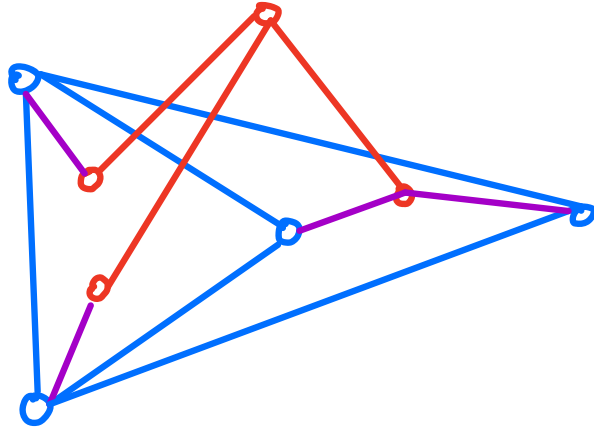


Topology of Augmented Bergman complexes

arXiv:
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2021
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1. Review **matroids** M
 - independent sets $I(M)$
 - flats $F(M)$
2. **Shellability**
3. Augmented Bergman complex Δ_M
4. Two kinds of shellings of Δ_M
and **corollaries**

1. Review matroids M

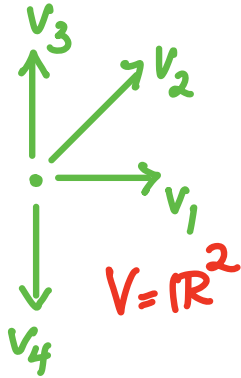
A matroid M of rank r on ground set $E = \{1, 2, \dots, n\}$ abstracts vectors v_1, v_2, \dots, v_n spanning an r -dimensional vector space V over some field k

EXAMPLE

$n=4$

$k=\mathbb{R}$

$r=2$

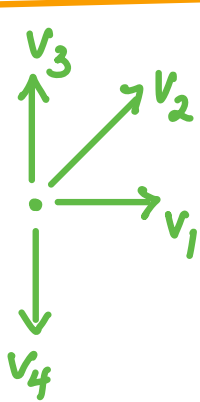


$$\begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \\ \left[\begin{array}{cccc} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 \end{array} \right] \end{array}$$

an $r \times n$ full rank matrix having v_i as its columns

The **matroid** M associated to v_1, v_2, \dots, v_n forgets their coordinates, but records the subscripts of (linearly) **independent sets**

$$\mathcal{I}(M) \stackrel{\text{DEF'N}}{:=} \left\{ I \subseteq \{1, 2, \dots, n\} : \{v_i\}_{i \in I} \text{ are linearly independent} \right\}$$



$$\rightsquigarrow \mathcal{I}(M) = \left\{ \emptyset, \begin{array}{l} 1, \\ 2, \\ 3, \\ 4 \end{array}, \begin{array}{l} 12, \\ 13, \\ 14, \\ 23, \\ 24 \end{array} \right\}$$

Note: $34 \notin \mathcal{I}(M)$ since $\{v_3, v_4\}$ are **dependent**
 $ijk \notin \mathcal{I}(M) \forall i, j, k$

$\mathcal{I}(M)$ always satisfies these independent set axioms:

$$(I_0) \quad \emptyset \in \mathcal{I}(M)$$

$$(I_1) \quad I \subseteq J \text{ and } J \in \mathcal{I}(M) \Rightarrow I \in \mathcal{I}(M)$$

$$(I_2) \quad I, J \in \mathcal{I}(M) \text{ and } \#I < \#J \\ \Rightarrow \exists j \in J \setminus I \text{ with } I \cup \{j\} \in \mathcal{I}(M)$$

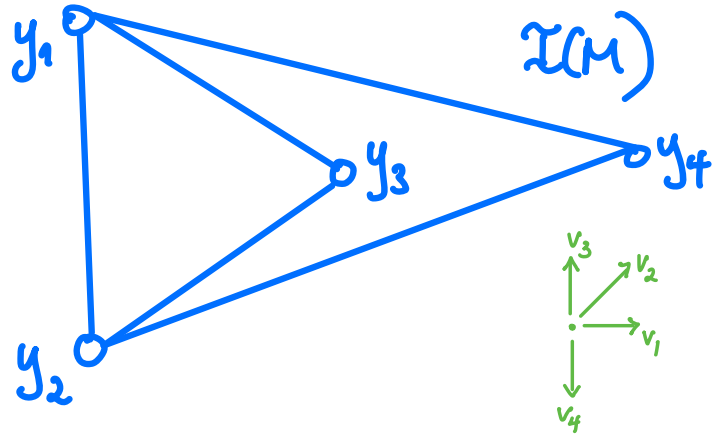
and this is our first definition of a matroid M :

a collection $\mathcal{I}(M)$ of subsets of $E = \{1, 2, \dots, n\}$ satisfying axioms $(I_0), (I_1), (I_2)$.

- (I0) $\emptyset \in \mathcal{I}(M)$
 (I1) $I \subseteq J$ and $J \in \mathcal{I}(M) \Rightarrow I \in \mathcal{I}(M)$
 (I2) $I, J \in \mathcal{I}(M)$ and $\#I < \#J$
 $\Rightarrow \exists j \in J \setminus I$ with $I \cup \{j\} \in \mathcal{I}(M)$

Axiom (I2) implies that all inclusion-maximal independent sets, called the **bases** $\mathcal{B}(M)$, have same cardinality r , called the **rank** $r(M)$.

Axioms (I0), (I1) say $\mathcal{I}(M)$ is an **abstract simplicial complex** on vertices $\{y_1, y_2, \dots, y_n\}$

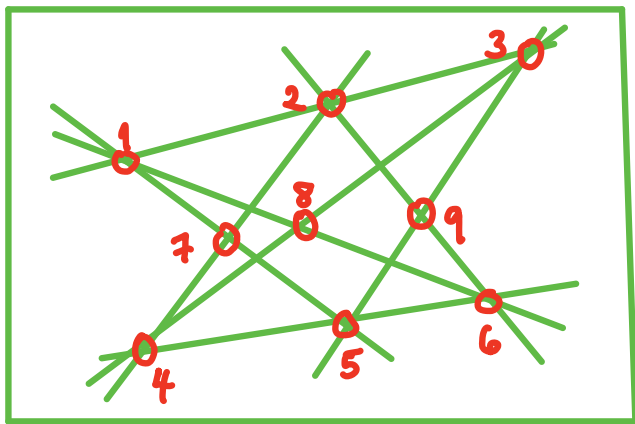


In other words, $\mathcal{I}(M)$ is a **pure** simplicial complex of dimension $r(M) - 1$.

Not all matroids M are representable by vectors v_1, v_2, \dots, v_n

EXAMPLE The non-Pappus matroid M on $E = \{1, 2, \dots, 9\}$ of rank 3 has

$\mathcal{I}(M) = \{ \text{all } I \subset \{1, 2, \dots, 9\} \text{ with } \#I \leq 3, \text{ except the collinear triples shown} \}$



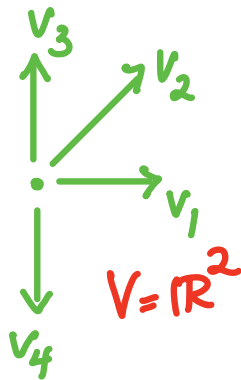
$789 \in \mathcal{I}(M)$ violates Pappus's Theorem

but does not violate axioms (I0), (I1), (I2).

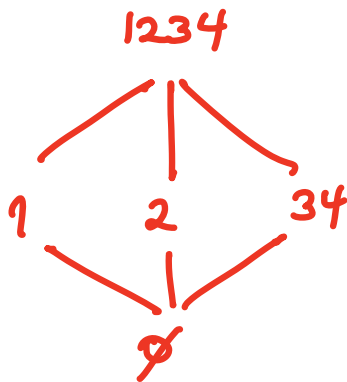
An alternate axiomatization of M uses the flats $\mathcal{F}(M)$ which are (when M is represented by v_1, v_2, \dots, v_n in V)

$$\mathcal{F}(M) := \left\{ F \subseteq \{1, 2, \dots, n\} : \{v_i\}_{i \in F} = W \cap \{v_1, v_2, \dots, v_n\} \text{ for some subspace } W \text{ of } V \right\}$$

EXAMPLE



flats $\mathcal{F}(M) = \{ \emptyset, 1, 2, 34, 1234 \}$



the poset $\mathcal{F}(M)$ ordered via inclusion

We could have defined a matroid M on $E = \{1, 2, \dots, n\}$ as a collection $\mathcal{F}(M)$ of subsets $F \subseteq E$, satisfying

the **flat axioms**:

$$(F0) \quad E = \{1, 2, \dots, n\} \in \mathcal{F}(M)$$

$$(F1) \quad F, G \in \mathcal{F}(M) \Rightarrow F \cap G \in \mathcal{F}(M)$$

$$(F2) \quad F \in \mathcal{F}(M) \text{ and } i \in E \setminus F \Rightarrow \\ \exists! G \in \mathcal{F}(M) \text{ covering } F \text{ with } i \in G.$$

$(F0), (F1) \Rightarrow$ the poset $\mathcal{F}(M)$ is a **lattice**, with $F \wedge G = F \cap G$.

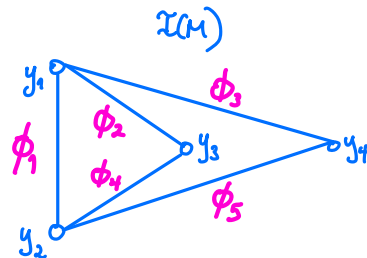
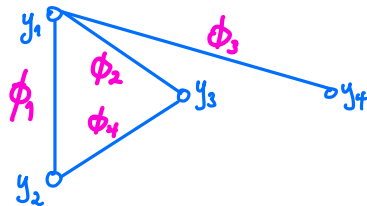
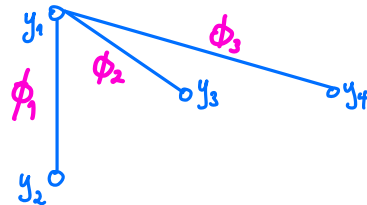
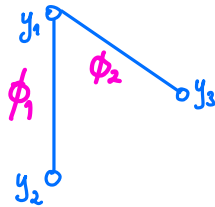
$(F2) \Rightarrow \mathcal{F}(M)$ is actually a **geometric lattice**.

\nearrow
atomic + upper semimodular

2. Shellability

DEFIN: A pure $(r-1)$ -dimensional simplicial complex Δ is **shellable** if we can order its **facets** $\phi_1, \phi_2, \dots, \phi_t$ in a **shelling order**:

$\forall j \geq 2$, ϕ_j intersects the subcomplex generated by $\phi_1, \phi_2, \dots, \phi_{j-1}$ in a pure $(r-2)$ -dim'l subcomplex



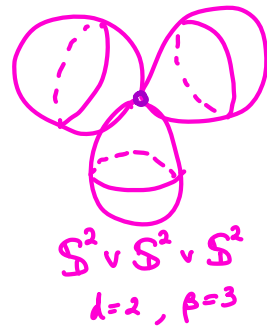
Shelling determines the homotopy type of Δ

DEF'N: Call ϕ_j a **homology facet** in the shelling $\phi_1, \phi_2, \dots, \phi_t$ if ϕ_j intersects the subcomplex gen'd by $\phi_1, \phi_2, \dots, \phi_{j-1}$ in the entire boundary $\text{Bd } \phi_j$

PROPOSITION: When Δ is pure d -dimensional and shellable,

then $\|\Delta\| \approx \underbrace{\mathbb{S}^d \vee \mathbb{S}^d \vee \dots \vee \mathbb{S}^d}_{\beta\text{-fold 1-point wedge of } d\text{-spheres } \mathbb{S}^d}$

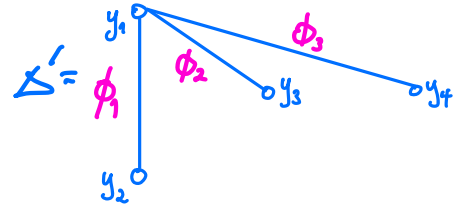
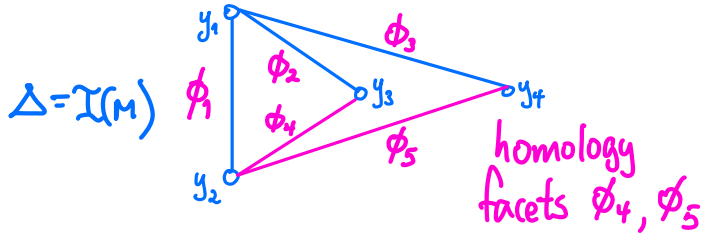
geometric realization of Δ homotopy equivalent



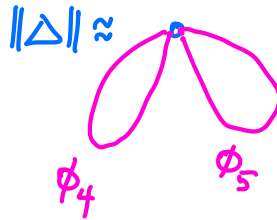
where $\beta := \#$ of **homology facets** ϕ_j in **any** shelling order

In fact, whenever Δ is **shellable**,

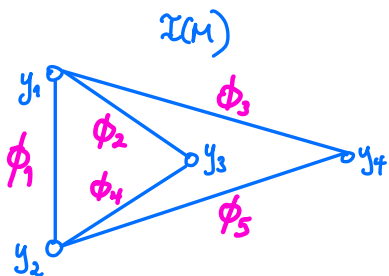
then $\Delta' := \Delta - \{ \text{homology facets } \phi_j \}$ is **contractible**:



contract Δ' to a point



THEOREM (Provan-Billera 1980) For a matroid M , the independent set complex $\mathcal{I}(M)$ is shellable, via lexicographic order on the bases $\mathcal{B}(M)$.



ϕ_1 ϕ_2 ϕ_3 ϕ_4 ϕ_5
 $12 <_{\text{lex}} 13 <_{\text{lex}} 14 <_{\text{lex}} 23 <_{\text{lex}} 24$

Furthermore, the number of homology facets is

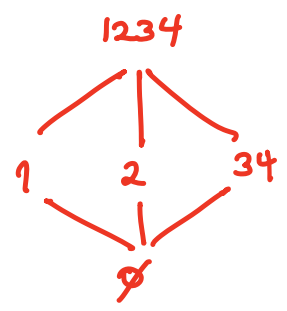
$$\beta = T_M(0,1) = \text{Tutte polynomial } T_M(x,y) \text{ evaluated at } x=0, y=1$$

$$= \# \text{ bases } B \in \mathcal{B}(M) \text{ of internal activity zero}$$

COROLLARY: $\|\mathcal{I}(M)\| \approx \underbrace{\mathbb{S}^{r(n)-1} \vee \dots \vee \mathbb{S}^{r(n)-1}}_{T_M(0,1) \text{-fold wedge}}$

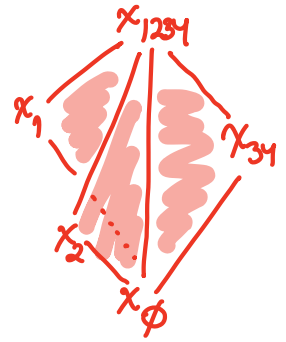
The flats $F(M)$ as a poset P gives us another simplicial complex, the **order complex** $\Delta P :=$ simplicial complex with **vertex set** $\{x_p\}_{p \in P}$ and **simplices/faces** the **totally ordered subsets** $\{x_{p_1}, x_{p_2}, \dots, x_{p_k}\}$ if $p_1 < p_2 < \dots < p_k$ in P

flat poset $F(M)$



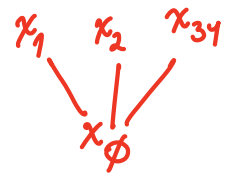
order complex

$\Delta F(M)$



contractible

$\text{Cone}(\Delta_M) \stackrel{\text{DEF}}{=} \Delta(F(M) - \{E\})$



contractible

Bergman complex $\Delta_M \stackrel{\text{DEF}}{=} \Delta(F(M) - \{\emptyset, E\})$



$S^0 \vee S^0$

a 2-fold wedge of 0-spheres

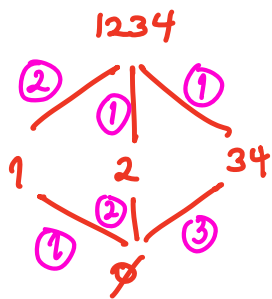
THEOREM
(Garsia 1980)

For a matroid M , all three of $\begin{cases} \Delta F(M) \\ \Delta(F(M) - \{E\}) \\ \Delta(F(M) - \{\emptyset, E\}) \end{cases} =: \underline{\Delta}_M$

are shellable, via **lexicographic order** on the edge-label sequences on maximal chains $\emptyset \subset F_1 \subset F_2 \subset F_3 \subset \dots \subset F_{r(M)-1} \subset E$ in $\mathcal{F}(M)$

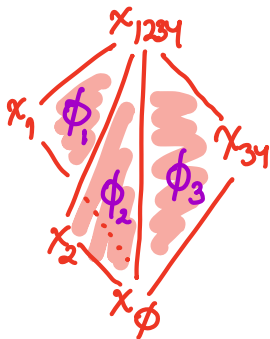
edge-labels: $(\min F_1, \min(F_2 - F_1), \min(F_3 - F_2), \dots, \min(E - F_{r(M)-1}))$

$\mathcal{F}(M)$

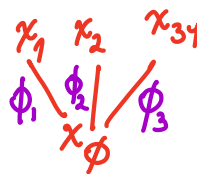


$(1,2) <_{\text{lex}} (2,1) <_{\text{lex}} (3,1)$
 $\phi_1 \quad \phi_2 \quad \phi_3$

$\Delta F(M)$



$\Delta(F(M) - \{E\})$



$\Delta(F(M) - \{\emptyset, E\})$



Furthermore, the number of homology facets is

$$\beta = T_M(1,0) = \text{Tutte polynomial } T_M(x,y) \text{ evaluated at } x=1, y=0$$

$$= \# \text{ bases } B \in \mathcal{B}(M) \text{ of external activity zero}$$

COROLLARY: $\|\underline{\Delta}(M)\| \approx \underbrace{\$^{r(n)-2} \vee \dots \vee \$^{r(n)-2}}_{T_M(1,0)\text{-fold wedge}}$

Bergman complex

$$\Delta_n := \Delta(F(M) - \{\bar{\phi}, \epsilon\})$$

3. Augmented Bergman complex Δ_M

In a monumental pair of 2020 papers, Braden-Huh-Matherne-Prandfoot-Wang introduced a hybrid.

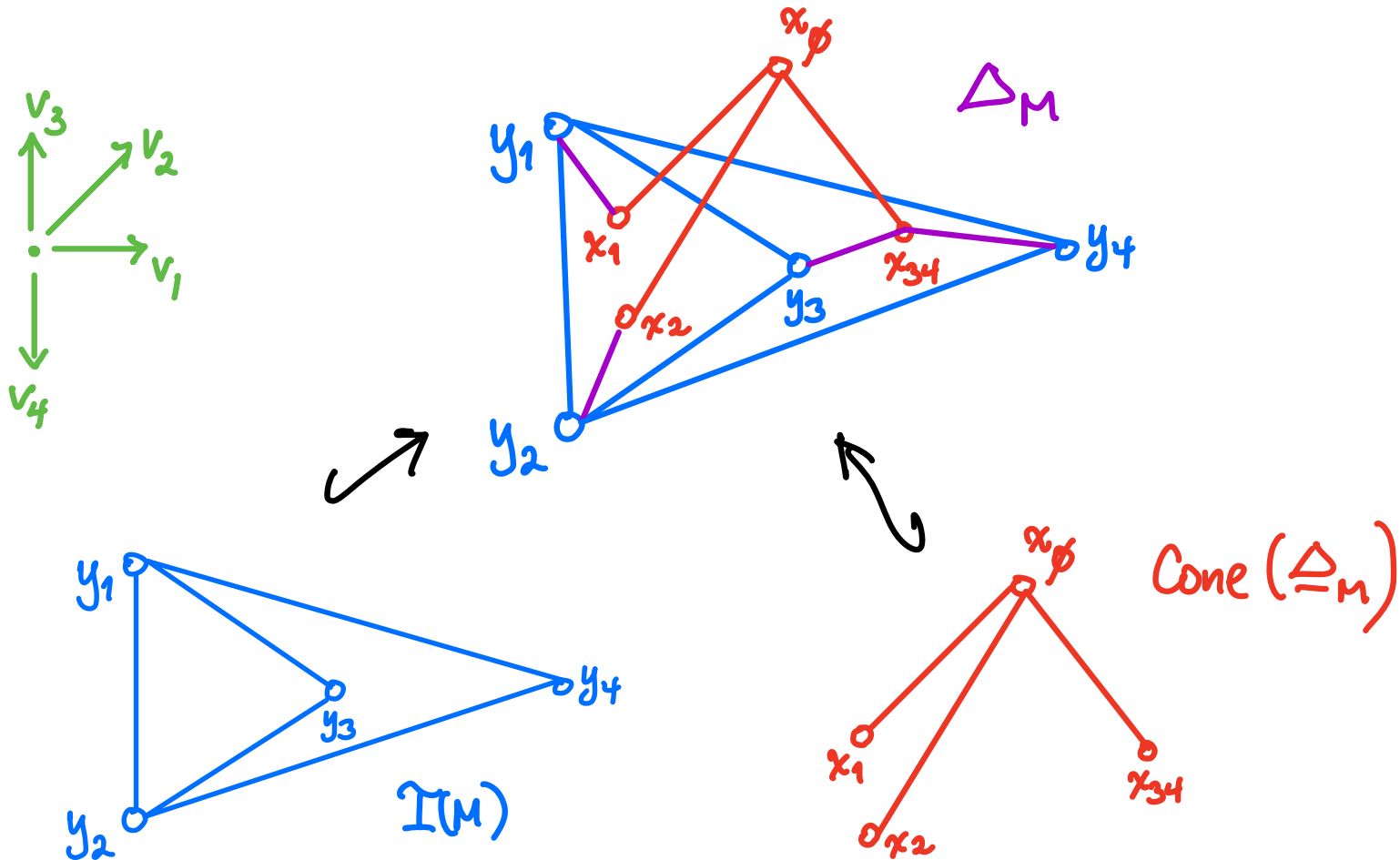
DEF'N: The augmented Bergman complex Δ_M

has vertex set $\{y_1, y_2, \dots, y_n\} \cup \{x_F\}$ $\begin{matrix} \emptyset \subseteq F \subsetneq E \\ \text{proper flats } F \in \mathcal{F}(M) \end{matrix}$

with simplices/faces $\{y_i\}_{i \in I} \cup \{x_{F_1}, x_{F_2}, \dots, x_{F_\ell}\}$

- when
- $I \in \mathcal{I}(M)$ is independent
 - F_1, F_2, \dots, F_ℓ are proper flats
 - $I \subseteq F_1 \subsetneq F_2 \subsetneq \dots \subsetneq F_\ell (\neq E)$

Δ_M is pure of dimension $r(M)-1$, containing both $I(M)$ and $\text{Cone}(\Delta_M)$ as subcomplexes:



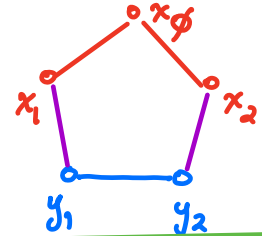
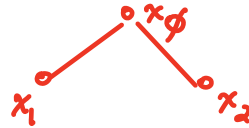
SPECIAL CASE: Boolean matroid M of rank n

$I(M)$
 = $(n-1)$ -simplex
 $2^{\{1,2,\dots,n\}}$

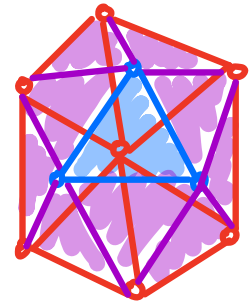
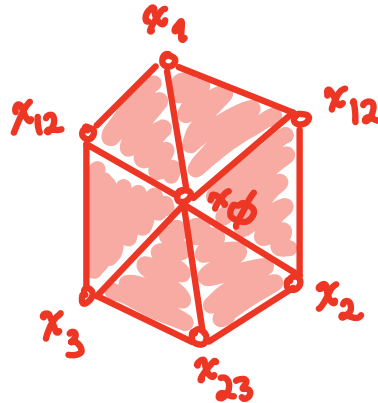
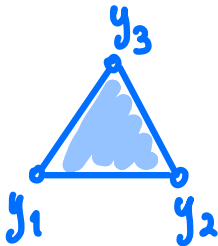
$\text{Cone}(\Delta_M)$
 = barycentric
 subdivision of
 $(n-1)$ -simplex

Δ_M
 = boundary of
 stellatedron

$n=2$



$n=3$



Why did BHMPW introduce Δ_n ?

Its Stanley-Reisner ring has an amazing Artinian quotient by certain linear forms

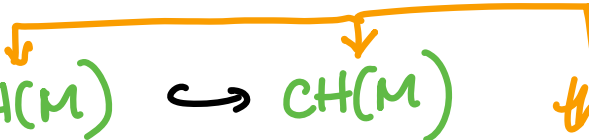
$$CH(M) = \mathbb{R}[y_1, \dots, y_n, x_F]_{\substack{\text{Plats} \\ F \subseteq E}}$$

= augmented Chow ring of M

$$\left(\begin{array}{l} x_F x_G, F \not\subseteq G, G \not\subseteq F \\ y_i x_F, i \notin F \\ y_i - \sum_{i \notin F} x_F, i=1,2,\dots,n \end{array} \right)$$

in which the y_1, \dots, y_n generate a subalgebra $H(M)$
= graded Möbius algebra of M

with a crucial $H(M)$ -submodule $IH(M)$
= intersection cohomology of M
and remarkable properties...

- $H(M) \leftrightarrow IH(M) \leftrightarrow CH(M)$

 these satisfy Kähler package
- Hilbert series for $H(M)$ interprets rank sizes k_k of $F(M)$
 and Kähler package for $IH(M) \Rightarrow$ Dowling-Wilson's Top Heavy Conj. (1974)
- Hilbert series for $IH(M)$ interprets Z -polynomial for M
 and Kähler package for $IH(M) \Rightarrow$ unimodality for Z
- Hilbert series for $IH(M)/(y_1, \dots, y_n)IH(M)$
 interprets Kazhdan-Lusztig polynomial for M
 \Rightarrow nonnegativity of K-L polynomial!

They used this *weaker* property of Δ_M than shellability:

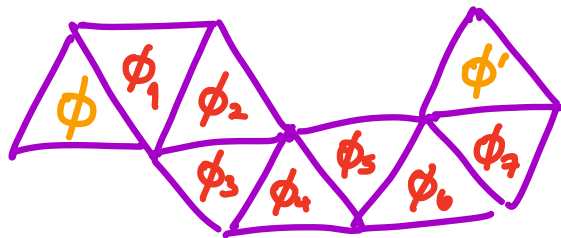
PROPOSITION: For any matroid M ,

(BHMPW)
2020

Δ_M is gallery-connected,

that is, any two facets ϕ, ϕ' are connected by a gallery of facets $\phi = \phi_0, \phi_1, \phi_2, \dots, \phi_{t-1}, \phi_t = \phi'$

with each $\phi_i \cap \phi_{i+1}$ of dimension $r(M) - 2$
(= codimension 1)



4. Two kinds of shellings of Δ_M and **corollaries**

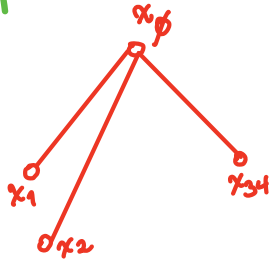
THEOREM (UMN REU 2021) For any matroid M ,
the augmented Bergman complex has

two families of shellings:

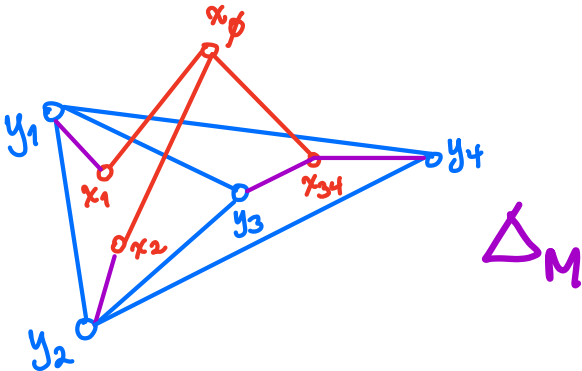
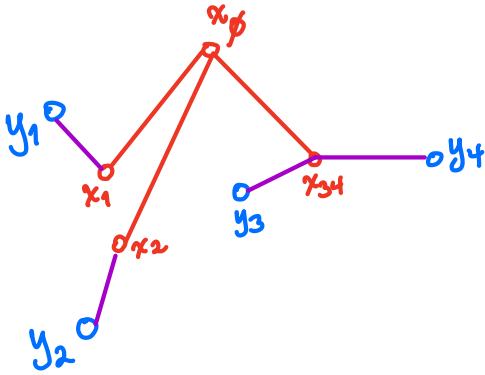
(i) some that shell the facets of $\text{Cone}(\Delta_M)$ first,
and facets of $\mathcal{I}(M)$ last.

(ii) some that shell the facets of $\mathcal{I}(M)$ first,
and facets of $\text{Cone}(\Delta_M)$ last.

Type (i) shellings

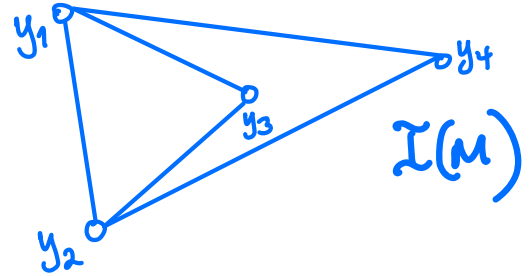


$\text{Cone}(\Delta_M)$

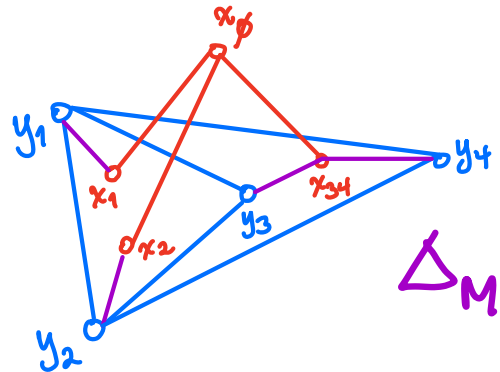
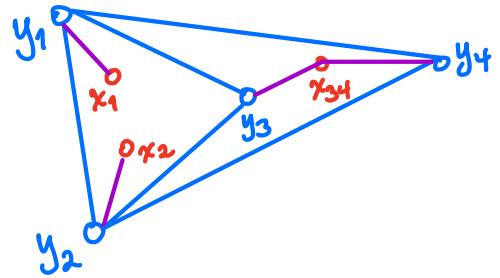


Δ_M

Type (ii) shellings



$I(M)$



Δ_M

COROLLARY: The augmented Bergman complex Δ_M
 (UMN REU 2021) has $\|\Delta_M\| \approx \underbrace{\mathbb{S}^{r(M)-1} \vee \dots \vee \mathbb{S}^{r(M)-1}}_{\beta\text{-fold wedge}}$

where β has two expressions:

(i) $\beta = T_M(1,1) = \#\mathcal{B}(M)$
 because the homology facets in type (i)
 shellings are $\{\gamma_i\}_{i \in \mathcal{B}}$ indexed by bases B of M .

(ii) $\beta = \sum_{\text{flats } F \in \mathcal{F}(M)} T_{M/F}(0,1) T_{M/F}(1,0)$
 counting type (ii) shelling homology facets.

REMARK: The equality

$$T_M(1,1) = \sum_{\text{flats } F} T_{M|_F}(0,1) T_{M/F}(1,0)$$

appeared in work of Étienne-Las Vergnas 1998,
rediscovered in Kook-R.-Stanton 2000,

and is a specialization of a convolution formula

$$T_M(x,y) = \sum_{\text{flats } F} T_{M|_F}(0,y) T_{M/F}(x,0)$$


for Tutte polynomials.

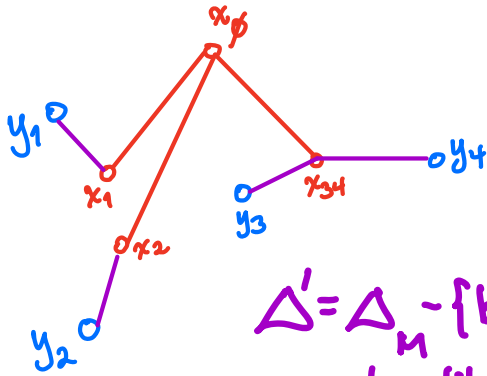
The type (i) shellings show **contractibility** of
 $\Delta' = \Delta_M - \{\text{facets } \{y_i\}_{i \in B} : \text{bases } B \in \mathcal{B}(M)\}$

Since **matroid automorphisms** set-wise stabilize the collection of basis facets, one can conclude:

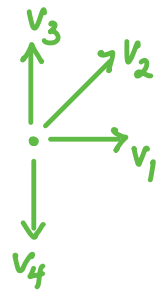
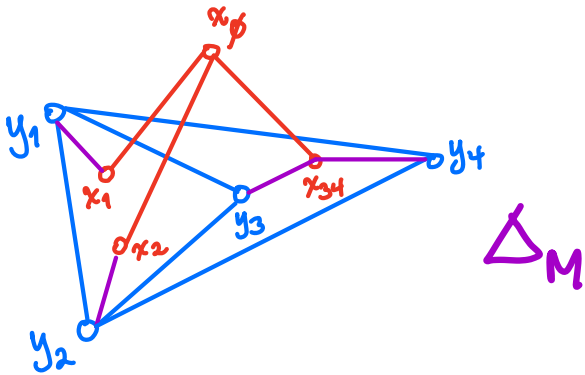
COROLLARY: The group $\text{Aut}(M)$ acts on
 $H_{r(M)-1}(\Delta_M, \mathbb{Z})$ as a **signed permutation representation**,
same as on $C_{r(M)-1}(\mathcal{L}(M), \mathbb{Z})$:

$$\sigma([b_1, b_2, \dots, b_r]) = [b_{\sigma(1)}, \dots, b_{\sigma(r)}] \text{ for bases } B = \{b_1, \dots, b_r\} \in \mathcal{B}(M)$$

 **oriented simplex**

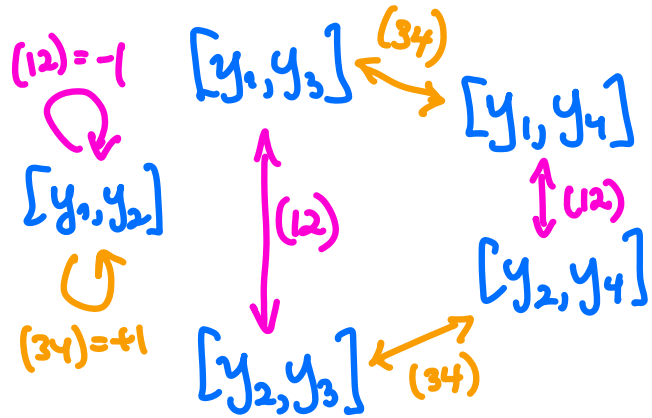


$\Delta' = \Delta_M - \{\text{bases}\}$
is contractible



$$\text{Aut}(M) = \{e, (12), (34), (12)(34)\}$$

$$H_1(\Delta_M) = \mathbb{Z}^5$$



REMARK: Neither \mathcal{I}_M nor $\underline{\Delta}_M$ have simple descriptions for their homology representations in general.

Notable special cases:

matroid M	$H_{r(M)-1}(\mathcal{I}_M)$	$H_{r(M)-2}(\underline{\Delta}_M)$
Boolean	trivial rep of S_n	sign rep of S_n
q -Boolean = \mathbb{F}_q -vector space	known virtually, not so explicit	Steinberg rep of $GL_n(\mathbb{F}_q)$
braid arrangement = complete graphic	an S_n -rep that restricts nicely to S_{n-1} (Kook 1996)	Lie rep of S_n

REMARK: Can generalize $\text{Aut}(M)$ -rep description to

arbitrary closure operators $2^E \xrightarrow{f} 2^E$

defining indep. sets $I : f(I - \{i\}) \neq f(I) \forall i \in I$

bases $B : B$ indep. and $f(B) = E$

flats $F : f(F) = F$

and augmented Bergman complex Δ_f

with vertices $\{y_1, \dots, y_n\} \cup \{x_F\}$ proper flats $F \neq E$

simplices $\{y_i\}_{i \in I} \cup \{x_{F_1}, \dots, x_{F_\ell}\}$

- with
- I indep.
 - F_1, \dots, F_ℓ flats
 - $I \subseteq F_1 \subseteq \dots \subseteq F_\ell$

$\mathcal{L}(f)$, $\underline{\Delta}_f$, Δ_f are not shellable in general.

Nevertheless:

THEOREM
(UMN REU 2021) $\|\Delta_f\| \approx \bigvee_{\text{bases } B} \#B-1$

and $\text{Aut}(M)$ acts on $H_*(\Delta_f)$ as a signed permutation rep on oriented chains $[b_1, b_2, \dots, b_r]$ indexed by bases $B = \{b_1, \dots, b_r\}$.

Again $\Delta' := \Delta_f - \left\{ \{y_i\}_{i \in B} : B \text{ a basis} \right\}$
is contractible.

Thanks
for
your
attention!