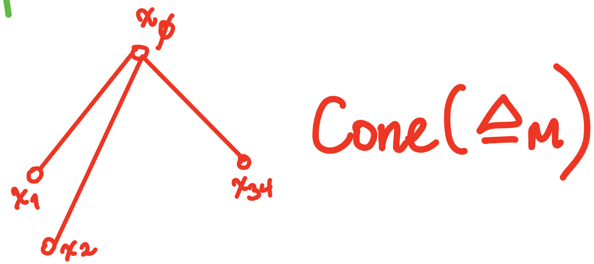
5. Two kinds of shellings of Δ_M and corollaries

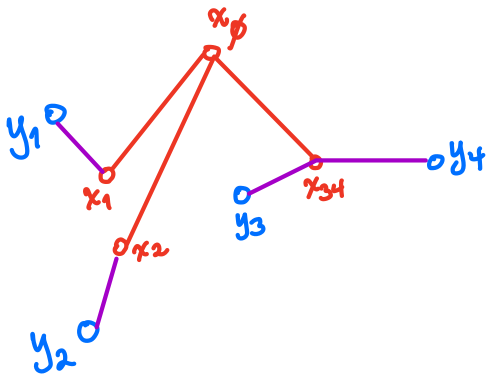
THEOREM (UMN REU 2021) For any matroid M, the augmented Bergman complex has two families of shellings:

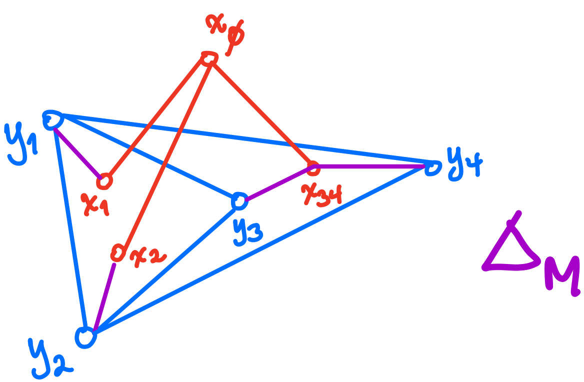
(i) some that shell the facets of $\mathrm{Cone}(\Delta_M)$ first, and facets of $\mathcal{I}(M)$ last.

(ii) some that shell the facets of $\mathcal{I}(M)$ first, and facets of $\mathrm{Cone}(\Delta_M)$ last.

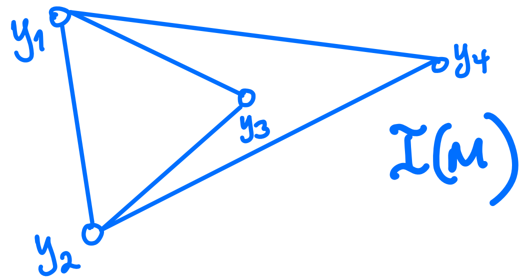
Type (i) shellings

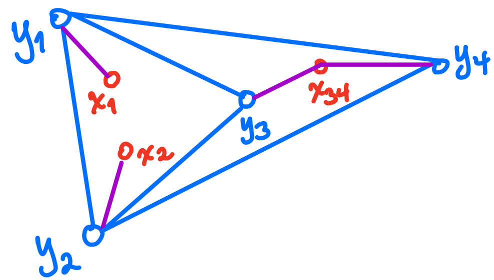


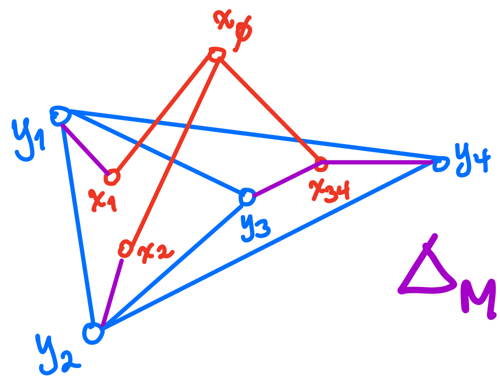




Type (ii) shellings







COROLLARY: (UMN REU 2021) The augmented Bergman complex Δ_M has $\|\Delta_M\| \approx \underbrace{S^{r(M)-1} \vee \cdots \vee S^{r(M)-1}}_{\beta\text{-fold wedge}}$

where β has two expressions:

(i) $\beta = T_M(1,1) = \#\mathcal{B}(M)$

because the homology facets in type (i) shellings are $\{y_i\}_{i \in B}$ indexed by bases B of M.

(ii) $\beta = \sum_{\text{flats } F \in \mathcal{F}(M)} T_{M|_F}(0,1)\, T_{M/F}(1,0)$

counting type (ii) shelling homology facets.

REMARK: The equality

$$T_M(1,1) = \sum_{\text{flats } F} T_{M|_F}(0,1)\, T_{M/F}(1,0)$$

appeared in work of Étienne–Las Vergnas 1998,
rediscovered in Kook–R.–Stanton 2000,

and is a specialization of a convolution formula

$$T_M(x,y) = \sum_{\text{flats } F} T_{M|_F}(0,y)\, T_{M/F}(x,0)$$

for Tutte polynomials.

The type (i) shellings show contractibility of

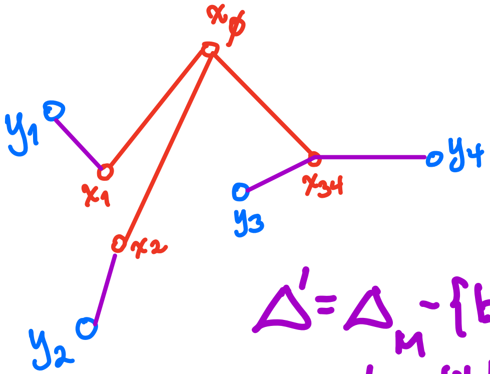
$$\Delta' = \Delta_M - \{\text{facets } \{y_i\}_{i\in B} : \text{bases } B \in \mathcal{B}(M)\}$$

Since matroid automorphisms set-wise stabilize the collection of basis facets, one can conclude:

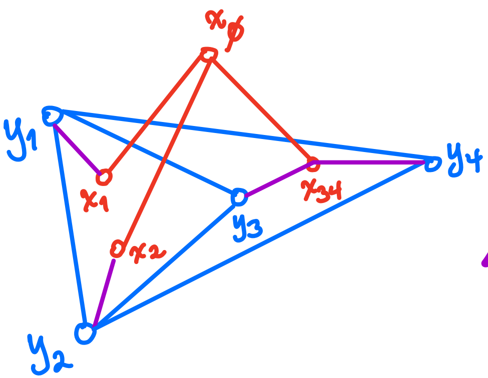
COROLLARY: The group $\mathrm{Aut}(M)$ acts on $H_{r(M)-1}(\Delta_M, \mathbb{Z})$ as a signed permutation representation, same as on $C_{r(M)-1}(\mathcal{I}(M), \mathbb{Z})$:

$$\sigma([b_1, b_2, \ldots, b_r]) = [b_{\sigma(1)}, \ldots, b_{\sigma(r)}] \text{ for bases } B = \{b_1, \ldots, b_r\} \in \mathcal{B}(M)$$

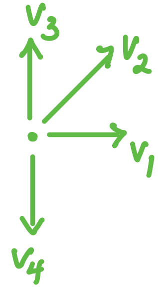
oriented simplex



$\Delta' = \Delta_M - \{\text{bases}\}$
is contractible

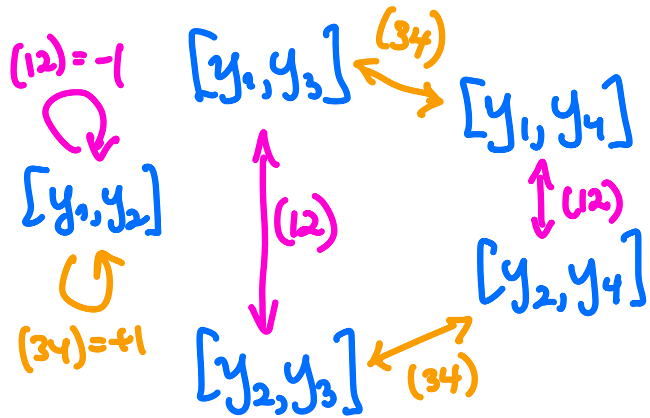


Δ_M



$$\mathrm{Aut}(M) = \{e, (12), (34), (12)(34)\}$$

$$H_1(\Delta_M) = \mathbb{Z}^5$$



REMARK: Neither I_M nor Δ_M have simple descriptions for their homology representations in general.

Notable special cases:

matroid M	$H_{r(M)-1}(I_M)$	$H_{r(M)-2}(\Delta_M)$
Boolean	trivial rep of S_n	sign rep of S_n
q-Boolean = $\mathbb{F}_q$-vector space	known virtually, not so explicit	Steinberg rep of $GL_n(\mathbb{F}_q)$
braid arrangement = complete graphic	an S_n-rep that restricts nicely to S_{n-1} (Kook 1996)	Lie rep of S_n

REMARK:

THEOREM:

(Amzi Jeffs 2022)

posted to arXiv last week!

Augmented Bergman complexes Δ_M are not only shellable, but vertex-decomposable.

(a known property for $I(M)$ and $\underline{\Delta}_M$)

Thanks
for
your
attention!