The Art Gallery Theorem

Vic Reiner, Univ. of Minnesota

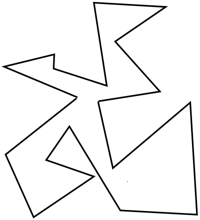
Augsburg College, March 9, 2016,
and St. John's University, Sept. 15, 2015

1. The players
2. The theorem
3. The proof from THE BOOK
4. Variations

The one who asked the question: Victor Klee

Klee's question posed to V. Chvátal

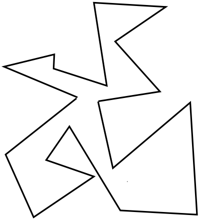
Given the floor plan of a weirdly shaped art gallery having N straight sides, how many guards will we need to post, in the worst case, so that every bit of wall is visible to a guard?



Can one do it with $N/3$ guards?

Klee's question posed to V. Chvátal

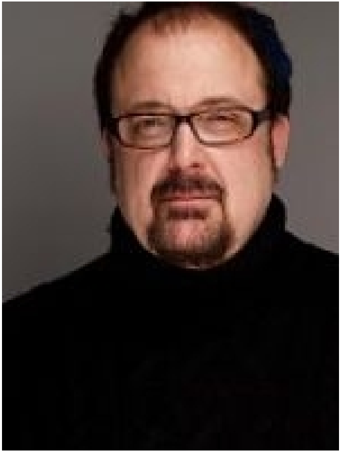
Given the floor plan of a weirdly shaped art gallery having N straight sides, how many guards will we need to post, in the worst case, so that every bit of wall is visible to a guard?



Can one do it with $N/3$ guards?

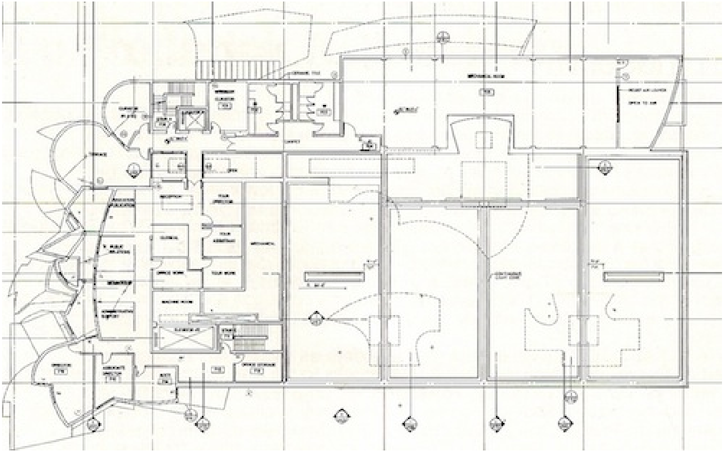
Vasek Chvátal: Yes, I can prove that!

Steve Fisk: OK, but I have a proof from THE BOOK!

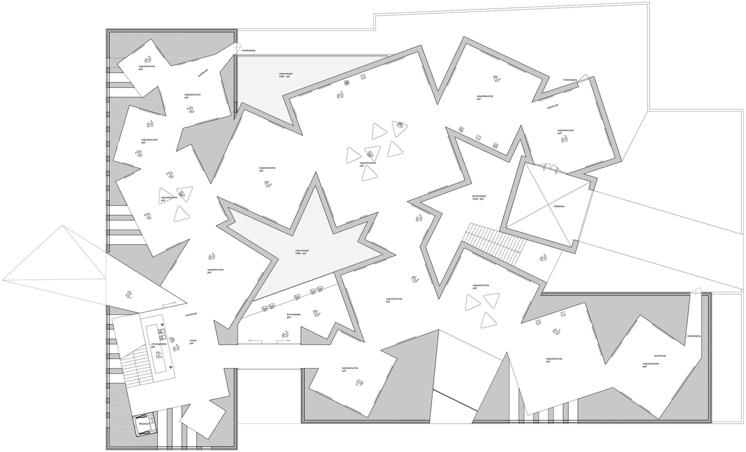


Our weirdly-shaped art museum: The Weisman

The floor plan

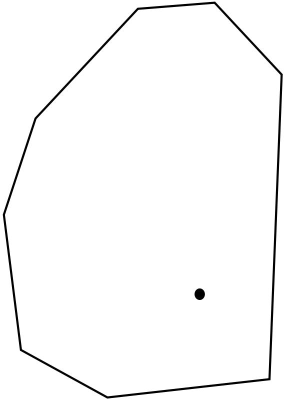


They can get crazier!

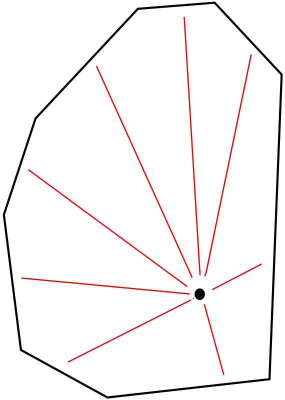


Enough fooling around– let's understand Klee's question!

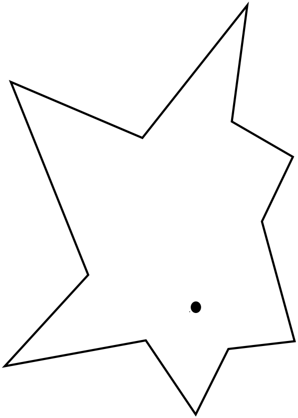
A convex gallery needs only one guard



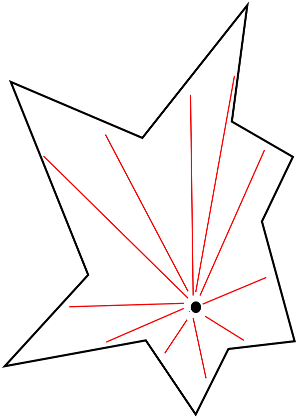
A convex gallery needs only one guard



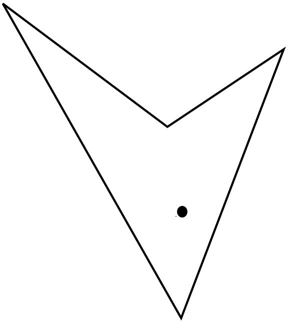
A star-shaped gallery needs only one guard



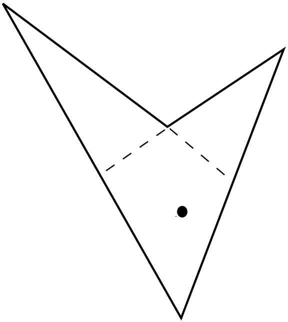
A star-shaped gallery needs only one guard



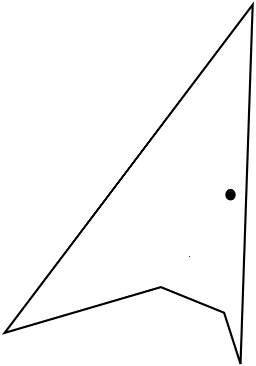
A 4-sided gallery needs only one guard



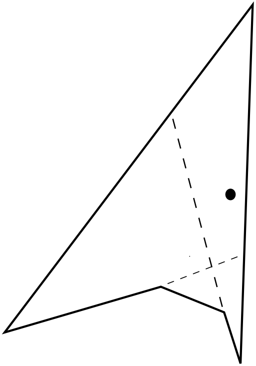
A 4-sided gallery needs only one guard



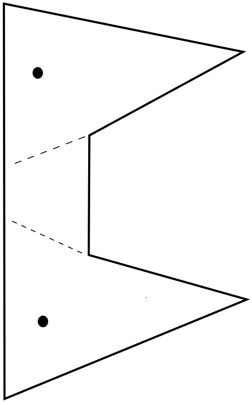
A 5-sided gallery needs only one guard



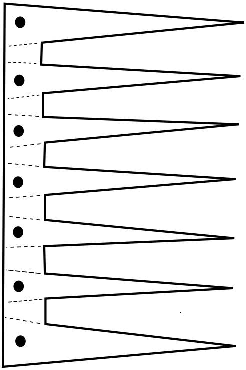
A 5-sided gallery needs only one guard



A 6-sided gallery might need two guards



N-sided galleries might need $N/3$ guards: the comb



N = 21
N/3 = 7

Klee's question for Chvátal

Question (V. Klee, 1973)

How many guards does an N-sided gallery need?
Is the comb the worst case?

Theorem (V. Chvátal, 1973, shortly thereafter)

Yes, the combs achieve the worst case:
every N-sided gallery needs at most $N/3$ guards.

(Of course, you can still have star-shaped galleries with a huge number of sides N, but they'll only need one guard.)

Klee's question for Chvátal

Question (V. Klee, 1973)

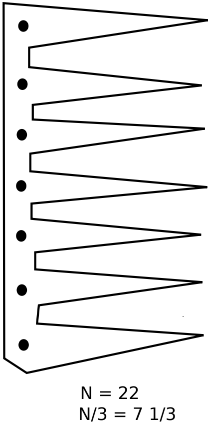
How many guards does an N-sided gallery need?
Is the comb the worst case?

Theorem (V. Chvátal, 1973, shortly thereafter)

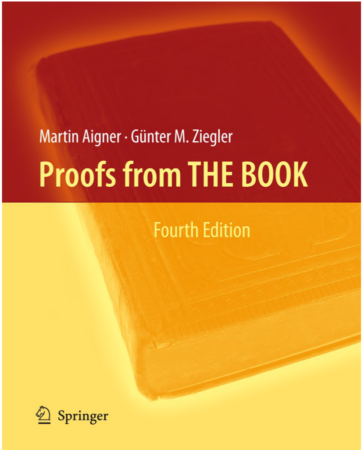
Yes, the combs achieve the worst case:
every N-sided gallery needs at most $N/3$ guards.

(Of course, you can still have star-shaped galleries with a huge number of sides N, but they'll only need one guard.)

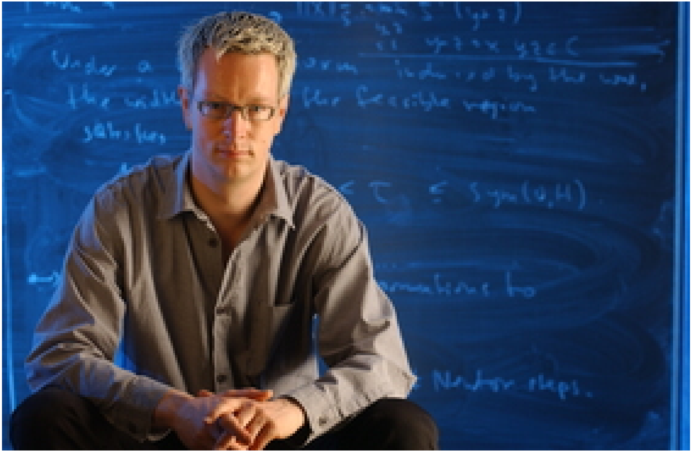
What if N isn't divisible by 3?



Steve Fisk's wonderful 1978 proof appears in this book

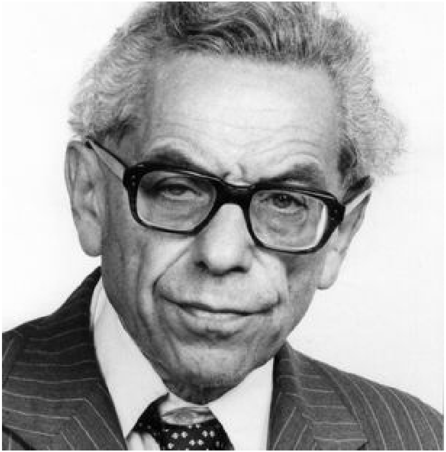


... by Gunter Ziegler ...



... and Martin Aigner

... aided and inspired by Paul Erdős

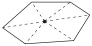


Guess which museum appears on page 231?

How to guard a museum **Chapter 35**

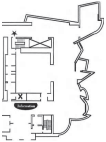
Here is an appealing problem which was raised by Victor Klee in 1973. Suppose the manager of a museum wants to make sure that at all times every point of the museum is watched by a guard. The guards are stationed at fixed posts, but they are able to turn around. How many guards are needed?

We picture the walls of the museum as a polygon consisting of n sides. Of course, if the polygon is *convex*, then one guard is enough. In fact, the guard may be stationed at any point of the museum. But, in general, the walls of the museum may have the shape of any closed polygon.



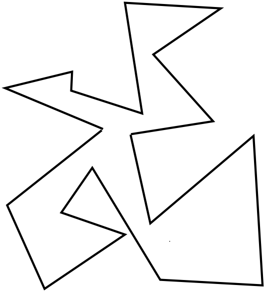
A convex exhibition hall

Consider a comb-shaped museum with $n = 3m$ walls, as depicted on the right. It is easy to see that this requires at least $m = \frac{n}{3}$ guards. In fact, there are n walls. Now notice that the point 1 can only be observed by a guard stationed in the shaded triangle containing 1, and similarly for the other points $2, 3, \ldots, m$. Since all these triangles are disjoint we conclude that at least m guards are needed. But m guards are also enough, since they can be placed at the top lines of the triangles. By cutting off one or two walls at the end, we conclude that for any n there is an n-walled museum which requires $\lfloor \frac{n}{3} \rfloor$ guards.

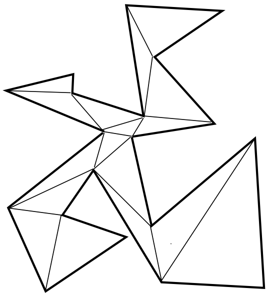


A real life art gallery…

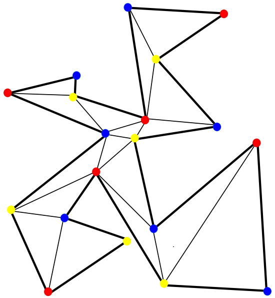
Fisk's proof from THE BOOK that $N/3$ guards suffice



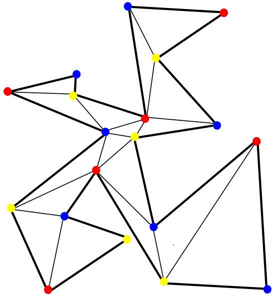
First triangulate the gallery without new vertices



Then properly 3-color its vertices

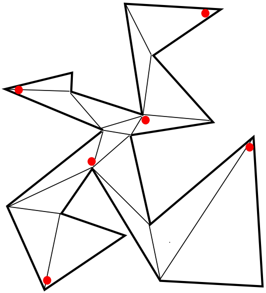


The least popular color gets used at most $N/3$ times

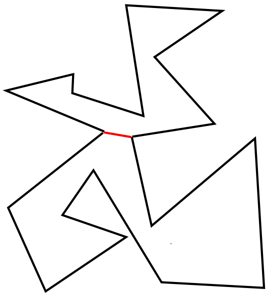


N=19 sides, so 19 vertices.
6 red, 7 blue, 6 yellow vertices

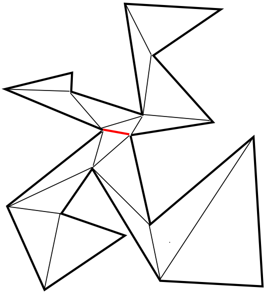
Post guards near the least popular color vertices



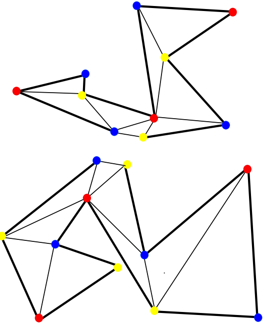
How to triangulate without new vertices? Induct!



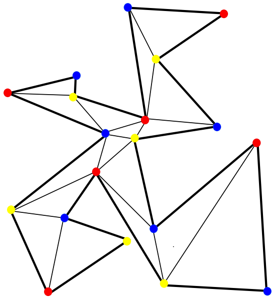
How to triangulate without new vertices? Induct!



How to 3-color the vertices? Induct!



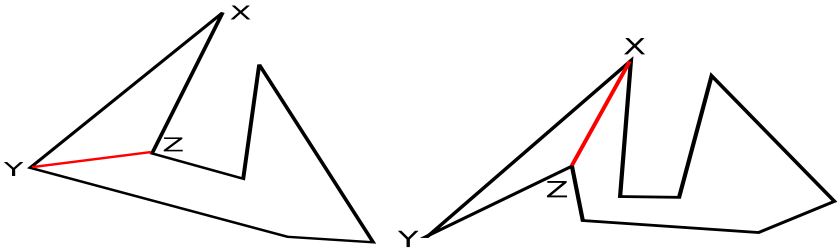
One can always glue the colorings back together



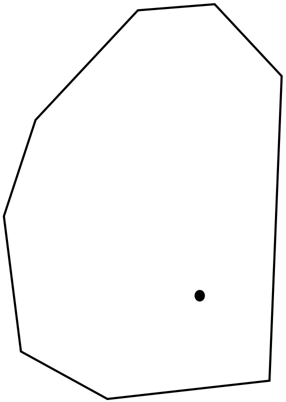
How to get the red dividing line to start inducting? The flashlight argument!

The flashlight argument

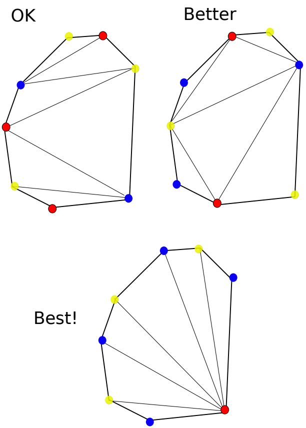
Starting at a vertex X, shine a flashlight along the wall to an adjacent vertex Y, and swing it in an arc until you first hit another vertex Z. Then either XZ or YZ works as the red dividing line.



How good is Fisk's method for convex galleries?



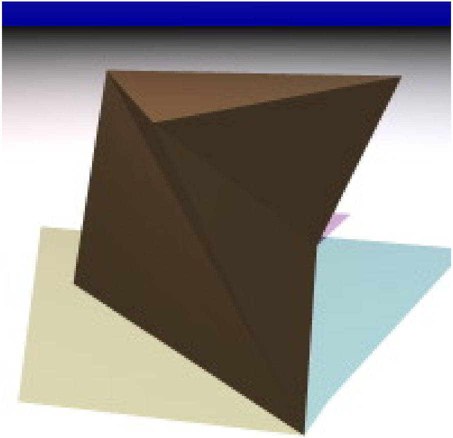
It depends on the triangulation



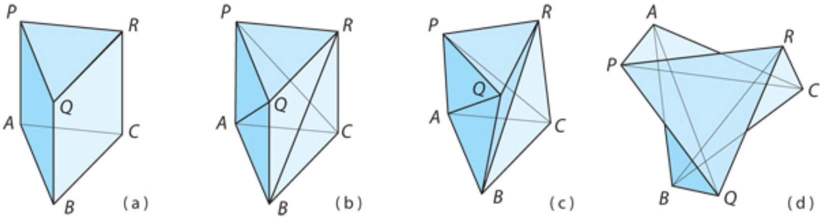
A couple of variations one might wonder about

- Three dimensional galleries?
- Only right-angled walls in two dimensions?

3D explains why we worried about how to triangulate!

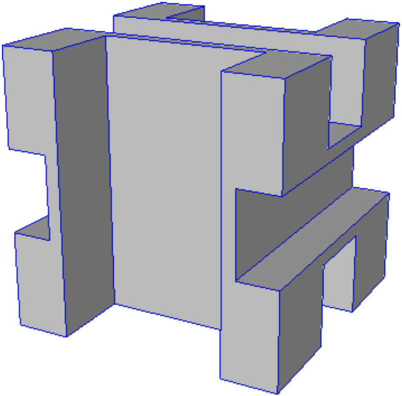


Schoenhardt's (1928) untriangulable sphere in 3D!



It has no interior tetrahedra that can be formed by any four out of its six vertices!

Even worse: Seidel's (1987) Octoplex



Its center point is visible from none of the vertices!

Existence of such examples makes the 3-D theory harder.

What about when the walls meet only at right angles?

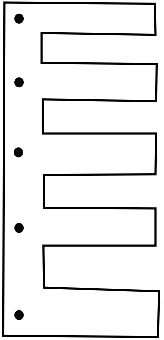
J. Kahn, M. Klawe, and D. Kleitman proved this result in a 1983 paper titled "Traditional galleries require fewer watchmen"

Theorem

For right-angled galleries with N sides, $N/4$ guards suffice.

One might guess how they feel about the Weisman Museum.

Right-angled combs again achieve the worst case



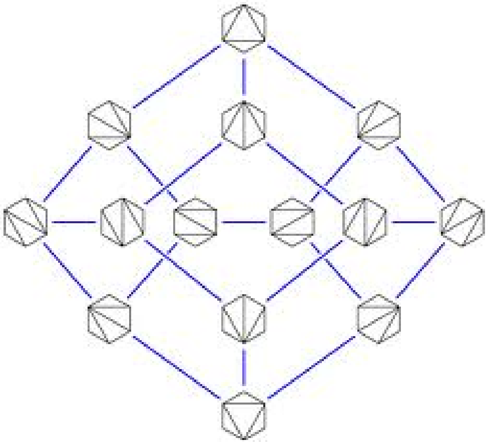
N=20 sides
N/4 = 5 guards

Does the Art Gallery Theorem have real applications?

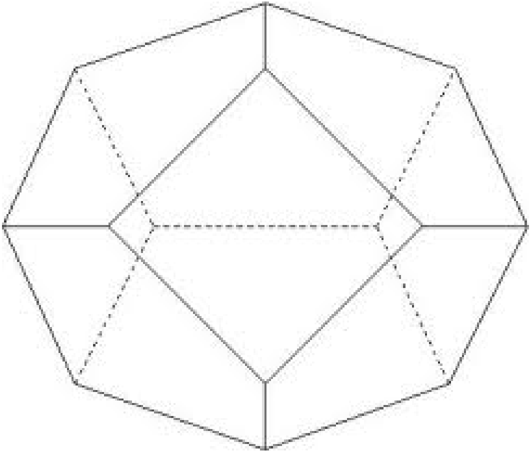
Not directly that I know. But related ideas from the areas of discrete geometry and combinatorics get used in designing algorithms for

- searching terrains,
- robot-motion planning,
- motorized vacuum cleaners (!)

The set of triangulations of a polygon is interesting!



A polyhedron beloved to me: the associahedron



Thanks for listening!

Bibliography

- T.S. Michael's
  - article: "Guards, Galleries, Fortresses, and the Octoplex" (College Math. Journal 42, no. 3, March 2011, pp.191-200)
  - book: "How to Guard an Art Gallery and Other Discrete Mathematical Adventures".
- Norman Do's "Mathellaneous" article on Art Gallery Theorems, (Australian Math. Soc. Gazette, Nov. 2004).
- Art Gallery Theorems, by J. O'Rourke
- Proofs from THE BOOK, by M. Aigner and G. Ziegler
- Triangulations: Structures for algorithms and applications, by J. De Loera, J. Rambau and F. Santos.