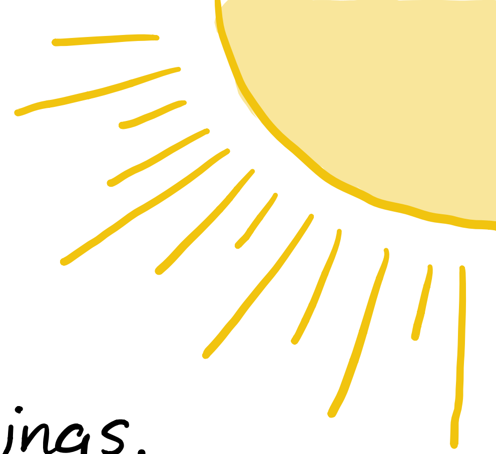
Add a bit of Color to your face...

rings.



Ashleigh Adams : UC Davis

Vic Reiner : University of Minnesota – Twin Cities

Graduate Student Combinatorics Conference

April 24, 2:45, Room E

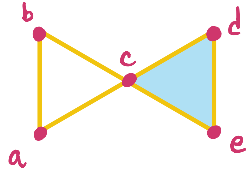
1. Review Stanley-Reisner rings, f-vectors, h-vectors, & Hilbert series

2. Colorful System of Parameters & a Colorful Hochster formula

3. Universal System of Parameters

4. How they relate!

Example

$\Delta :=$ 

$$= \{ \emptyset, \; a, b, c, d, e, \; ab, ac, bc, cd, ce, de, \; cde \}$$

Stanley-Reisner ring

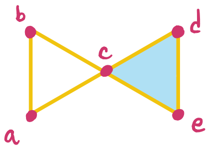
$$S := k[a,b,c,d,e]$$

$$I_\Delta := \langle ad, ae, bd, be, abc \rangle$$

$$k[\Delta] := S/I_\Delta$$

dimension of a face F: $\dim(F) = \#F - 1$

Ex. $\dim(ac) = \#ac - 1 = 1$

$\Delta :=$ 

$= \{ \emptyset,$
$a, b, c, d, e,$
$ab, ac, bc, cd, ce, de,$
$cde \}$

__f-vector__ : $\underline{f} = (f_{-1}, f_0, f_1, f_2)$

$f_i = \#\, i\text{-dim faces}$

$= (1, 5, 6, 1)$

-1-dim faces | 0-dim faces | 1-dim faces | 2-dim faces

__h-vector__ : $\underline{h} = (h_0, h_1, h_2, h_3)$

$(*)$

$= (1, 2, -1, -1)$

__Hilbert Series__ :

$$\mathrm{Hilb}_{k[\Delta]}(t) := \sum_{d=0}^{\infty} \dim_k\left(k[\Delta]_d\right) \cdot t^d$$

$$= \sum_{i=0}^{d} f_{i-1} \left(\frac{t}{1-t}\right)^{d} \overset{*}{=}: \sum_{i=0}^{d} \frac{h_i t^i}{(1-t)^d}$$

$$= \frac{1 + 2t - 1t^2 - 1t^3}{(1-t)^3}$$

We can also compute the Hilbert series from a finite minimal free resolution of $k[\Delta] = S/I_\Delta$ over S:

Syzygies: 0^{th} 1^{st} 2^{nd} 3^{rd}

$$0 \leftarrow k[\Delta] \leftarrow S^1 \leftarrow \begin{matrix} S(-2)^4 \\ \oplus \\ S(-3)^1 \end{matrix} \leftarrow \begin{matrix} S(-3)^4 \\ \oplus \\ S(-4)^2 \end{matrix} \leftarrow \begin{matrix} S(-4)^1 \\ \oplus \\ S(-5)^1 \end{matrix} \leftarrow 0$$

$$\mathrm{Hilb}_{k[\Delta]}(t) = \mathrm{Hilb}(S,t) \cdot \left(1t^0 - (4t^2 + 1t^3) + (4t^3 + 2t^4) - (1t^4 + 1t^5)\right)$$

$$= \frac{1}{(1-t)^5}\left(1t^0 - 4t^1 + 3t^3 + 1t^4 - 1t^5\right)$$

$$= \frac{1t^0 + 2t^1 - 1t^2 - 1t^3}{(1-t)^3}$$

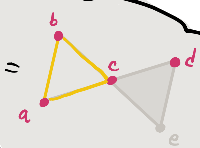
$S(-d) :=$ free S-module with 1 basis element in degree d

How can we construct the (minimal) free resolution of $\Bbbk[\Delta]$ from before?

We first consider the vertex-selected subcomplexes of Δ:

$$\Delta_V := \{ F \in \Delta : F \subseteq V \}$$

Ex. $\Delta_{\{a,b,c,d\}} =$



Then the

$$\# S(-\#V) = \dim_{\Bbbk} \tilde{H}^{\#V-1-i}(\Delta_V ; \Bbbk)$$

"in" the i-th syzygy

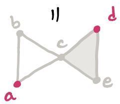
THM Hochster (1977)

Syzygies: 0th 1st 2nd 3rd

$$0 \leftarrow k[\Delta] \leftarrow S^1 \leftarrow \begin{matrix} S(-2)^4 \\ \oplus \\ S(-3)^1 \end{matrix} \leftarrow \begin{matrix} S(-3)^4 \\ \oplus \\ S(-4)^2 \end{matrix} \leftarrow \begin{matrix} S(-4)^1 \\ \oplus \\ S(-5)^1 \end{matrix} \leftarrow 0$$

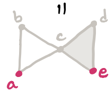
1. $\tilde{H}^{2-1-1}(\Delta_{\{a,d\}}) = \Bbbk^1$

||



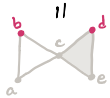
2. $\tilde{H}^{2-1-1}(\Delta_{\{a,e\}}) = \Bbbk^1$

||



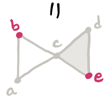
3. $\tilde{H}^{2-1-1}(\Delta_{\{b,d\}}) = \Bbbk^1$

||



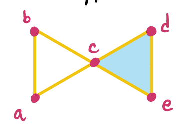
4. $\tilde{H}^{2-1-1}(\Delta_{\{b,e\}}) = \Bbbk^1$

||



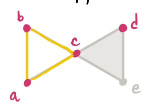
1. $\tilde{H}^{5-1-3}(\Delta_{\{abcde\}}) = \Bbbk^1$

||



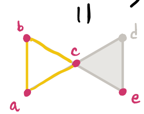
1. $\tilde{H}^{4-1-2}(\Delta_{\{a,b,c,d\}}) = \Bbbk^1$

||



2. $\tilde{H}^{4-1-2}(\Delta_{\{a,b,c,e\}}) = \Bbbk^1$

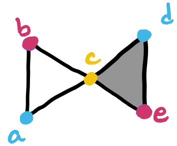
||

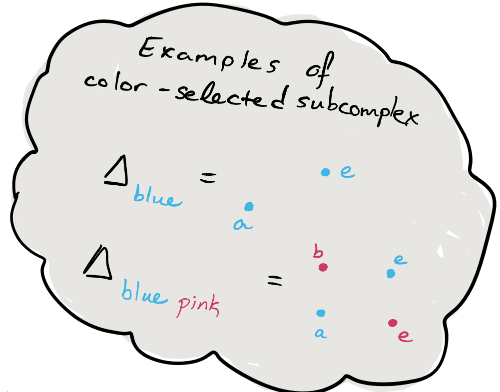


What if we consider a subcomplex defined by a <u>proper d-coloring</u> of the 1-skeleton of Δ instead?

$d = \dim(\Delta)+1$

Step 1: Give Δ a proper d-coloring. ($d = 3$)

$\Delta =$ 



Step 2: Compute a <u>colorful system of parameters</u>

$$\theta_c := \sum_{\substack{\text{vertices } i \\ \text{of color } c}} x_i \ , \quad \rightsquigarrow \quad \begin{aligned} \theta_1 &= a+d \\ \theta_2 &= b+e \\ \theta_3 &= c \end{aligned}$$

Step 3: Write the resolution of $\Bbbk[\Delta]$ over the polynomial ring

$$A := \Bbbk[z_1, \dots, z_{\#colors}] \rightsquigarrow \Bbbk[z_1, z_2, z_3]$$

where $z_i \curvearrowright \Bbbk[\Delta]$ by multiplication by θ_i.

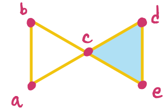
Step 3:

$$\underline{\theta} = (\underset{\substack{\| \\ a+d}}{\theta_1}, \underset{\substack{\| \\ b+e}}{\theta_2}\ \underset{\substack{\| \\ c}}{\theta_3})$$

$$\begin{array}{ccccccccc}
 & \text{syzygy:} & & 0^{th} & & 1^{st} & & \\
0 \leftarrow & \Bbbk[\Delta] & \leftarrow & A^1 & \longleftarrow & A(-2)^1 & \leftarrow & 0 \\
 & & & \oplus & & \oplus & & \\
 & & & A(-2)^1 & & A(-3)^1 & &
\end{array}$$

$$\boxed{\begin{aligned} \mathrm{Hilb}_{\Bbbk[\Delta]}(t) &= \mathrm{Hilb}_A(t) \cdot \left(t^0 + (t^1 + t^2) - (t^2 + t^3)\right) \\ &= \frac{1}{(1-t^3)} \cdot \left(1 + 2 - t^2 - t^3\right) \end{aligned}}$$

But why is this minimal free-resolution shorter?

$\Delta =$ 

$S = \mathbb{k}[\overbrace{a,b,c,d,e}^{\text{vertices}}]$

$A = \mathbb{k}[\underbrace{z_1, z_2, z_3}_{\text{colors}}]$

$$\begin{array}{ccccccc} \operatorname{depth}(S) - & \operatorname{depth}(\mathbb{k}[\Delta]) = & 5 & - & 2 & = & 3 \\ \vee & \| & \vee & & \| & & \vee \\ \operatorname{depth}(A) - & \operatorname{depth}(\mathbb{k}[\Delta]) = & 3 & - & 2 & = & 1 \end{array}$$

Auslander-Buchsbaum (1959)

$S := \mathbb{k}[\overbrace{v_1, \dots, v_n}^{\text{vertices}}]$, $\quad \overset{\#\text{vertices}}{\overset{\|}{\operatorname{depth}(S)}} - \operatorname{depth}(k[\Delta]) =$ length of the resolution of $k[\Delta]$ as an S-module

$$\geq \qquad\qquad \|$$

$A := \mathbb{k}[\underbrace{z_1, \dots z_k}_{\text{colors}}]$, $\quad \underset{\#\text{colors}}{\underset{\|}{\operatorname{depth}(A)}} - \operatorname{depth}(k[\Delta]) =$ length of the resolution of $k[\Delta]$ as an A-module

Punchline: We can produce a "shorter" minimal free-resolution for $\Bbbk[\Delta]$.

("Colorful Hochster formula")

If we first consider the color-selected subcomplexes of Δ:

$$\Delta_C := \{ F \in \Delta : \text{colors of } F \subseteq C \}$$

Then the

$$\# S(-\# C) = \dim_{\Bbbk} \tilde{H}^{\# V - 1 - i}(\Delta_C ; \Bbbk)$$

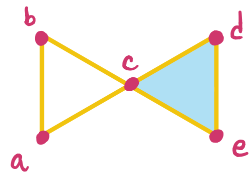
"in" the i-th syzygy

THM A., Reiner

But what if we consider an even "more" canonical system of parameters?

DEF The <u>universal system of parameters</u> is a sequence $\theta_1, \dots, \theta_d$ in $\Bbbk[\Delta]$

$$\theta_i = \sum_{\substack{\text{faces } F \in \Delta \\ \# F = i}} \underline{x}_F \, , \quad \rightsquigarrow \quad \Delta :=$$



$\theta = a+b+c+d+e$ } vertices

$\theta_2 = ab+ac+bc+cd+ce+de$ } edges

$\theta_3 = abc$ } 2-dim face

The universal system of parameters are invariant under symmetries

Universal* System of Parameters

Found in work by:

1. De Concini, Eisenbud, + Procesi 1972
   (on algebras with straightening laws)
2. Garsia + Stanton 1984
   (invariant theory of permutation groups)
3. D.E. Smith 1990
   (sheaves of posets)
4. Herzog + Moradi* 2020

DEF

Krull dim of $\Bbbk[\Delta]$

$\geq$

* depth$(\Bbbk[\Delta])$ *

$=$

max length of a regular sequence in $\Bbbk[\Delta]$

* D.E. Smith proved this for any pure simplicial complex *

For any finite regular cell complex (in our case, any simplicial complex Δ)

$$\operatorname{depth}(k[\Delta]) = \max\{\delta : (\theta_1, \ldots, \theta_\delta) \text{ forms a regular sequence on } k[\Delta]\}$$

where $\underline{\theta} = (\theta_1, \ldots, \theta_d)$ is the universal system of parameters for $k[\Delta]$ with $\delta \leq d$.

Theorem (A., Reiner)

PUNCHLINE (Prop. A. Reiner)

We can write the resolution of $\mathbb{k}[\Delta]$ over the polynomial ring

$$A := \mathbb{k}[z_1, \dots, z_d] \rightsquigarrow \mathbb{k}[z_1, z_2, z_3]$$

because $\mathbb{k}[\Delta]$ is finitely generated as an A-module.

where $z_i \curvearrowright \mathbb{k}[\Delta]$ by multiplication by θ_i,

PUNCHLINE #1

$k[\Delta]$ can be finitely generated as a $k[\underline{z}]$-module

θ_i colorful

PUNCHLINE #2

$k[\mathrm{sd}\,\Delta]$ can be finitely generated as a $k[\underline{z}]$-module

θ_i universal

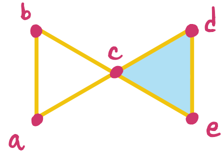
$\Longrightarrow$ We can write down a (finite) minimal free resolution for

$\underline{\theta}$ -colorful

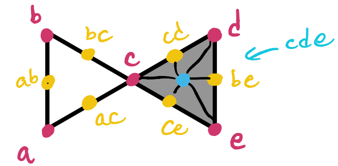
- $k[\Delta]$ over $k[\underline{z}]$

$\underline{\theta}$ -universal

The Universal SOP and the Colorful SOP relate!

$\Delta =$ 

$\rightsquigarrow$

$Sd(\Delta) =$ 

$\theta_1 = a+b+c+d+e$

$\theta_2 = ab+ac+bc + d+ce+de$

$\theta_3 = abc$

$$A = [z_1, z_2, z_3]$$

$$0 \leftarrow \mathbb{k}[\Delta] \leftarrow \begin{matrix} A^1 \\ \oplus \\ A(-2)^4 \\ \oplus \\ A(-3)^5 \\ \oplus \\ A(-4)^2 \end{matrix} \leftarrow \begin{matrix} A(-4)^2 \\ \oplus \\ A(-5)^3 \\ \oplus \\ A(-6)^1 \end{matrix} \leftarrow 0$$

Conjecture (A., Reiner, 2020)

When we take the resolution of

$\Bbbk[\Delta]$ over the universal SOP

and we take the resolution of

$\Bbbk[\Delta]$ over the colorful SOP,

the two resolutions have *<u>the same</u>* shape!

If true, then we know the shape by the colorful Hochster formula!

Evidence — The conjecture is true when...

COROLLARY (A., Reiner)

1. $k[\Delta]$ is Cohen-Macaulay

PROPOSITION (A., Reiner)

2. Δ is a 1-dimensional complex (i.e., a graph with multiple edges but no self-loops).

PROPOSITION (A., Reiner)

univeral colorful

3. $\dim_k \mathrm{Tor}_i^{k[z]}(k[\Delta], k)_b \leq \dim_k \mathrm{Tor}_i^{k[z]}(k[\mathrm{Sd}\Delta], k)_b$

(i.e. the conjecture gives a correct upper bound)

The story gets even better!

$\Delta :=$ simplicial complex with $\dim(\Delta) = d-1$

For $\Bbbk[\Delta]$ with universal parameters $(\theta_1, \dots, \theta_d)$

For $\Bbbk[\mathrm{sd}\,\Delta]$ with colorful parameters $(\theta_1, \dots, \theta_d)$

$\Bbbk[z_1, \dots, z_d]$

$\boxed{\text{QUESTION}}$ (S. Murai)

Is $\Bbbk[\Delta] \cong \Bbbk[\mathrm{sd}\,\Delta]$ as $\mathcal{A}$-modules?

Evidence

1. We haven't found a counterexample.

Note

Everything we've done for a simplicial complex can be generalized for a simplicial poset P and its corresponding face ring $\Bbbk[P]$.

- In this case, P is a regular cell complex Γ, and $\operatorname{Sd}\Gamma$ is a simplicial complex so that the colorful SOP exist.

Thank You!

arxiv: 2007.13021

Macaulay2 package: ResolutionsOfStanleyReisnerRings

These slides can be found on

ashleigh-adams.com