Algebraic Combinatorics

Using algebra to help one count

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Outline

1. What is Algebraic Combinatorics?
2. A general counting problem
3. Four properties
4. An algebraic approach

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What is Algebraic Combinatorics?

- **Combinatorics** is the study of finite or discrete objects, and their structure.
- **Counting** them is **enumerative** combinatorics.
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Example: enumerating subsets up to symmetry

We'll explore an interesting family of examples:

Enumerating **subsets**, up to **symmetry**.

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Let $[n] := \{1, 2, \ldots, n\}$,
permuted by the **symmetric group** $\mathfrak{S}_n$ on n letters.

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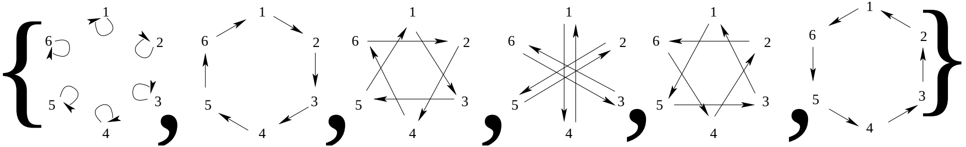
EXAMPLE: G=cyclic symmetry, with $n = 6$

$$G=$$



Counting G-orbits of subsets

Let's count the set

$$2^{[n]} := \{ \text{ all } \textbf{subsets} \text{ of } [n] \}$$

or equivalently,

black-white colorings of $[n]$,

but only **up to equivalence** by elements of G.

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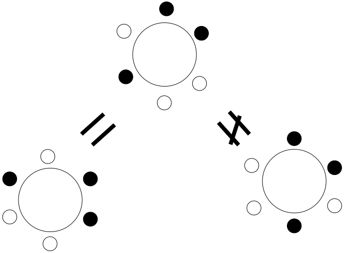
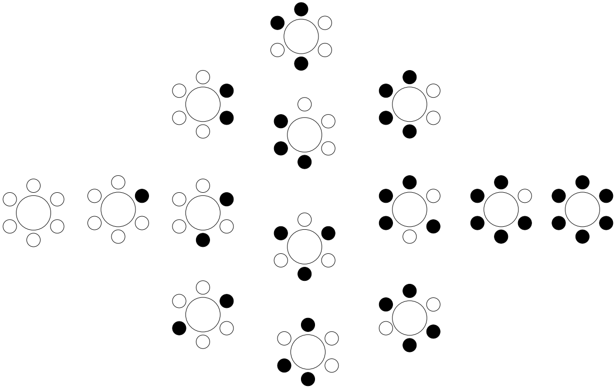
EXAMPLE: black-white necklaces



For G the cyclic group of rotations as above, G-orbits of colorings of $[n]$ are sometimes called **necklaces**.

All the black-white necklaces for $n = 6$



In this case, $|2^{[n]}/G| = 14$.

More refined counting of G-orbits

Let's even be more refined: count the sets

$$\binom{[n]}{k} := \{ \text{ all k-element } \textbf{subsets} \text{ of } [n] \ \}$$

or equivalently,

black-white colorings of $[n]$ with k blacks,

but again only **up to equivalence** by elements of G.

I.e. we want to understand

$$\begin{aligned} c_k &:= \left| \binom{[n]}{k} / G \right| \\ &= \text{number of } G\text{-orbits of black-white} \\ &\quad\ \text{colorings of } [n] \text{ with } k \text{ blacks.} \end{aligned}$$

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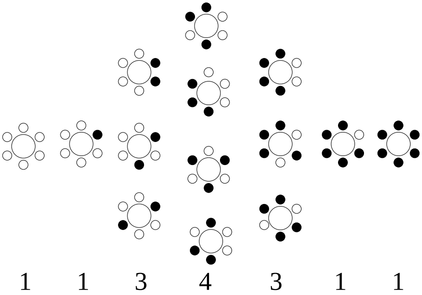
The refined necklace count for $n = 6$



Here $(c_0, c_1, c_2, c_3, c_4, c_5, c_6) = (1, 1, 3, 4, 3, 1, 1)$.

The basic question

QUESTION: What can we say in general about the sequence

$$c_0, c_1, c_2, \ldots, c_n?$$

AN ANSWER: They share many properties with the case where G is the **trivial** group, where the c_k are the **binomial coefficients**

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The binomial coefficients

Recall what binomial coefficient sequences

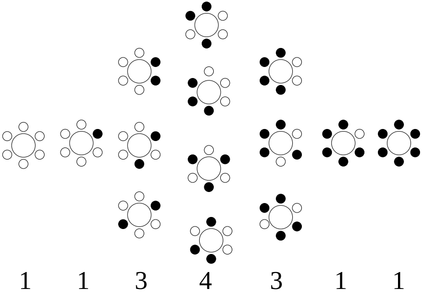
$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n-1}, \binom{n}{n}$$

look like:

$$\begin{array}{rccccccccccccc}
n=0: & & & & & & & 1 & & & & & & \\
n=1: & & & & & & 1 & & 1 & & & & & \\
n=2: & & & & & 1 & & 2 & & 1 & & & & \\
n=3: & & & & 1 & & 3 & & 3 & & 1 & & & \\
n=4: & & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
n=5: & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
n=6: & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

PROPERTY 1 (the easy one)

SYMMETRY: For any permutation group G, one has $c_k = c_{n-k}$



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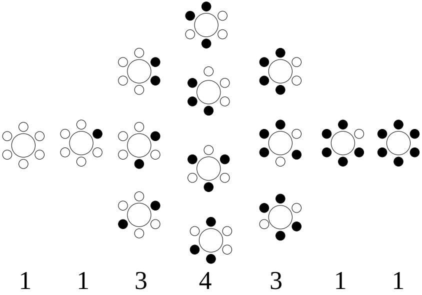

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PROPERTY 2 (the hardest one)

UNIMODALITY: (Stanley 1982)

$$c_0 \leq c_1 \leq \ldots \leq c_{\frac{n}{2}} \geq \cdots \geq c_{n-1} \geq c_n$$

e.g.

$$1 \leq 1 \leq 3 \leq 4 \geq 3 \geq 1 \geq 1$$

Nontrivial, but fairly easy with some **algebra**.
Currently **only** known in general via various algebraic means.

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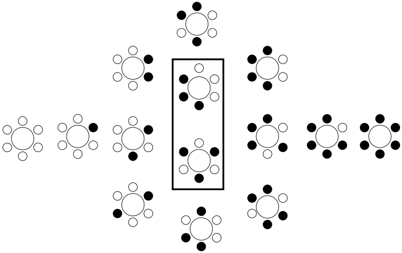
PROPERTY 3 (not so hard, but a bit surprising)

ALTERNATING SUM: (de Bruijn 1959)
$c_0 - c_1 + c_2 - c_3 + \cdots$ counts
self-complementary G-orbits.

e.g. there are $1 - 1 + 3 - 4 + 3 - 1 + 1 = 2$
self-complementary black-white necklaces for $n = 6$:



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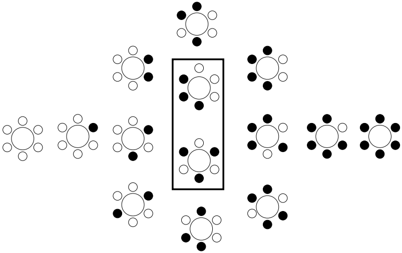

Wait! How was **that** like binomial coefficients?

It's easy to see that

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots = (1 + (-1))^n = 0$$

and there are **no self-complementary subsets** S of $[n]$.

PROPERTY 4 (not so hard, but also a bit surprising)

GENERATING FUNCTION: (Redfield 1927, Polya 1937)

$$c_0 + c_1 q + c_2 q^2 + c_3 q^3 + \cdots + c_n q^n$$

is the **average** over all g in G of the very simple products

$$\prod_{\text{cycles } C \text{ of } g} (1 + q^{|C|})$$

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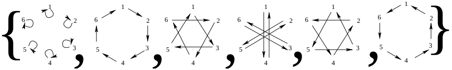

$$
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 & & & & \times\frac{1}{6}\ \downarrow & & & \\
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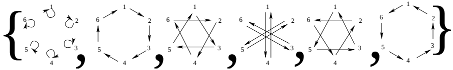

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Linearize!

In the algebraic approach, instead of thinking of numbers like $|2^{[n]}/G|$ and $c_k = |\binom{[n]}{k}/G|$ as **cardinalities** of **sets**, one tries to re-interpret them as **dimensions** of **vector spaces**.

Hopefully these vector spaces are natural enough that one can prove

- **equalities** of cardinalities via vector space **isomorphisms**,
- **inequalities** via vector space **injections** or **surjections**,
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Tensor products and colorings

Let $V = \mathbb{C}^2$ have a $\mathbb{C}$-basis

$$\begin{array}{cccc} \{ & w, & b & \} \\ & \| & \| & \\ & \textbf{white} & \textbf{black} & \end{array}$$

Then

$$V^{\otimes n} := \underbrace{V \otimes \cdots \otimes V}_{n \text{ tensor positions}}$$

has its tensor positions labelled by $[n]$, and has a $\mathbb{C}$-basis $\{e_S\}$ indexed by

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A typical basis tensor e_S

E.g. For $n = 6$ and the subset $S = \{1, 4, 5\}$, one has the basis element of $V^{\otimes 6}$

$$e_{\{1,4,5\}} = \begin{array}{ccccccccccc} b & \otimes & w & \otimes & w & \otimes & b & \otimes & b & \otimes & w \\ 1 & & 2 & & 3 & & 4 & & 5 & & 6 \end{array}$$

or for short, just

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Quick tensor product reminder

Recall tensor products are **multilinear**, that is, linear in each tensor factor.

E.g. for any constants c_1, c_2 in $\mathbb{C}$ one has

$$\begin{aligned} & b \otimes w \otimes (\mathbf{c_1} \cdot \mathbf{w} + \mathbf{c_2} \cdot \mathbf{b}) \otimes b \otimes b \otimes w \\ &= \mathbf{c_1} \cdot (b \otimes w \otimes \mathbf{w} \otimes b \otimes b \otimes w) \\ &\quad + \mathbf{c_2} \cdot (b \otimes w \otimes \mathbf{b} \otimes b \otimes b \otimes w) \end{aligned}$$

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The subspace of G-invariants

The subgroup G of $\mathfrak{S}_n$ acts on $V^{\otimes n}$ by **permuting the tensor positions**.

Consider the subspace of G-**invariants**

$$\left(V^{\otimes n}\right)^G.$$

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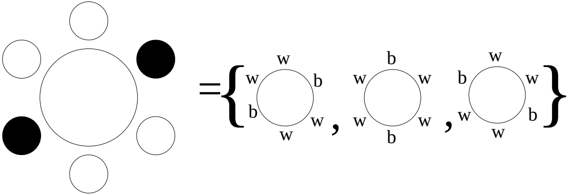

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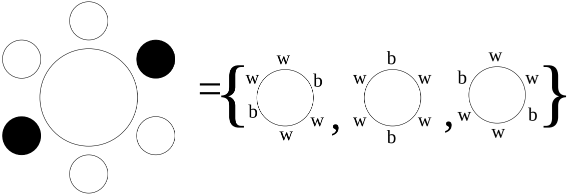

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Interpreting the c_k's

Better yet, if one defines subspaces

$$V_k^{\otimes n} := \mathbb{C}\text{-span of } \{e_S \text{ with } |S| = k\}$$

then

- one has a direct sum decomposition $V^{\otimes n} = \bigoplus_{k=0}^{n} V_k^{\otimes n}$,
- the group G acts on each $V_k^{\otimes n}$, and
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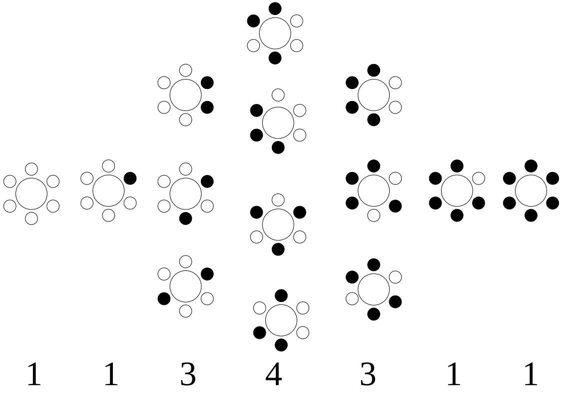
This gives a good framework for understanding the c_k.
We've naturally **linearized** this picture:



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$$\left(V_k^{\otimes n}\right)^G \to \left(V_{n-k}^{\otimes n}\right)^G.$$

Silly proof of SYMMETRY (cont'd)

Any $\mathbb{C}$-linear map

$$t : V \longrightarrow V$$

gives rise to a $\mathbb{C}$-linear map

$$t : V^{\otimes n} \longrightarrow V^{\otimes n}$$

acting **diagonally**, i.e. the same in each tensor position.

Schur-Weyl duality

Such maps **commute** with the *G*-action permuting the tensor positions.

$$\begin{tikzcd}
v_1 \otimes v_2 \otimes v_3 \arrow[r, mapsto, "t"] \arrow[d, "g = (12)"] & t(v_1) \otimes t(v_2) \otimes t(v_3) \arrow[d, "g = (12)"] \\
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Silly proof of SYMMETRY (cont'd)

Let $t : V \rightarrow V$ **swap** the basis elements $\{w, b\}$, so on tensors it also swaps them, e.g.

$$t(bwbbwb) = wbwwbw.$$

Note that $t^2 = 1$, so t gives a $\mathbb{C}$-linear isomorphism

$$V_k^{\otimes n} \rightarrow V_{n-k}^{\otimes n}$$

which restricts to a $\mathbb{C}$-linear isomorphism

$$\left(V_k^{\otimes n}\right)^G \rightarrow \left(V_{n-k}^{\otimes n}\right)^G,$$

as desired to show $c_k = c_{n-k}$. QED

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Not-so-silly proof of Property 3: ALTERNATING SUM

We want to show that

$$c_0 - c_1 + c_2 - c_3 + \cdots$$

counts **self-complementary** G-orbits.

Begin with this observation:

PROPOSITION: The number of **self-complementary** G-orbits is the **trace** of the color-swapping map t from before, when it acts on $(V^{\otimes n})^G$.

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Not-so-silly proof (cont'd)

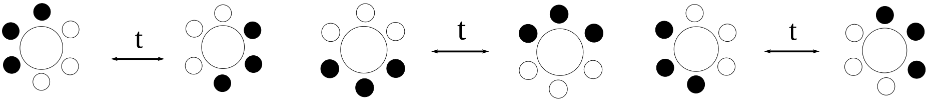
Proof.

- t permutes the basis of $(V^{\otimes n})^G$ indexed by G-orbits of black-white colorings, and
- t **fixes** such a basis element if and only if this G-orbit is **self-complementary**.QED

For example, with $n = 6$ and G=cyclic rotation, t fixes this basis element of $\left(V^{\otimes 6}\right)^G$

$$wwwbbb + bwwwbb + bbwwwb + bbbwww + wbbbww + wwbbbw$$

as it is a sum over the t-stable G-orbit shown below:



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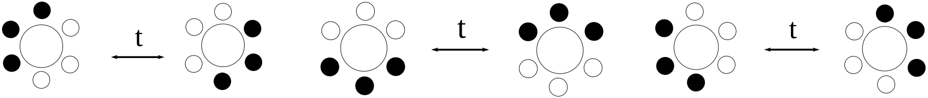

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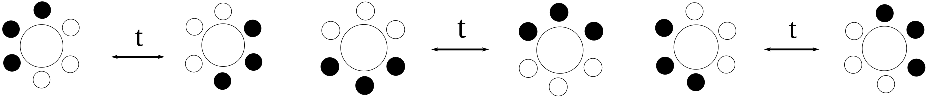

Not-so-silly proof (cont'd)

What does this have to do with $c_0 - c_1 + c_2 - \cdots$?

Well, inside $GL(V)$,

$$t = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and } s = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

- are both diagonalizable and have eigenvalues $+1, -1$,
- so they must be **conjugate** within $GL(V)$,
- so t, s must act on $V^{\otimes n}$ and on $(V^{\otimes n})^G$ by $\mathbb{C}$-linear maps which are conjugate.

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Not-so-silly proof (cont'd)

Recall that $Tr(AB) = Tr(BA)$ implies
conjugate transformations have the **same trace**:

$$Tr(PAP^{-1}) = Tr(P^{-1} \cdot PA) = Tr(A).$$

Thus s, t must act with the **same trace** on $(V^{\otimes n})^G$.

We know from the previous Proposition that this trace for t is the number of self-complementary G-orbits.

So it suffices to apply the following fact with $q = -1$...

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In particular, for $q = -1$, the element $s = s(-1)$ acts with trace

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Proof.

- $s(q)$ fixes w.
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$$s(q)(bwbbbw) = \mathbf{q}b \otimes w \otimes \mathbf{q}b \otimes \mathbf{q}b \otimes \mathbf{q}b \otimes w = \mathbf{q}^4 \cdot bwbbbw.$$

- Hence $s(q)$ scales all of $V_k^{\otimes n}$ by q^k,
- and therefore scales all of $\left(V_k^{\otimes n}\right)^G$ by q^k.
- So $s(q)$ acts on $\left(V^{\otimes n}\right)^G = \bigoplus_k \left(V_k^{\otimes n}\right)^G$ with trace $\sum_k c_k q^k$.

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A proof of Property 2: UNIMODALITY

We want to show that for $k < \frac{n}{2}$, one has

$$c_k \leq c_{k+1}$$

So we'd like a $\mathbb{C}$-linear **injective** map

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Maybe we should look for an injective map

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The only natural injection

There is only one **obvious candidate** for such an injection $U_k : V_k^{\otimes n} \hookrightarrow V_{k+1}^{\otimes n}$, namely define

$$U_k(e_S) := \sum_{\substack{T \supset S: \\ |T|=k+1}} e_T$$

E.g. for $n = 6$, $k = 2$, one has

$$U_2(bwbwww) = b\mathbf{b}bwww + bwb\mathbf{b}ww + bwbw\mathbf{b}w + bwbww\mathbf{b}$$

Easy to check U_k commutes with $\mathfrak{S}_n$ permuting positions. But why is U_k **injective**?

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A cute injectivity argument

There are several arguments for this, but here's a cute one.

PROPOSITION: For $k < \frac{n}{2}$, the operator $U_k^t U_k$ on $V_k^{\otimes n}$ turns out to be **positive definite**, i.e. all its (real) eigenvalues are **strictly** positive.

In particular,

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Quick review of positive (semi-)definiteness

Recall that a real symmetric matrix $A = A^t$

- always has only **real eigenvalues**,
- is **positive semidefinite** if they're all nonnegative, or equivalently, $\mathbf{x}^t A \mathbf{x} \geq 0$ for all vectors $\mathbf{x}$,
- is **positive definite** if they're all positive, or equivalently, if $\mathbf{x}^t A \mathbf{x} > 0$ for all **nonzero** vectors $\mathbf{x}$,
- is always positive semidefinite when $A = B^t B$ for some rectangular matrix B, since

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PROOF that $U_k^t U_k$ is positive definite.

- Check (on each e_S) that

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- Hence

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- Hence the sum $U_k^t U_k$ is positive **definite**. QED

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- Hence the sum $U_k^t U_k$ is positive **definite**. QED

A cute injectivity argument (cont'd)

PROOF that $U_k^t U_k$ is positive definite.

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A proof of Property 4: GENERATING FUNCTION

(To be flipped through at lightning speed during the talk; read it **later**, if you want!)

We want to show

$$\sum_{k=0}^{n} c_k q^k = \frac{1}{|G|} \sum_{g \in G} \left(\prod_{\text{cycles } C \text{ of } g} (1 + q^{|C|}) \right)$$

Such **averages** over the group are ubiquitous due to the following easily-checked fact.

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An idempotent projector

PROPOSITION: When a finite group G acts linearly on a vector space W over a field in which $|G|$ is invertible (nonzero), the map $W \xrightarrow{\pi} W$ given by

$$w \mapsto \frac{1}{|G|} \sum_{g \in G} g(w)$$

is

- **idempotent**, i.e. $\pi^2 = \pi$, and
- π *projects* onto the subspace of *G-invariants* W^G.

Trace of idempotent = dimension of image

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Putting two idempotent facts together

Apply these two facts to the idempotent projector $\pi = \frac{1}{|G|}\sum_{g \in G} g$ onto the G-fixed subspace of each $W = V_k^{\otimes n}$:

$$\sum_k c_k q^k = \sum_k \dim_{\mathbb{C}} \left(V_k^{\otimes n}\right)^G q^k = \sum_k Tr\left(\pi|_{V_k^{\otimes n}}\right) q^k$$

$$= \frac{1}{|G|} \sum_{g \in G} \left(\sum_k Tr(g|_{V_k^{\otimes n}}) q^k \right).$$

It only remains to show

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note that

- any g in G **permutes** the basis for $V_k^{\otimes n}$ indexed by black-white colorings,
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Proof by example

E.g. $g = (12)(34)(567)$ in $\mathfrak{S}_7$ fixes these colorings/tensors:

12	34	567	
ww	ww	www	1
bb	ww	www	$+q^2$
ww	bb	www	$+q^2$
ww	ww	bbb	$+q^3$
bb	bb	www	$+q^2 \cdot q^2$
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For **combinatorial** purposes, it is definitely **worth learning more algebra**, including (but not limited to)

- Linear, multilinear algebra,
- Group theory,
- Representation theory,
- Commutative algebra, Hopf algebras, ...

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