Sharp representation stability for configurations of points in $\mathbb{R}^d$

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OUTLINE:

1. Rep'n stability review

2. Church's Thm. on $\mathrm{Conf}(n, X)$

3. Sharpening for $X = \mathbb{R}^d$

4. The crux of the proof

5. Wiltshire-Gordon's Conjectures and a related result

1. Representation Stability review

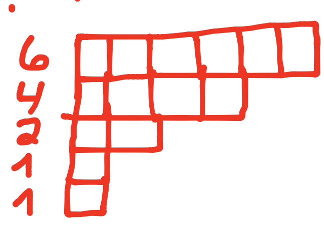
$\mathfrak{S}_n$ = symmetric group on n letters

has (complex, finite dimensional) irreducible representations $\{\chi^\lambda\}$ indexed by partitions of n

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell)$$

$$|\lambda| = \lambda_1 + \ldots + \lambda_\ell = n$$

e.g. $\lambda = 64211$



EXAMPLE: $n=3$

$$\chi^{\ydiagram{3}} = \text{trivial } \mathfrak{S}_3\text{-rep'n on } \mathbb{C}^1$$

$$\chi^{\ydiagram{1,1,1}} = \text{sgn } \mathfrak{S}_3\text{-rep'n on } \mathbb{C}^1$$

$$\chi^{\ydiagram{2,1}} = \underset{\substack{\circlearrowleft \\ \mathfrak{S}_3 \text{ permuting coordinates}}}{\mathbb{C}^3} \Big/ \mathbb{C}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

DEF: Say $\mathfrak{S}_n$-reps $\{V_n\}_{n=1,2,\ldots}$

stabilize by n_0 if the

unique decomposition

$$V_{n_0} = \sum_{\substack{\text{partitions} \\ \lambda \text{ of } n_0}} c_\lambda \chi^{\boxed{\lambda}}$$

determines all the rest for $n \geq n_0$ via

$$V_n = \sum_{\substack{\text{partitions} \\ \lambda \text{ of } n_0}} c_\lambda \chi^{\boxed{\lambda}\,\underbrace{\square\cdots\square}_{n-n_0}}$$

Say $\{V_n\}$ stabilizes sharply at n_0 if this n_0 is smallest with this property.

EXAMPLE: $\{V_n = \mathbb{C}^n\}$ stabilizes sharply at $n_0 = 2$

$\circlearrowleft$
$\mathfrak{S}_n$ permuting coordinates

$$V_n = \mathbb{C}^n = \mathbb{C}\begin{bmatrix}1\\1\\ \vdots\\1\end{bmatrix} \oplus \begin{bmatrix}1\\1\\ \vdots\\1\end{bmatrix}^{\perp}$$

$$= \begin{cases} \chi^{(n)} + \chi^{(n-1,1)} & \text{for } n \geq 2 = n_0 \\ \chi^{(1)} & \text{for } n = 1 \end{cases}$$

2. Church's example

DEF: X a topological space

$\mathrm{Conf}(n,X)$ = configuration space of n labelled/ordered distinct points in X

$$= \{(x_1,\dots,x_n) : x_i \neq x_j \text{ for } 1 \leq i < j \leq n\}$$

$$= X^n \setminus \underbrace{\bigcup_{1\leq i<j\leq n} \{x_i = x_j\}}_{\text{thick diagonal in } X^n}$$

$\mathfrak{S}_n$ acts on $\mathrm{Conf}(n,X)$ permuting coordinates and on $H^i(\mathrm{Conf}(n,X))$ with $\mathbb{C}$-coefficients.

THM (Church 2011)

Let X be a connected, orientable d-manifold with $d \geq 2$, and $H^\bullet(X)$ finite-dimensional

Fixing $i \geq 0$, $\{V_n = H^i(\mathrm{Conf}(n,X))\}$ as $\mathfrak{S}_n$-rep'ns

- vanish unless $d-1$ divides i
- stabilize by $n_0 = \begin{cases} 2i & \text{if } d \geq 3 \\ 4i & \text{if } d = 2 \end{cases}$.

EXAMPLE: $i=1 \quad d=2$

$$H^1(\mathrm{Conf}(n,\mathbb{R}^2)) \circlearrowleft \mathfrak{S}_n$$

$$= \begin{cases} 0 & n=1 \\ \chi^{(2)} & n=2 \\ \chi^{(2,1)} + \chi^{(3)} & n=3 \\ \chi^{(2,2)} + \chi^{(3,1)} + \chi^{(4)} & n=4 \ (=n_0) \\ \chi^{(n-2,2)} + \chi^{(n-1,1)} + \chi^{(n)} & \text{for } n \geq 5 \end{cases}$$

3. Sharpening for $X = \mathbb{R}^d$

THM (Hersh-R. 2014):

Let $d \geq 2$. Fixing $i \geq 0$,

$\{H^i(\mathrm{Conf}(n, \mathbb{R}^d))\}$ as $\mathfrak{S}_n$-rep'ns

- vanish unless $d-1$ divides i
- stabilizes sharply at $n_0 = \begin{cases} \frac{3}{d-1} \cdot i & \text{if } d \text{ odd} \\ 1 + \frac{3}{d-1} i & \text{if } d \text{ even} \end{cases}$

(cf. previous $\begin{Bmatrix} 2i & \text{if } d \geq 3 \\ 4i & \text{if } d = 2 \end{Bmatrix}$)

Why might we care about the $X = \mathbb{R}^d$ case?

A couple of reasons...

(1) Church's method used Totaro's description of E_2-page in Leray spec. seq. for

$$\mathrm{Conf}(n, X) \hookrightarrow X^n:$$

$$E_2^{*,*} = \bigoplus_{\substack{\text{set partitions} \\ \sigma \text{ of } \{1,2,\dots,n\}}} H^\bullet(\underbrace{\mathrm{Conf}(\sigma, \mathbb{R}^d)}_{\substack{\text{points distinct} \\ \text{within blocks} \\ \text{of } \sigma}}) \otimes H^\bullet(\underbrace{X^\sigma}_{\substack{\text{points equal} \\ \text{within blocks} \\ \text{of } \sigma}})$$

Q: Among all d-manifolds X $\left(\begin{array}{l} d \geq 2, \\ \text{conn.,} \\ \text{orientable,} \\ \dim H^\cdot(X) \text{ finite} \end{array}\right)$, does $\left\{ H^i(\mathrm{Conf}(n, X)) \right\}$ stabilize earliest for $X = \mathbb{R}^d$?

② We know the $X = \mathbb{R}^2 = \mathbb{C}^1$ case is important since

$$\mathrm{Conf}(n, \mathbb{R}^2) = K(PB_n, 1)$$

for the pure braid group PB_n:

$$1 \to \underset{\text{pure braid group}}{PB_n} \longrightarrow \underset{\text{braid group}}{B_n} \longrightarrow S_n \longrightarrow 1$$

So $H^i(\mathrm{Conf}(PB_n, 1)) = \underset{\text{group cohomology}}{H^i(PB_n)}$

(And it also plays a crucial role in the **Church-Ellenberg-Farb** work on statistics on monic squarefree polynomials $f(T)$ in $\mathbb{F}_q[T]$.)

4. The crux of the proof

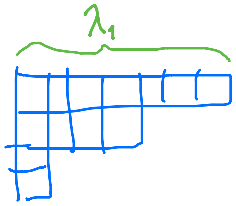
MAIN STABILITY LEMMA (Hemmer 2011):

For an $\mathfrak{S}_m$-rep'n χ, define $\mathfrak{S}_n$-rep'ns

$$M_n(\chi) := \begin{cases} 0 & \text{if } n < m \\ (\chi \otimes \mathbb{1})\uparrow_{\mathfrak{S}_m \times \mathfrak{S}_{n-m}} & \text{if } n \geq m \end{cases}.$$

Then $\{M_n(\chi^\lambda)\}$ stabilizes sharply at

$$n_0 = \underbrace{|\lambda|}_{\text{number of cells}} + \underbrace{\lambda_1}_{\text{largest part}}$$

$\lambda =$ 

COROLLARY: For a finite sum

$$\sum_{\mu} \underbrace{c_\mu}_{\in \mathbb{Z}_{>0}} \chi^\mu$$

with μ possibly of different sizes,

$$\left\{ M_n \left(\sum_{\mu} c_\mu \chi^\mu \right) \right\}$$

stabilizes

sharply at $n_0 = \max\{|\mu| + \mu_1\}$.

EXAMPLE:

We'll see that

$$H^1(\mathrm{Conf}(n,\mathbb{R}^2)) = M_n\big(\chi^{\underset{\mu}{(2)}}\big)$$

explaining why it stabilized at $n_0 = 4$
$= 2+2$
$= |\mu| + \mu_1$

$$H^2(\mathrm{Conf}(n,\mathbb{R}^2)) = M_n\big(\chi^{(2,1)} + \chi^{(3,1)}\big)$$

will stabilize at

$$n_0 = 7 = \max\{\underset{(2,1)}{3+2}, \underset{(3,1)}{4+3}\}$$

So we need $H^i(\mathrm{Conf}(n,\mathbb{R}^d))$ expressed in the form $M_n(\text{———})$.

THM (Orlik-Solomon 1980 for $d=2$, Sundaram-Welker 1997 for all d)

$H^i(\mathrm{Conf}(n,\mathbb{R}^d))$ vanishes unless $i=(d-1)i'$ in which case it is isomorphic to

$$\begin{cases} PBW^{i'} & \text{if } d \text{ odd} \\ WH^{i'} & \text{if } d \text{ even} \end{cases}$$

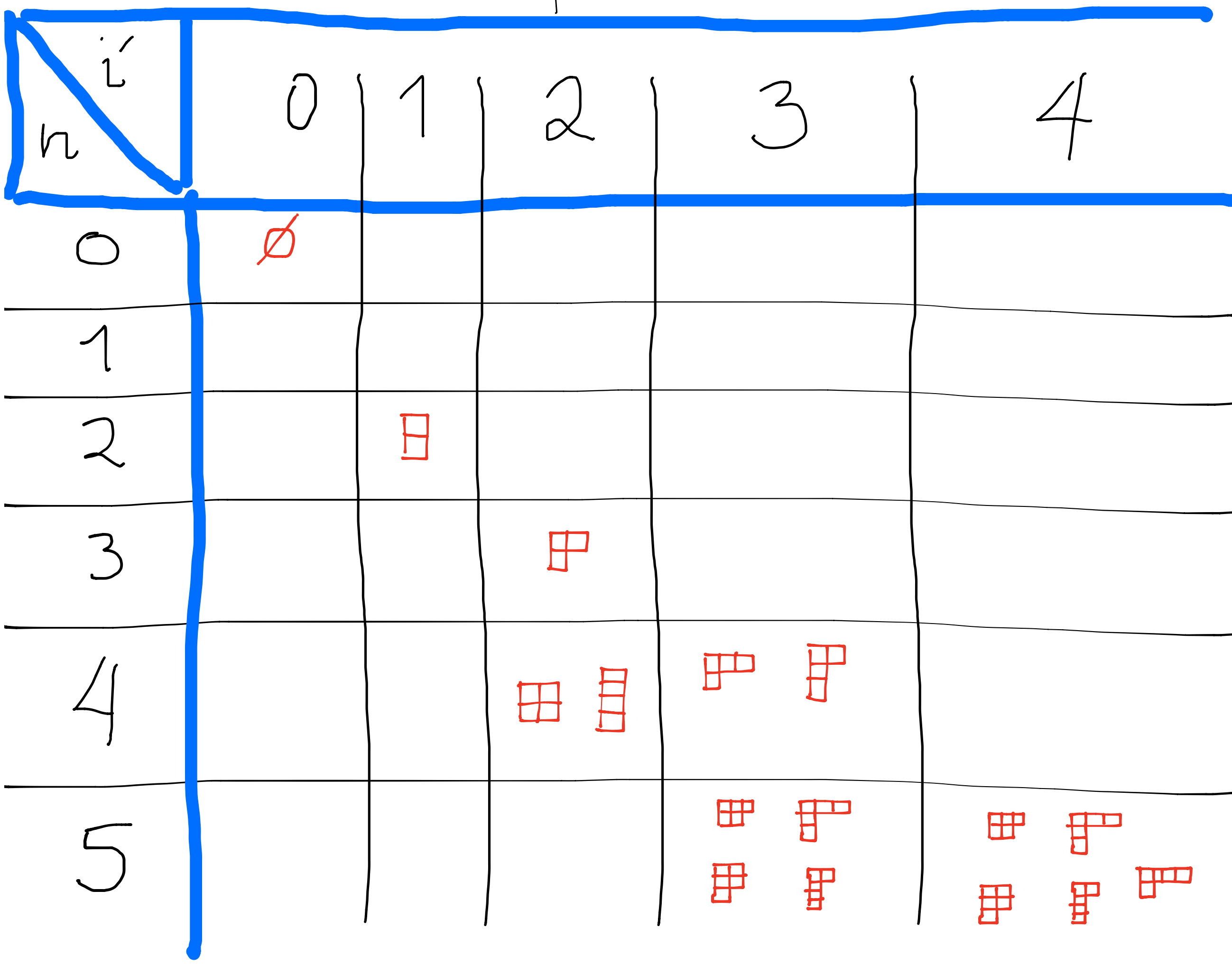
$PBW^{i'}$, $WH^{i'}$ will be described more explicitly.

But the **crux** is that their irreducible expansions $\sum_{\mu} c_\mu \chi^\mu$ only involve μ with

$|\mu| \leq 2i'$ and $\mu_1 \leq \begin{cases} i' & \text{if } d \text{ odd} \\ 1+i' & \text{if } d \text{ even} \end{cases}$

(Church-Farb) (New!)

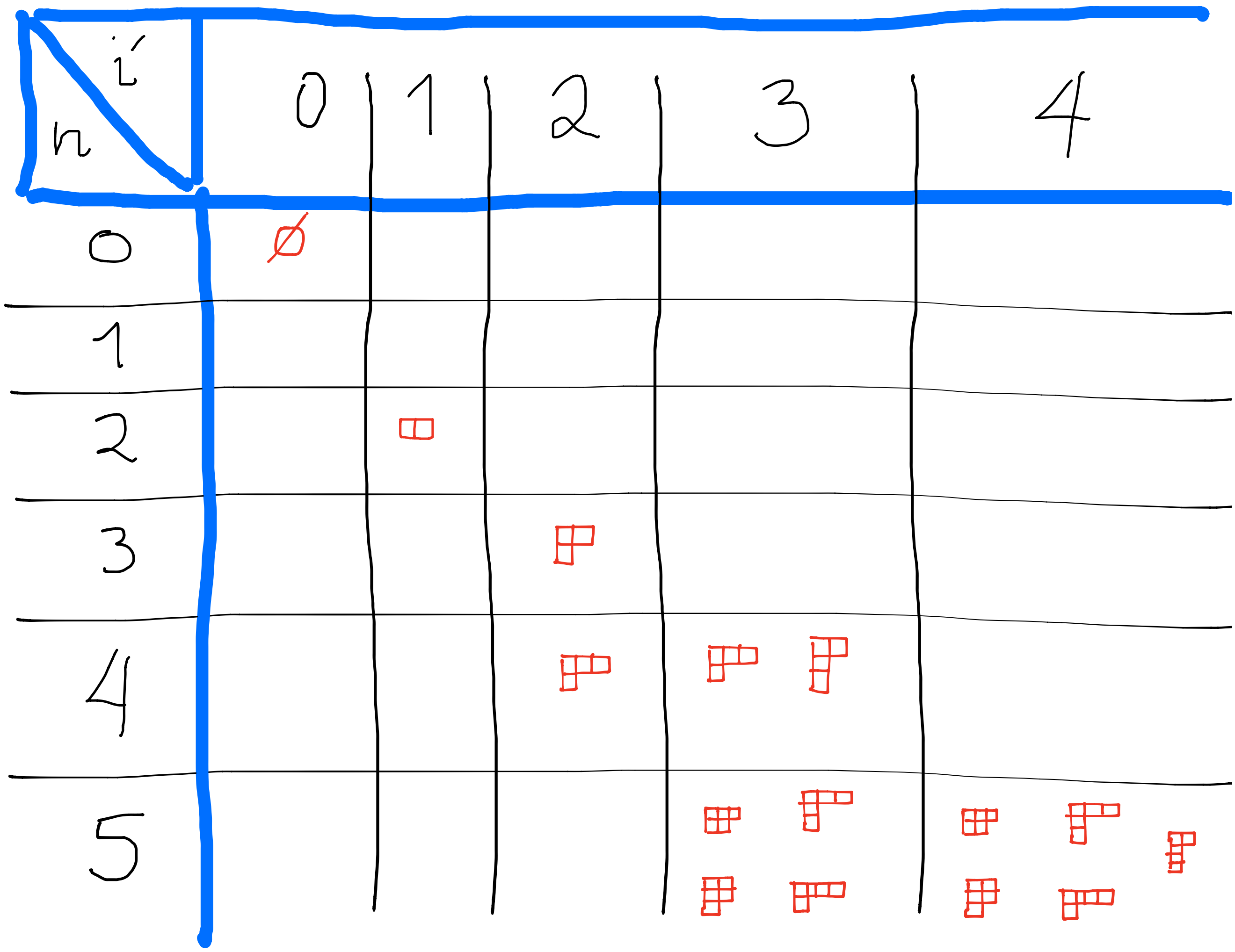
Irreducible expansions of $PBW^{i'}$



Note μ in column i' have

$$\mu_1 \leq i'$$

Irreducible expansions of $WH^{i'}$



Note μ in column i' have

$$\mu_1 \leq 1 + i'$$

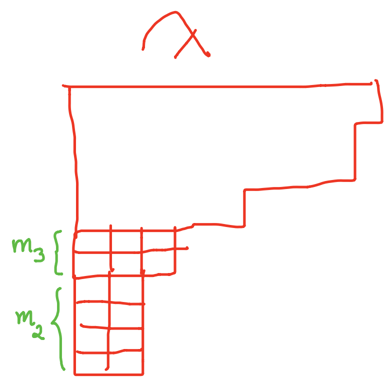
So what are $PBW^{i'}$, $WH^{i'}$?

$$\sum_{\lambda} \begin{cases} PBW_\lambda & d \text{ odd} \\ WH_\lambda & d \text{ even} \end{cases}$$

as λ ranges over all partitions with

- $\operatorname{rank}(\lambda) = \sum_j (\lambda_j - 1) = i'$.
- no parts of size 1 in λ

If $\lambda = 2^{m_2} 3^{m_3} 4^{m_4} \cdots$



then

$$PBW_\lambda = h_{m_2}[\pi_2] * h_{m_3}[\pi_3] * h_{m_4}[\pi_4] * \cdots$$

$$WH_\lambda = e_{m_2}[\pi_2] * h_{m_3}[\pi_3] * e_{m_4}[\pi_4] * \cdots$$

Here

- $\chi_1 * \chi_2$ is induction product:
$$\chi_1 \otimes \chi_2 \Big\uparrow_{\mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2}}^{\mathfrak{S}_{n_1+n_2}}$$

- π_n is the $\mathfrak{S}_n$-rep'n on the top homology of the proper part of the poset of set partitions of $\{1, 2, \ldots, n\}$

- $h_m[\chi] = \underbrace{\chi \otimes \cdots \otimes \chi}_{m \text{ factors}} \Big\uparrow_{\mathfrak{S}_m[\mathfrak{S}_n]}^{\mathfrak{S}_{mn}}$

- $e_m[\chi] = \underbrace{\chi \otimes \cdots \otimes \chi}_{m \text{ factors}} \otimes \operatorname{sgn} \Big\uparrow_{\mathfrak{S}_m[\mathfrak{S}_n]}^{\mathfrak{S}_{mn}}$

This is enough to bound the μ_1's in the irreducible expansions $\sum_\mu c_\mu \chi^\mu \ldots$

- THM: (Hanlon, Stanley 1982) $\pi_n = e^{2\pi i/n} \Big\uparrow_{\mathbb{Z}/n\mathbb{Z}}^{\mathfrak{S}_n}$

  and in particular, π_n has μ_1 bounded by $\begin{cases} n-1 & \text{for } n \geq 3 \\ n & \text{for } n=2 \end{cases}$

- If χ_1, χ_2 have μ_1 bounded by ℓ_1, ℓ_2 then $\chi_1 * \chi_2$ has μ_1 bounded by $\ell_1 + \ell_2$

- If χ has μ_1 bounded by ℓ then $h_m[\chi], e_m[\chi]$ have μ_1 bounded by $m\ell$

DIGRESSION...

Why the notation PBW_λ?

$$V = \mathbb{C}^n$$

$T(V)$ = tensor algebra = free assoc. alg. on V

$\mathcal{L}(V)$ = free Lie algebra on V

$$T(V) = \underbrace{\mathcal{U}}_{\text{universal enveloping algebra}}(\mathcal{L}(V)) \underset{\text{Poincaré-Birkhoff-Witt Thm.}}{\cong} \underbrace{\mathrm{Sym}}_{\text{symmetric algebra}}(\mathcal{L}(V))$$

$$= \mathrm{Sym}\Big(\bigoplus_{d \geq 0} \underbrace{\mathcal{L}_d(V)}_{\text{Lie bracketings of degree } d}\Big)$$

$$\cong \bigoplus_{\lambda = 1^{m_1} 2^{m_2} 3^{m_3} \cdots} \underbrace{\mathrm{Sym}^{m_1}(\mathcal{L}_1(V)) \otimes \mathrm{Sym}^{m_2}(\mathcal{L}_2(V)) \otimes \cdots}_{\text{Schur-Weyl dual to } PBW_\lambda}$$

Why the notation WH_λ ?

For d even and $i=(d-1)i'$,

$H^i(\mathrm{Conf}(n,\mathbb{R}^d)) =$

$(i')^{th}$ Whitney Homology of the poset of set partitions σ of $\{1,2,\dots,n\}$

$$:= \bigoplus_{\substack{\sigma \text{ having} \\ n-i' \text{ blocks}}} \tilde{H}_{i'-2}(\underbrace{(\hat{0},\sigma)}_{\substack{\text{simplicial complex of chains} \\ \text{strictly between the discrete} \\ \text{set partition } \hat{0} \text{ and } \sigma}})$$

$$= \bigoplus_{\substack{\text{partitions } \lambda \\ \text{of } n \text{ having} \\ n-i' \text{ parts}}} \underbrace{\bigoplus_{\substack{\sigma \text{ having} \\ \text{block} \\ \text{sizes } \lambda}} \tilde{H}_{i'-2}((\hat{0},\sigma))}_{\cong\, WH_\lambda}$$

5. J. Wiltshire-Gordon's Conjectures on WH^i

We'd like to know WH^i, WH_λ

PBW^i, PBW_λ

explicitly decomposed into irreducibles, but we don't.

In fact, decomposing

$$PBW_\lambda = \sum_\mu c_\mu^\lambda \chi^\mu$$

is a problem going back to Thrall 1942.

Wiltshire-Gordon made some conjectures on WH^i, having analogues for PBW^i.

THM (Wiltshire-Gordon's Conj 1):

$$\underbrace{WH^i_n}_{\substack{\text{the } \mathfrak{S}_n\text{-rep'n} \\ \text{component} \\ \text{of } WH^i}} \Big\downarrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} = \left(WH^{i-1}_{n-1} \Big\downarrow^{\mathfrak{S}_{n-1}}_{\mathfrak{S}_{n-2}} + WH^{i-1}_{n-2} \right) \Big\uparrow^{\mathfrak{S}_{n-1}}_{\mathfrak{S}_{n-2}}$$

EXAMPLE: $i=3,\ n=4$

$$WH^3_4 \Big\downarrow^{\mathfrak{S}_4}_{\mathfrak{S}_3} = \left(WH^2_3 \Big\downarrow^{\mathfrak{S}_3}_{\mathfrak{S}_2} + WH^2_2 \right) \Big\uparrow^{\mathfrak{S}_3}_{\mathfrak{S}_2}$$

$$\left(\chi^{(3,1)} + \chi^{(2,1,1)} \right) \Big\downarrow^{\mathfrak{S}_4}_{\mathfrak{S}_3} \qquad\qquad \left(\chi^{(2,1)} \Big\downarrow^{\mathfrak{S}_3}_{\mathfrak{S}_2} + 0 \right) \Big\uparrow^{\mathfrak{S}_3}_{\mathfrak{S}_2}$$

$$= \chi^{(3)} + \chi^{(2,1)} + \chi^{(2,1)} + \chi^{(1,1,1)}$$

$$= \left(\chi^{(2)} + \chi^{(1,1)} \right) \Big\uparrow^{\mathfrak{S}_3}_{\mathfrak{S}_2}$$

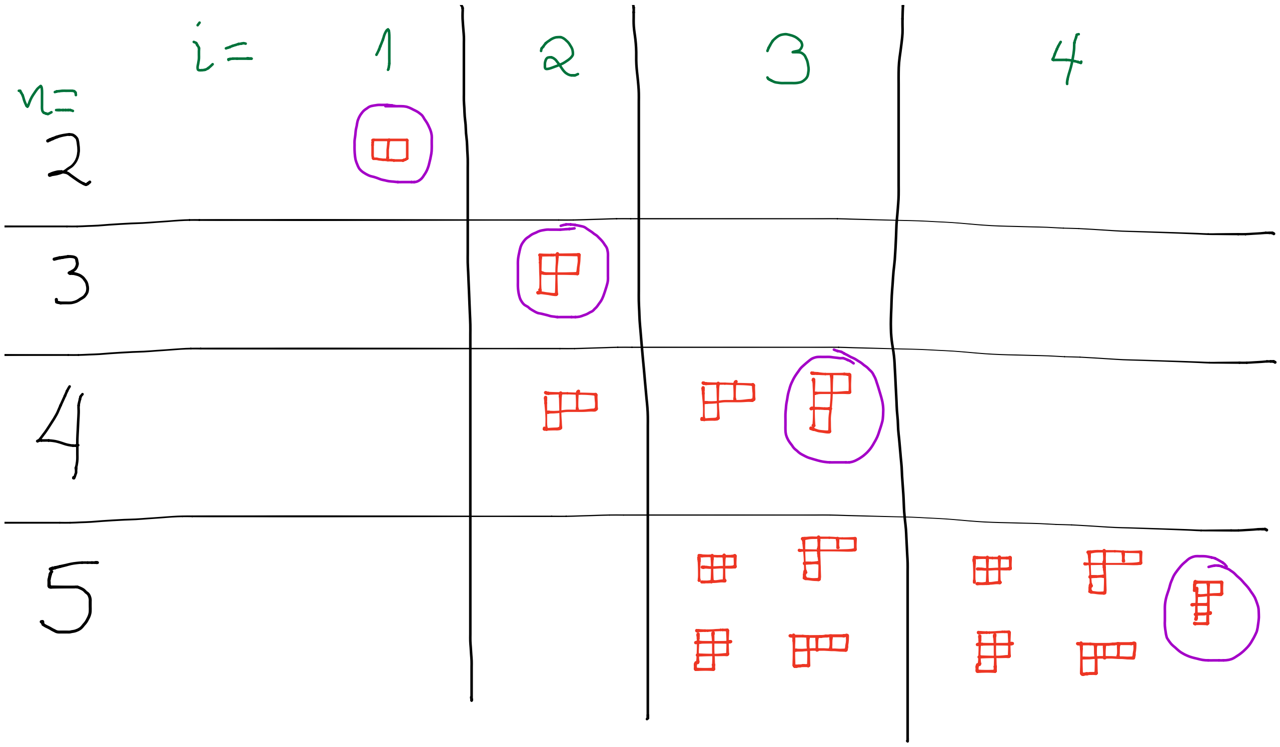
$$= \chi^{(3)} + \chi^{(2,1)} + \chi^{(2,1)} + \chi^{(1,1,1)}$$

THM (Wiltshire-Gordon's Conj 2):

$$\sum_i (-1)^i WH^i_n = (-1)^{n-1} \chi^{(2,1^{n-2})}$$

as virtual $\mathfrak{S}_n$-rep'ns.

(Actually he conjectured that there should be a cochain complex structure $(WH^\bullet_n, d)$ with cohomology concentrated at $i=n-1$, carrying $\chi^{(2,1^{n-2})}$. We have a candidate cochain complex ...)



i= 1 2 3 4
n=
2
3
4
5

Both conjectures follow from a known generating function, that collates as a symmetric function

$$\sum_{\lambda} WH_{\lambda} \, x^{\mathrm{rank}(\lambda)} \, y^{|\lambda|}$$

into an infinite product, involving the power sum symmetric functions

$p_1, p_2, p_3, \ldots$

where $p_r = x_1^r + x_2^r + x_3^r + \cdots$

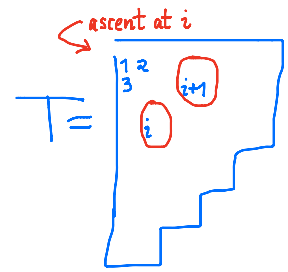
- CONJ 1 arises roughly from taking $\partial/\partial p_1$ in the generating function, corresponding to $(-)\big\downarrow^{\tilde{G}_n}_{\tilde{G}_{n-1}}$

- CONJ 2 arises from setting $x = -1$.

We do know another result of a similar flavor.

THM (Webb–R. 2004, related to Désarménien–Wachs 1988)

$$\sum_i \mathrm{PBW}_n^i = \sum_{\substack{\text{standard Young}\\ \text{tableaux } T \text{ of size } n \text{ having}\\ \text{first ascent even}}} \chi^{\mathrm{shape}(T)}$$



(and an analogous result for $\sum_i \mathrm{WH}_n^i$)

EXAMPLE: $n=4$

$$\sum_i \mathrm{PBW}_4^i = \mathrm{PBW}_4^2 + \mathrm{PBW}_4^3$$

$$= \left(\chi^{(2,2)} + \chi^{(1,1,1,1)}\right) + \left(\chi^{(3,1)} + \chi^{(2,1,1)}\right)$$

$$\boxed{\begin{array}{cc}1&3\\2&4\end{array}} \quad \boxed{\begin{array}{c}1\\2\\3\\4\end{array}} \quad \boxed{\begin{array}{ccc}1&3&4\\2&&\end{array}} \quad \boxed{\begin{array}{cc}1&3\\2&\\4&\end{array}}$$

$$\cancel{\begin{array}{cc}1&2\\3&4\end{array}} \qquad \cancel{\begin{array}{ccc}1&2&4\\3&&\end{array}} \quad \cancel{\begin{array}{cc}1&4\\2&\\3&\end{array}}$$

$$\cancel{\begin{array}{ccc}1&2&3\\4&&\end{array}} \quad \cancel{\begin{array}{cc}1&2\\3&\\4&\end{array}}$$

PROBLEM:

Refine these tableau models for the irreducible expansions of $\sum_i WH_n^i$

$$\sum_i PBW_n^i$$

to models for WH_n^i, PBW_n^i,

WH_λ, PBW_λ.

THANK

YOU!