General linear groups as reflection group "wannabes"

Vic Reiner
Univ. of Minnesota

Aisenstadt Lectures
CRM Univ. de Montreal
March 14-16, 2017

Three reflection group counting stories where the general linear group GL_n wants in on the game...

TALK 1: Cycling subsets
(yesterday)

TALK 2: Catalan numbers
(today!)

TALK 3: Factorizations into reflections

Catalan numbers

- Catalan, q-Catalan numbers and a Cyclic sieving phenomenon
- Reflection group version
- $GL_n(\mathbb{F}_q)$ again

CATALAN & q-CATALAN

Recall Catalan numbers

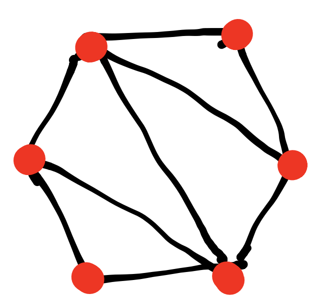
$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)(2n-1)\cdots(n+2)}{n(n-1)\cdots 2}$$

count many things, including triangulations of an $(n+2)$-gon

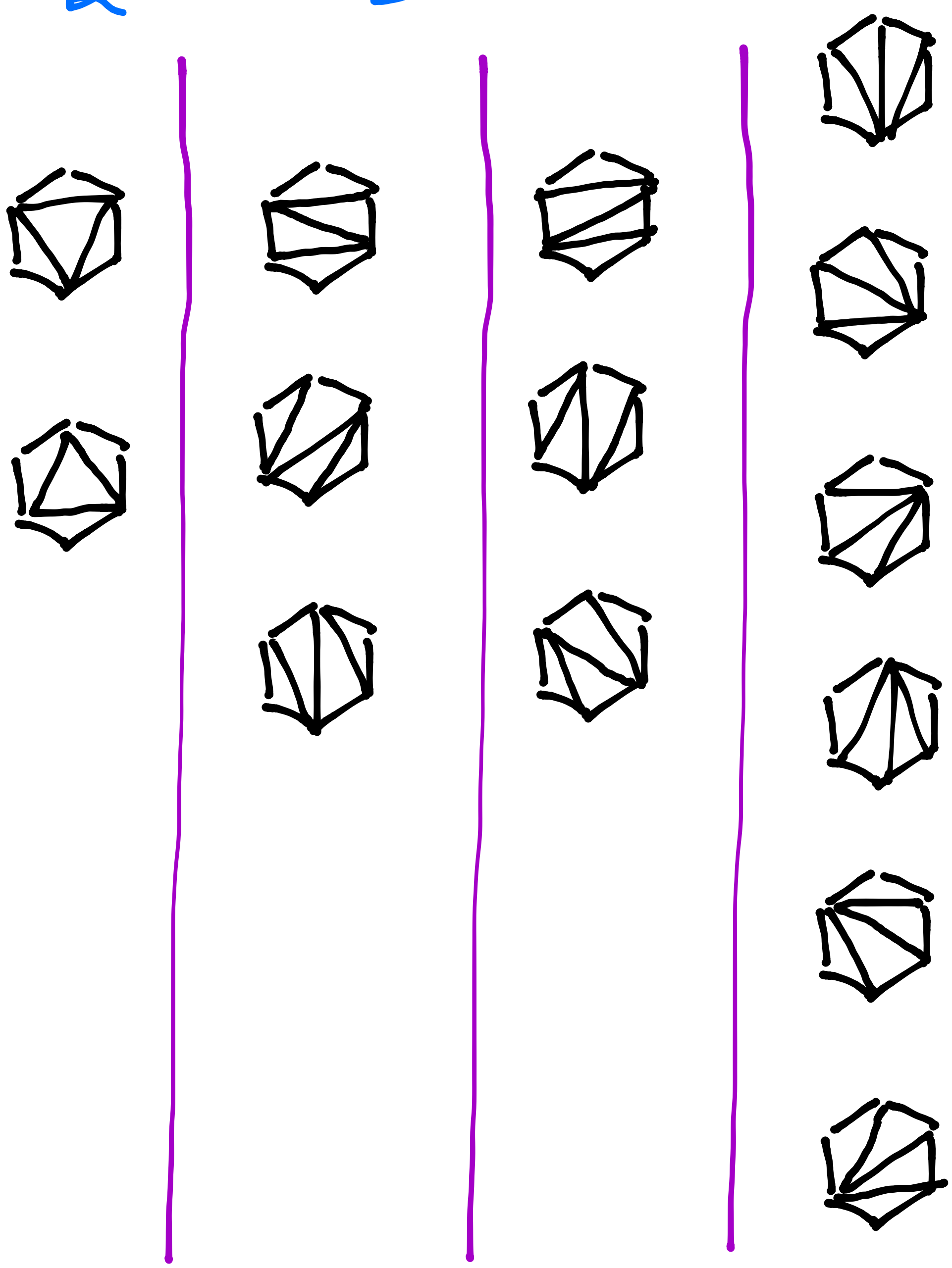
EXAMPLE $\quad n=4$

$$C_4 = \frac{1}{5}\binom{8}{4} = \frac{8\cdot 7\cdot 6}{4\cdot 3\cdot 2} = 14$$

counts these:



$C_4 = 14$
$= 2 + 3 + 3 + 6$



These have an obvious cyclic group acting by rotations, and a (non-obvious!) cyclic sieving phenomenon (CSP):

- they are a finite set X,
- with a cyclic group $C = \{1, c, c^2, \ldots, c^{m-1}\}$ of order m acting on X,
- and a polynomial $X(q)$, with

$$\#\{x \in X : c^d(x) = x\} = X(q)\Big|_{q = \left(e^{\frac{2\pi i}{m}}\right)^d}$$

THEOREM (R-Stanton-White)

MacMahon's q-Catalan number

$$C_n(q) := \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q$$

specialized to $q = \left(e^{\frac{2\pi i}{n+2}}\right)^d$

counts the triangulations

having $\frac{n+2}{d}$-fold symmetry.

Recall $[n]_q := 1 + q + q^2 + \ldots + q^{n-1}$

and $\begin{bmatrix} n \\ k \end{bmatrix}_q := \dfrac{[n]!_q}{[k]!_q \, [n-k]!_q}$

where $[n]!_q := [n]_q [n-1]_q \cdots [2]_q [1]_q$

Thus $C_n(q) := \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q$

$$= \frac{[2n]_q [2n-1]_q \cdots [n+2]_q}{[n]_q [n-1]_q \cdots [2]_q}$$

EXAMPLE:

$$C_4(q) = \frac{1}{[5]_q} \begin{bmatrix} 8 \\ 4 \end{bmatrix}_q = \frac{[8]_q [7]_q [6]_q}{[4]_q [3]_q [2]_q}$$

$$= 1 + q^2 + q^3 + 2q^4 + q^5 + 2q^6 + q^7 + 2q^8 + q^9 + q^{10} + q^{12}$$

REMARK: $C_n(q)$ is a polynomial in q, with nonnegative integer coefficients, i.e., $C_n(q) \in \mathbb{N}[q]$.

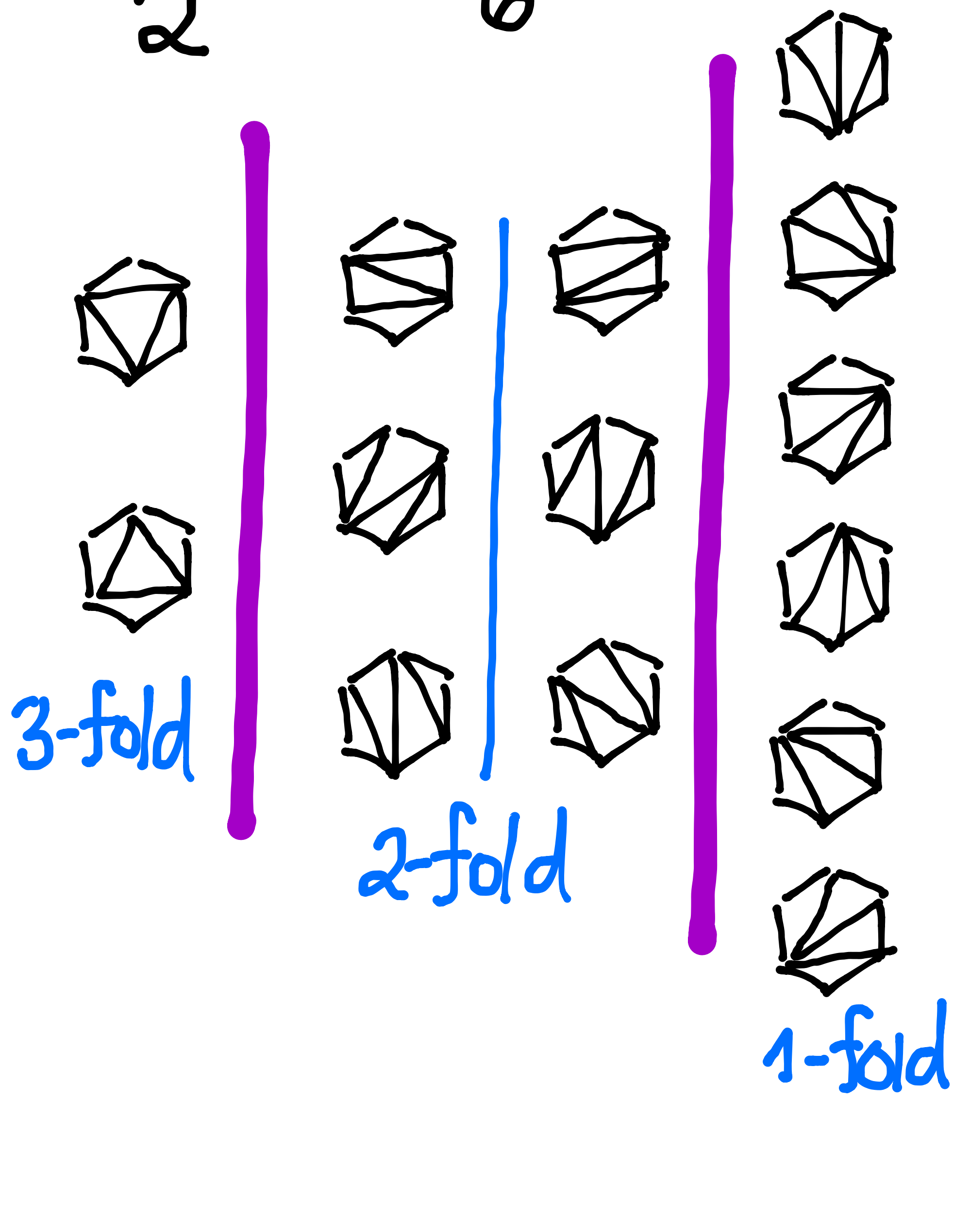
$$1+q^{2}+q^{3}+2q^{4}+q^{5}+2q^{6}+q^{7}+2q^{8}+q^{9}+q^{10}+q^{12}$$

$q=e^{\frac{2\pi i}{6}}$ $\rightsquigarrow$ 0

$q=e^{\frac{2\pi i}{3}}$ $\rightsquigarrow$ 2

$q=-1$ $\rightsquigarrow$ 6

$q=1$ $\rightsquigarrow$ 14



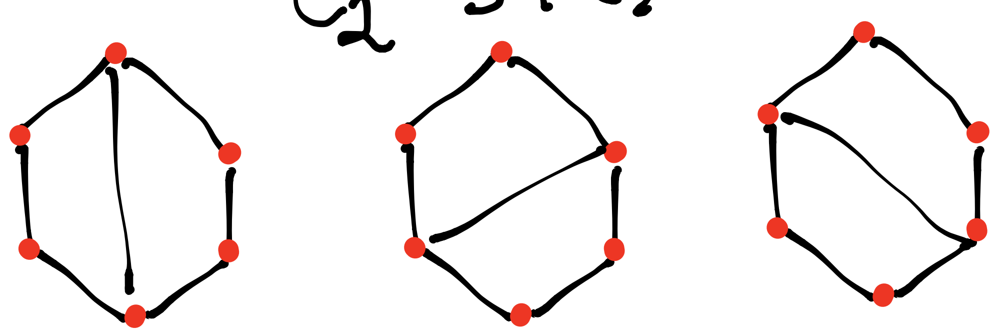
More generally, there are Fuss-Catalan numbers

$$C_n^{(m)} = \frac{1}{mn+1}\binom{(m+1)n}{n}$$

counting dissections of an $(mn+2)$-gon into $(m+2)$-gons, with a similar q-analogue, CSP.

EXAMPLE $m=2, n=2$

$$C_2^{(2)} = \frac{1}{5}\binom{3\cdot 2}{2} = 3$$



REMARK Just as $GL_n(\mathbb{F}_q)$ has an interpretation for q-binomials

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \#\left\{\begin{array}{l} k\text{-dimensional} \\ \mathbb{F}_q\text{-linear subspaces} \subset \mathbb{F}_q^n \end{array}\right\}$$

$$= \# \underbrace{Gr(k, \mathbb{F}_q^n)}_{\text{finite Grassmannian}} = \# GL_n(\mathbb{F}_q)/P_k$$

it also has one for q-Fuss-Catalans:

Rewrite it as

$$\frac{1}{[mn+1]_q}\begin{bmatrix} (m+1)n \\ n \end{bmatrix}_q = \frac{1}{[(m+1)n+1]_q}\begin{bmatrix} (m+1)n+1 \\ n \end{bmatrix}_q$$

$$= \frac{1}{[a+b]_q}\begin{bmatrix} a+b \\ a \end{bmatrix}_q$$

where $a = n$, $b = mn+1$, so $\gcd(a,b) = 1$

PROPOSITION: When $\gcd(a,b)=1$ and $\mathbb{F}_{q^{a+b}}^\times$ inside $GL_{a+b}(\mathbb{F}_q) \cong GL_{\mathbb{F}_q}(\mathbb{F}_q^{a+b})$ acts on $Gr(a, \mathbb{F}_q^{a+b})$, the subgroup $\mathbb{F}_q^\times \subset \mathbb{F}_{q^{a+b}}^\times$ acts trivially, but $\mathbb{F}_{q^{a+b}}^\times / \mathbb{F}_q^\times$ acts freely, with

$$\frac{1}{[a+b]_q} \begin{bmatrix} a+b \\ a \end{bmatrix}_q = \#\ \mathbb{F}_{q^{a+b}}^\times\text{-orbits on } Gr(a, \mathbb{F}_q^{a+b})$$

$$= \#\ \mathbb{F}_{q^{a+b}}^\times \backslash GL_n(\mathbb{F}_q) / P_{\lambda}$$

Don't know how to use this!

REFLECTION GROUP CATALANS

Recall a subgroup W of $GL_n(\mathbb{F})$ is a **reflection group** if it is generated by **reflections** t, that is, elements whose fixed space $(\mathbb{F}^n)^t = \{v \in \mathbb{F}^n : t(v) = v\}$ is a **hyperplane**.

Among them are the finite subgroups W of $GL_n(\mathbb{F})$ whose action on

$$S = \mathbb{F}[x_1, x_2, \ldots, x_n]$$

has the **W-invariants** polynomial

$$S^W = \mathbb{F}[f_1, f_2, \ldots, f_n]$$

DEFINITION: For a finite real reflection group W in $GL_n(\mathbb{R})$, acting irreducibly on $\mathbb{R}^n$,

let $S^W = \mathbb{R}[f_1, \ldots, f_n]$

with each f_i homogeneous

and of degrees $d_1 \le \ldots \le d_n =: h$

Coxeter number of W

- the W-Fuss Catalan number is

$$\mathrm{Cat}^{(m)}(W) := \prod_{i=1}^{n} \frac{mh + d_i}{d_i}$$

- the q-W-Fuss Catalan number is

$$\mathrm{Cat}^{(m)}(W, q) := \prod_{i=1}^{n} \frac{[mh + d_i]_q}{[d_i]_q}$$

EXAMPLE The symmetric group $\mathfrak{S}_n$ acts irreducibly on $\{x_1+\dots+x_n=0\} \subset \mathbb{R}^n$

and on $S = \mathbb{R}[x_1,\dots,x_n]/(x_1+\dots+x_n)$

with $S^{\mathfrak{S}_n} = \mathbb{R}[e_2(\bar{x}), e_3(\bar{x}), \dots, e_n(\bar{x})]$

where the i^{th} elementary symmetric polynomial $e_i(\bar{x})$ has degree i.

Thus the degrees are $2 \le 3 \le \dots \le n := h$

and $\mathrm{Cat}^{(1)}(\mathfrak{S}_n) = \prod_i \frac{h+d_i}{d_i}$

$$= \frac{(n+2)(n+3)\cdots(n+n)}{(2)(3)\cdots(n)}$$

$$= \frac{1}{n+1}\binom{2n}{n} = \text{Catalan number}$$

(usual)

EXAMPLE (continued)

More generally, one can check

$$\mathrm{Cat}^{(m)}(\mathfrak{S}_n, q) = \prod_i \frac{[mh+d_i]_q}{[d_i]_q}$$

$$= \frac{1}{[mn+1]_q} \begin{bmatrix} (m+1)n \\ n \end{bmatrix}_q$$

the q-Fuss-Catalan.

We conjectured a generalization of the CSP for rotating triangulations, proven by Eu and Fu 2008:

- triangulations
  $\rightsquigarrow$ clusters in finite type cluster algebras

- rotation
  $\rightsquigarrow$ Fomin & Zelevinsky's deformed Coxeter element of order $h+2$

- Fuss-Catalan dissections
  $\rightsquigarrow$ Fomin & Reading's generalized cluster complexes.

Still pretty mysterious...

THEOREM (Berest-Etingof-Ginzburg, Gordon 2003)

$\mathrm{Cat}^{(m)}(W)$ lies in $\mathbb{N}$, and $\mathrm{Cat}^{(m)}(W,q)$ lies in $\mathbb{N}[q]$. In fact,

$$\mathrm{Cat}^{(m)}(W,q) = \mathrm{Hilb}\left(\left(S/(\theta_1,\ldots,\theta_n) \right)^W, q \right)$$

where $\theta_1,\ldots,\theta_n$ are a

- homogeneous system of parameters of degree $mh+1$ in S,
- have $\mathbb{R}\theta_1+\ldots+\mathbb{R}\theta_n$ W-stable,
- with same W-rep'n as $\mathbb{R}x_1+\ldots+\mathbb{R}x_n$.

Why should such magical parameters $\theta_1, \dots, \theta_n$ exist??

In general, need subtle theory of rational Cherednik algebra $\mathbb{H}_c(W)$:

Verma module simple module

$$\begin{tikzcd}
M_c(\mathrm{triv}) \arrow[r, two heads] \arrow[d, phantom, "\cong"] & L_c(\mathrm{triv}) \arrow[d, phantom, "\cong \text{ if } c = m + \frac{1}{h}"] \\
S & S/(\theta_1, \dots, \theta_n)
\end{tikzcd}$$

Existence of the magical $(\theta_1, \ldots, \theta_n)$ was known earlier for $W = \mathfrak{S}_n$ (Haiman 1993, Dunkl 1998) but the arguments were tricky

---

For $W = W(B_n)$ = hyperoctahedral group
$= \{n \times n \text{ signed permutations}\}$
and $W = W(D_n) \subset W(B_n)$, it's easy:
$h = 2n$ or $2(n-1)$ is even
and one can take

$$(\theta_1, \ldots, \theta_n) = (x_1^{mh+1}, \ldots, x_n^{mh+1})$$

Now it's $GL_n(\mathbb{F}_q)$'s turn!

OBSERVATION: For $W = GL_n(\mathbb{F}_q)$ acting on $S = \mathbb{F}_q[x_1, \ldots, x_n]$,

$$(\theta_1, \ldots, \theta_n) = \left(x_1^{q^m}, \ldots, x_n^{q^m}\right)$$

- form a homogeneous system of parameters in S, of degree q^m
- with $\mathbb{F}_q\theta_1 + \ldots + \mathbb{F}_q\theta_n = \left\{ (c_1x_1 + \ldots + c_nx_n)^{q^m} : \underline{c} \in \mathbb{F}_q^n \right\}$
  W-stable
- carrying same W-rep'n as $\mathbb{F}_q x_1 + \ldots + \mathbb{F}_q x_n$!

Recall our THESIS:

$GL_n(\mathbb{F}_q)$ pretends to be a real reflection group with

- Coxeter number $h = q^n - 1$.
- Coxeter elements = Singer cycles

Why? Recall $S = \mathbb{F}_q[x_1, \ldots, x_n]$

has $S^{GL_n(\mathbb{F}_q)} = \mathbb{F}_q[\underbrace{f_1, \ldots, f_n}_{\text{Dickson polynomials}}]$

of degrees $q^n - q^{n-1} < \cdots < q^n - q^2 < q^n - q < \underbrace{q^n - 1}_{=: h}$

(and Singer cycles are Springer regular elements in $GL_n(\mathbb{F}_q)$ of order $h = q^n - 1$)

FURTHER EXAMPLE

Real reflection groups have magical systems of parameters $(\theta_1,\ldots,\theta_n)$ of degrees $mh+1$ relevant for Fuss-Catalan

$\left(\overset{m=1}{\rightsquigarrow} \; h+1 \text{ relevant for Catalan} \right)$

... while $GL_n(\mathbb{F}_q)$ has its magical systems of parameters

$$(\theta_1,\ldots,\theta_n) = (x_1^{q^m},\ldots,x_n^{q^m})$$

of degrees $q^m = (q^m-1)+1$

$\left(\overset{m=n}{\rightsquigarrow} \; (q^n-1)+1 = h+1 \right)$

This suggests, taking $S = \mathbb{F}_q[x_1, \ldots, x_n]$, that we should consider

$$\mathrm{Hilb}\left(\left(S/(\theta_1, \ldots, \theta_n) \right)^W, t \right)$$

$$= \mathrm{Hilb}\left(\left(S/(x_1^{q^m}, \ldots, x_n^{q^m}) \right)^{GL_n(\mathbb{F}_q)}, t \right)$$

as a reasonable $GL_n(\mathbb{F}_q)$-analogue of $\mathrm{Cat}^{(m)}(W, q)$.

But what does it look like?

It's not a product...

CONJECTURE (Lewis-R.-Stanton 2014)

$$\mathrm{Hilb}\left(\left(S/(x_1^{q^m},\ldots,x_n^{q^m}) \right)^{GL_n(\mathbb{F}_q)}, t \right)$$

$$= \sum_{k=0}^{\min(n,m)} t^{(n-k)(q^m-q^k)} \begin{bmatrix} m \\ k \end{bmatrix}_{q,t}$$

where recall

$$\begin{bmatrix} m \\ k \end{bmatrix}_{q,t} = (q,t)\text{-binomial} = \frac{m!_{q,t}}{k!_{q,t}\,(m-k)!_{q,t^{q^k}}}$$

$$= \frac{\mathrm{Hilb}(S^{P_k},t)}{\mathrm{Hilb}(S^{GL_m(\mathbb{F}_q)},t)}$$

CONJECTURE

$$\mathrm{Hilb}\left(\left(S/(x_1^{q^m}, \dots, x_n^{q^m}) \right)^{GL_n(\mathbb{F}_q)}, t \right) = \sum_{k=0}^{\min(n,m)} t^{(n-k)(q^m-q^k)} \begin{bmatrix} m \\ k \end{bmatrix}_{q,t}$$

has only been proven for

$$\begin{cases} n = \underbrace{0, 1}_{\text{trivial}}, \underset{\text{takes real work!}}{2} \\ m = \underbrace{0, 1}_{\text{easy}}, \underset{\text{recent work of P. Goyal}}{2} \end{cases}$$

It has a tantalizing consequence, using Gorenstein duality in $S/(x_1^{q^m}, \ldots, x_n^{q^m})$:

CONJECTURE: The divided power algebra $S^* = \mathrm{Div}(\mathbb{F}_q^n)$ has

$$\mathrm{Hilb}\left((S^*)^{GL_n(\mathbb{F}_q)}, t \right) = \sum_{k \geq 0} \frac{t^{n(q^k-1)}}{k!_{q,t}}$$

$$= 1 + \frac{t^{n(q-1)}}{1-t^{q-1}} + \frac{t^{n(q^2-1)}}{(1-t^{q^2-1})(1-t^{q^2-q})} + \frac{t^{n(q^3-1)}}{(1-t^{q^3-1})(1-t^{q^3-q})(1-t^{q^3-q^2})}$$

$$+ \cdots$$

No (good) idea yet how to prove it!

REMARK: The conjecture

$$\mathrm{Hilb}\left(\left(S/(x_1^{q^m}, \ldots, x_n^{q^m}) \right)^{GL_n(\mathbb{F}_q)}, t \right)$$

$$= \sum_{k=0}^{\min(n,m)} t^{(n-k)(q^m - q^k)} \begin{bmatrix} m \\ k \end{bmatrix}_{q,t}$$

would also be consistent with a (proven) CSP involving

$X = GL_n(\mathbb{F}_q)$-orbits in $(\mathbb{F}_{q^m})^n$

and action of

$C = \{1, c, c^2, \ldots, c^{q^m - 2}\} = \mathbb{F}_{q^m}^{\times}$ = Singer cycles in $GL_m(\mathbb{F}_q)$

Thanks again
for your attention !