General linear groups as reflection group "wannabes"

Vic Reiner
Univ. of Minnesota

Aisenstadt Lectures
CRM Univ. de Montreal
March 14-16, 2017

Three reflection group counting stories where the general linear group GL_n wants in on the game...

TALK 1: Cycling subsets
(Tuesday)

TALK 2: Catalan numbers
(yesterday)

TALK 3: Factorizations into reflections
(today!)

Short factorizations into...

- ...transpositions in $\mathfrak{S}_n$
  ...reflections in real reflection groups
  ...reflections in $GL_n(\mathbb{F}_q)$
- Frobenius's method for counting them
- Characterizing "short"

Short transposition factorizations

How many transpositions $t = (i,j)$ does it take to factor an n-cycle

$$c = (1, 2, 3, \ldots, n-1, n)$$
$$= (i_1 j_1)(i_2 j_2) \cdots (i_\ell j_\ell)$$
$$= t_1 t_2 \cdots t_\ell \;?$$

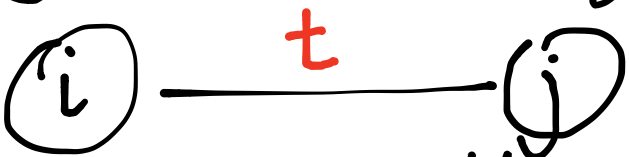
That is, what is the smallest ℓ?

Certainly $\ell = n-1$ is enough,

e.g. $c = (1,2)(2,3)\cdots(n-1,n)$ works

$$= t_1 \, t_2 \cdots t_{n-1}$$

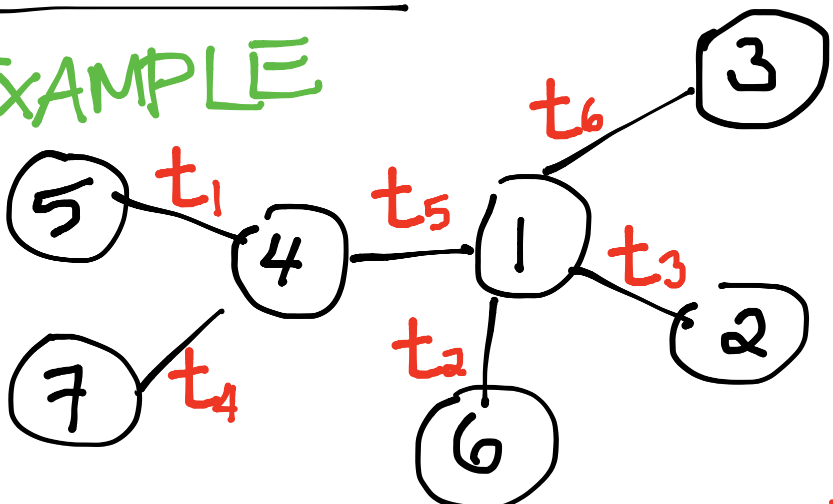
Also, one needs at least $n-1$ transpositions $t=(i,j)$ since when we picture them as edges



they must connect the n vertices

$1, 2, \ldots, n$

EXAMPLE



A tree, minimally connecting $n=7$ vertices with $n-1=6$ edges

$$t_1t_2t_3t_4t_5t_6=(45)(16)(12)(47)(14)(13)$$

$$=(1\,3\,7\,5\,4\,2\,6)$$

(= some n-cycle for $n=7$, not necessarily $(1,2,3,4,5,6,7)$)

How many short factorizations?

THEOREM (Hurwitz 1891)
The number of short factorizations

$$c = (1,2,\ldots,n) = t_1 t_2 \cdots t_{n-1}$$

is n^{n-2}, same as the number of trees on vertices $1, 2, \ldots, n$ (Cayley 1889)

EXAMPLE $n=3$

$$\left.\begin{aligned} c=(1,2,3) &= (12)(23) \\ &= (13)(12) \\ &= (23)(13) \end{aligned}\right\} \; 3 = 3^{3-2} \text{ short factorizations}$$

$$\left.\begin{array}{l} 1 - 2 - 3 \\ 1 - 3 - 2 \\ 2 - 1 - 3 \end{array}\right\} \; 3 = 3^{3-2} \text{ trees}$$

THEOREM (Hurwitz)

$$\#\left\{\begin{array}{c}\text{factorizations}\\ (1,2,\ldots,n)=t_1t_2\cdots t_{n-1}\end{array}\right\} = \#\left\{\begin{array}{c}\text{trees on}\\ ①,②,\ldots,ⓝ\end{array}\right\} \left(= n^{n-2}\right)$$

proof: (Dénes 1959; not Hurwitz's)

- All n-cycles are $\mathfrak{S}_n$-conjugate
- All transpositions are $\mathfrak{S}_n$-conjugate

$\Longrightarrow$ Each of the $(n-1)!$ different n-cycles has the same number (call it M) of short factorizations into transpositions.

So, $(n-1)!\,M = \#\left\{\begin{array}{c}\text{short factorizations}\\ \text{of all } n\text{-cycles}\end{array}\right\}$

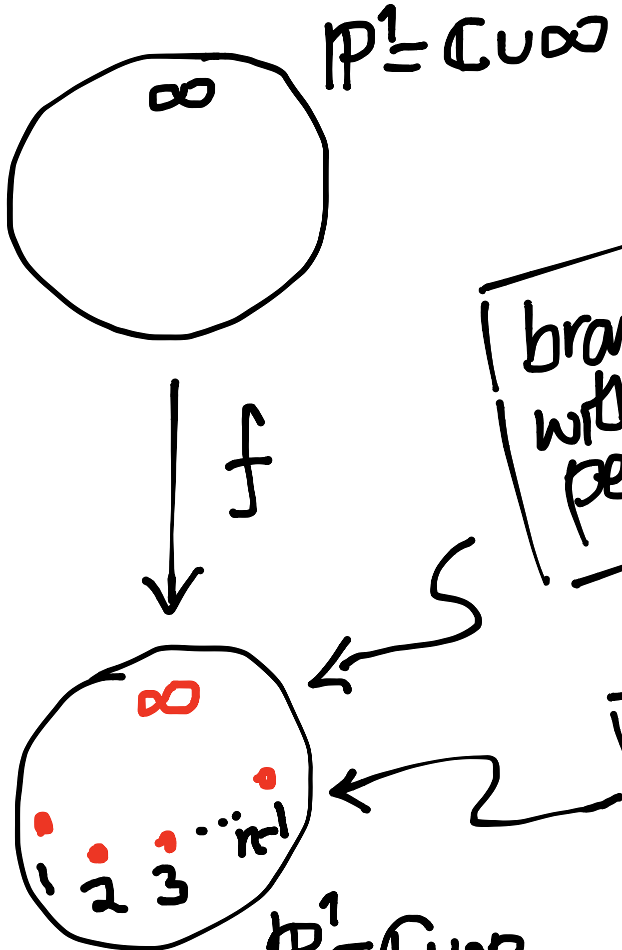
$$= \#\left\{\begin{array}{c}\text{trees on } ①,②,\ldots,ⓝ\\ \text{with edges labeled } t_1,\ldots,t_{n-1}\end{array}\right\}$$

$$= (n-1)!\ \#\{\text{trees on } ①,②,\ldots,ⓝ\}$$

Now cancel $(n-1)!$ $\square$

REMARK Why did Hurwitz care?

He was counting (up to a certain equivalence) degree n branched coverings $\mathbb{P}^1 \xrightarrow{f} \mathbb{P}^1$



branch point at ∞ with monodromy permutation an n-cycle

$n-1$ branch points, whose monodromy permutations are transpositions

Hurwitz's result generalizes.

THEOREM (Deligne 1970's, Bessis 2007, Michel 2014)

For a finite real reflection group W in $GL_m(\mathbb{R})$ acting irreducibly on $\mathbb{R}^m$, with Coxeter number h, any Coxeter element c has

$$\#\left\{\begin{array}{l}\text{short factorizations} \\ c = t_1 t_2 \cdots t_m \\ \text{into reflections } t_i\end{array}\right\} = \frac{h^m \cdot m!}{|W|}$$

$$\left(\underset{W=\mathfrak{S}_n}{\overset{\uparrow}{=}} \frac{n^{n-1}(n-1)!}{n!} = n^{n-2}\right)$$

We wish the beautiful Dénes proof generalized!

Recall our THESIS:

$GL_n(\mathbb{F}_q)$ pretends to be a real reflection group with

- Coxeter number $h = q^n - 1$.
- Coxeter elements = Singer cycles

EXAMPLE

Instead of Hurwitz's count of n^{n-2} short factorizations of an n-cycle into transpositions in $\mathfrak{S}_n$ ...

THEOREM (Lewis-R-Stanton 2014)

There are $(q^n-1)^{n-1}$ shortest factorizations of a Singer cycle

$$c = t_1 t_2 \cdots t_n$$

into reflections in $GL_n(\mathbb{F}_q)$

Crying out for a more direct or bijective or Dénes-style proof, instead of what we did.

Frobenius's method

It's a very reliable method, albeit not so insightful, for handling these factorization problems in a finite group G, knowing

- irreducible G-representations
- and their character values (or at least, some of them)

THEOREM (Frobenius 1896)
Given any G-conjugacy closed subsets $\mathcal{C}_1, \ldots, \mathcal{C}_\ell \subset G$

$$\#\left\{ \text{factorizations } c = c_1 c_2 \cdots c_\ell \text{ with } c_j \in \mathcal{C}_j \right\}$$

$$= \frac{1}{\#G} \sum_{\substack{\text{irreducible} \\ G\text{-characters} \\ \chi}} \frac{\chi(c^{-1})\,\chi(\mathcal{C}_1)\,\chi(\mathcal{C}_2)\cdots\chi(\mathcal{C}_\ell)}{\chi(e)^{\ell-1}}$$

where $\chi(\mathcal{C}) := \sum_{g \in \mathcal{C}} \chi(g)$

$$\frac{1}{\#G} \sum_{\chi} \frac{\chi(c^{-1})\chi(c_1)\chi(c_2)\cdots\chi(c_\ell)}{\chi(e)^{\ell-1}}$$

is evaluable...

- for n-cycles c in $\mathfrak{S}_n$ since most $\mathfrak{S}_n$-irreducibles χ^λ have $\chi^\lambda(c)=0$

- for Coxeter elements c in Weyl groups W via theory of Deligne-Lusztig characters (Michel)

- for Singer cycles c in $GL_n(\mathbb{F}_q)$ via q-analogues of $\mathfrak{S}_n$ character facts!

Characterizing "short"

Another way in which $GL_n(\mathbb{F})$ (over any field $\mathbb{F}$)

behaves like a finite real reflection group:

How can one tell when a reflection-factorization $w = t_1 t_2 \cdots t_\ell$ in $GL_n(\mathbb{F}) = GL(V)$ is shortest?

Given $w = t_1 t_2 \cdots t_n$,

Since t_i will fix a hyperplane H_i,

w will fix the space $H_1 \cap \ldots \cap H_l$

of dimension $\geq n-l$

Hence $V^w \supseteq H_1 \cap \ldots \cap H_l$

$$\dim(V^w) \geq n-l$$

$$\operatorname{codim}(V^w) \leq l$$

When does equality occur?

THM: Finite real reflection groups $W \leq GL_n(\mathbb{R})$ (Carter 1972) have $w = t_1 t_2 \cdots t_\ell$ is shortest

$\overset{(*)}{\iff}$ $\operatorname{codim}(V^w) = \ell$

Generally false for complex reflection groups

THM: A finite reflection group $W \leq GL_n(\mathbb{C})$ (Foster-Greenwood 2014) has $(*)$ $\iff$ either W real

or $W = G(d,1,n)$

$= \mathfrak{S}_n[\mathbb{Z}/d\mathbb{Z}]$

THM: (Huang-Lewis-R. 2015) General linear groups

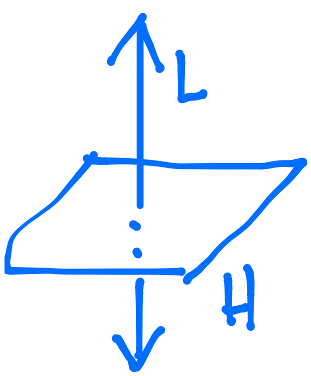
$W = GL_n(\mathbb{F})$ for any field $\mathbb{F}$

always have $(*)$.

Carter (1972) actually showed this:

THM: In a finite reflection group $W \leq GL_n(\mathbb{R})$ a reflection factorization $w = t_1 t_2 \cdots t_\ell$ is shortest

$\iff$ (a) the hyperplanes $\underset{V^{t_1}}{\overset{\shortparallel}{H_1}}, \ldots, \underset{V^{t_\ell}}{\overset{\shortparallel}{H_\ell}}$ have



$$\dim H_1 \cap \cdots \cap H_\ell = n - \ell$$

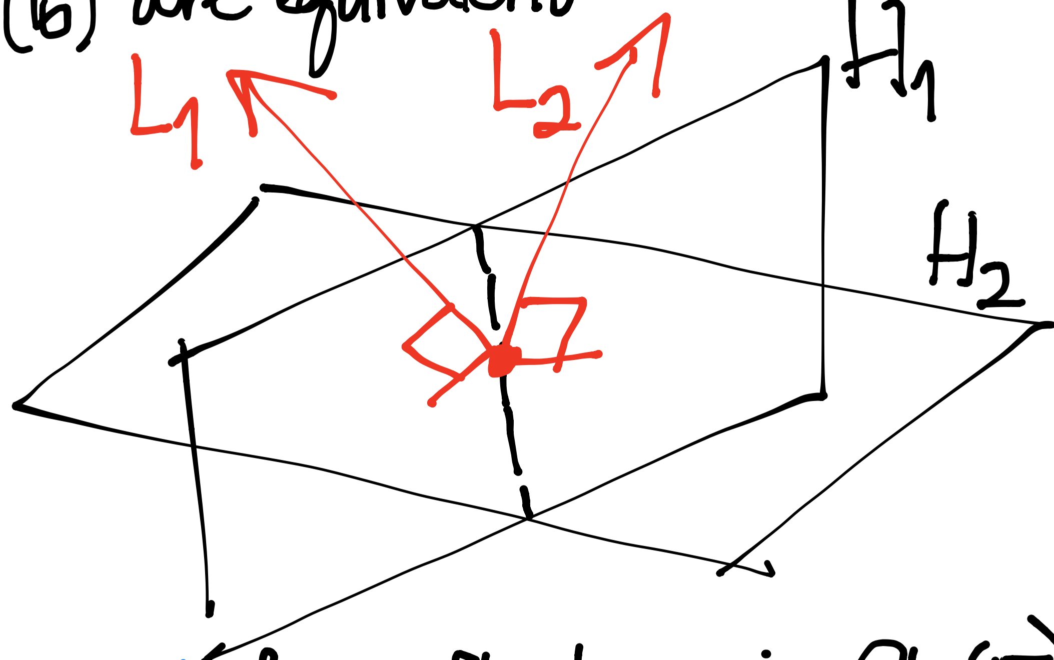
$\iff$ (b) the "root" lines $\underset{\mathrm{im}(t_1 - 1)}{\overset{\shortparallel}{L_1}}, \ldots, \underset{\mathrm{im}(t_\ell - 1)}{\overset{\shortparallel}{L_\ell}}$ have

$$\dim L_1 + \cdots + L_\ell = \ell$$

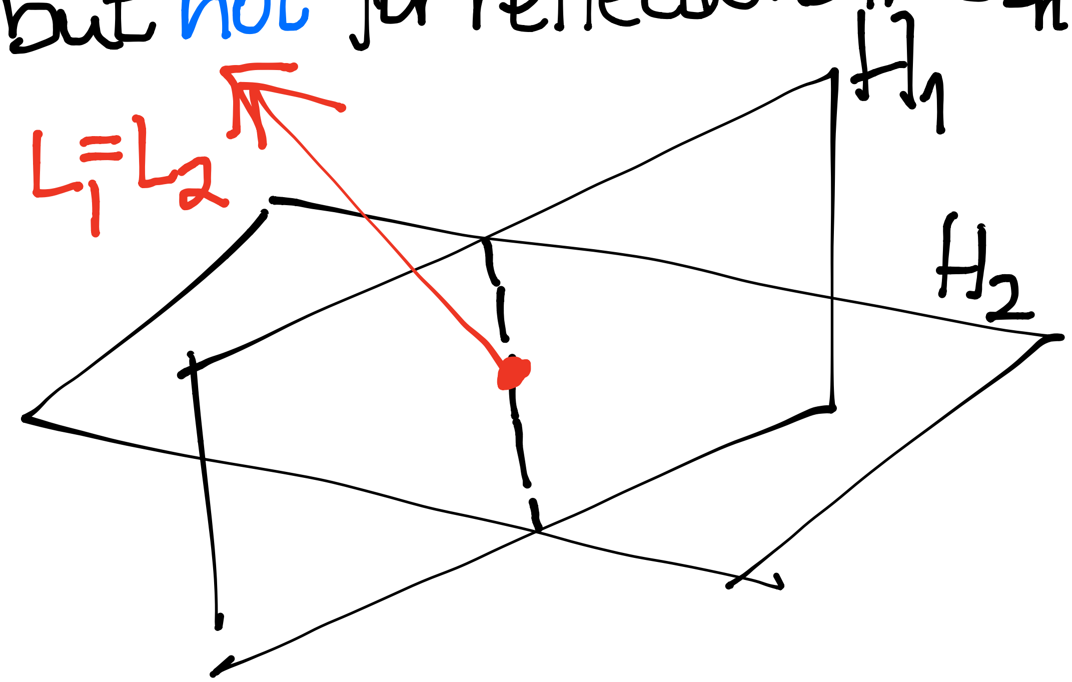
THM: (del Mas 2016) In $W = GL_n(\mathbb{F})$, a reflection factorization $w = t_1 t_2 \cdots t_\ell$ is shortest $\iff$

both $\left\{ \begin{array}{l} \text{(a) the hyperplanes } \underset{\underset{V^{t_1}}{\|}}{H_1}, \ldots, \underset{\underset{V^{t_\ell}}{\|}}{H_\ell} \text{ have } \\ \dim H_1 \cap \cdots \cap H_\ell = n - \ell \\ \text{— AND —} \\ \text{(b) the lines } \underset{\underset{\mathrm{im}(t_1 - 1)}{\|}}{L_1}, \ldots, \underset{\underset{\mathrm{im}(t_\ell - 1)}{\|}}{L_\ell} \text{ have} \\ \dim L_1 + \cdots + L_\ell = \ell \end{array} \right.$

For orthogonal or unitary reflections,
(a), (b) are equivalent:



but not for reflections in $GL_n(\mathbb{F})$:



So in this part of the story, $GL_n(\mathbb{F})$ emulates the real reflection groups, but also brings in its own novel features.

(Hooray!)

Thank you
once again
for your
attention,
and...

... one last

HUGE THANK YOU

to C.R.M.!