Hurwitz's factorization count and its deformations

Vic Reiner

Univ. of Minnesota

Helen Barton Lecture Series

UNC Greensboro August 26, 2020

PLAN

- Hurwitz's count & why
- Proof(s)
- q-Deformation 1 and a CSP
- q-Deformation 2 and $GL_n(\mathbb{F}_q)$

- Hurwitz's count & why
- Proof(s)
- q-Deformation 1 and a CSP
- q-Deformation 2 and $GL_n(\mathbb{F}_q)$


Adolf
Hurwitz
1859
-1919

THEOREM (Hurwitz 1891)

Inside the symmetric group $\mathfrak{S}_n = \{\text{permutations of } \{1,2,\ldots,n\}\}$ a fixed n-cycle, say $c = (1, 2, \ldots, n)$ has exactly

n^{n-2} factorizations $c = t_1 \cdot t_2 \cdots t_{n-1}$

$$= (i_1, j_1) \cdot (i_2\, j_2) \cdots (i_{n-1}\, j_{n-1})$$

into $n-1$ transpositions $t_k = (i_k\, j_k)$

THEOREM (Hurwitz 1891)

Inside the symmetric group $\mathfrak{S}_n$ a fixed n-cycle, say $c = (1,2,\dots,n)$ has exactly n^{n-2} factorizations $c = t_1 \cdot t_2 \cdots t_{n-1}$ into $n-1$ transpositions $t_k = (i_k\, j_k)$

EXAMPLES

$n=3$

$$c = (1,2,3) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$\left.\begin{aligned} &= (12)(23) \\ &= (13)(12) \\ &= (23)(13) \end{aligned}\right] \; 3^{3-2} = 3$$

factorizations $c = t_1 t_2$

$n=4$

$c = (1,2,3,4)$

$$\left.\begin{aligned} &= (12)(23)(34) \\ &= (14)(12)(23) \\ &= (34)(14)(12) \\ &= (23)(34)(14) \end{aligned}\right] \; 4$$

$$\left.\begin{aligned} &= (23)(13)(34) \\ &= (14)(23)(13) \\ &= (24)(14)(23) \\ &= (34)(24)(14) \\ &= (12)(34)(24) \\ &= (13)(12)(34) \\ &= (14)(13)(12) \\ &= (23)(14)(13) \\ &= (24)(23)(14) \\ &= (12)(24)(23) \\ &= (34)(12)(24) \\ &= (13)(34)(12) \end{aligned}\right] \; 12$$

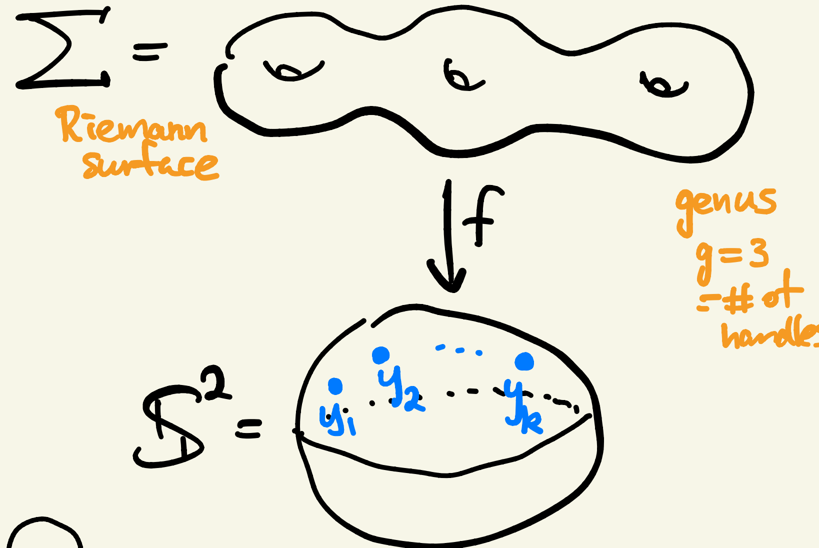
$4^{4-2} = 16 \; (= 4 + 12)$

factorizations $c = t_1 t_2 t_3$

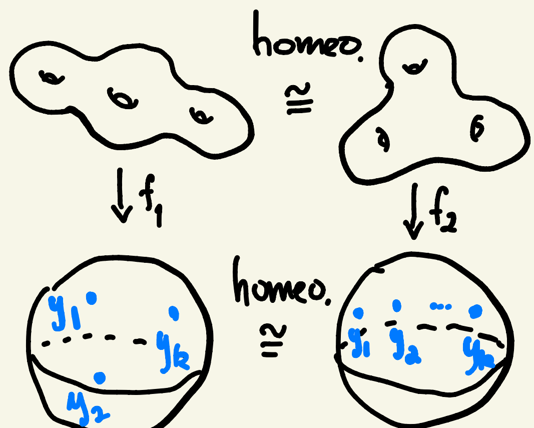
Why count this? Riemann & Hurwitz wanted to count

degree n ramified coverings

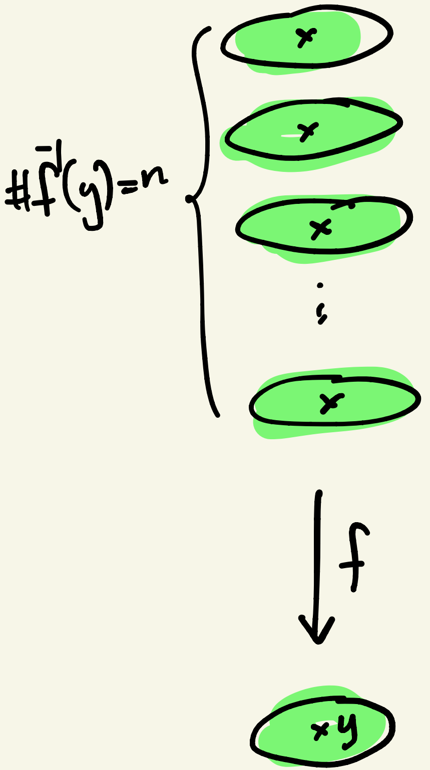
n-sheeted coverings of $S^2 - \{y_1, y_2, \dots, y_k\}$



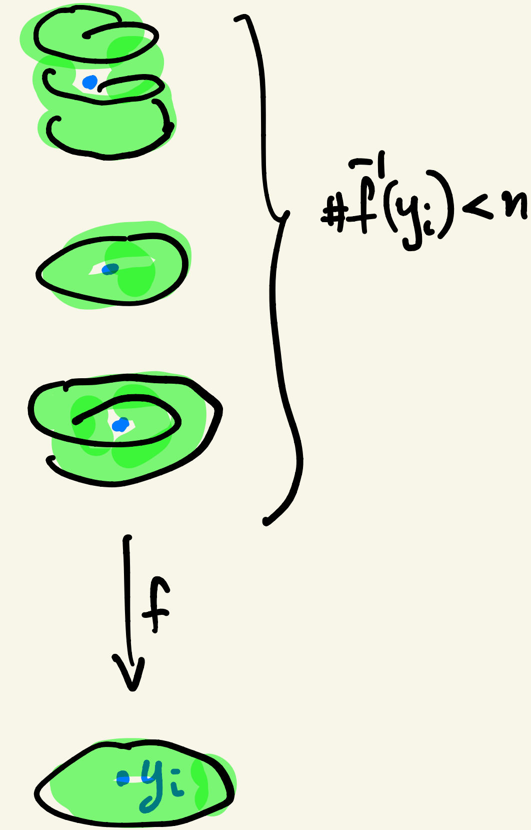
up to equivalence



Above most points y :



Above branch points y_i

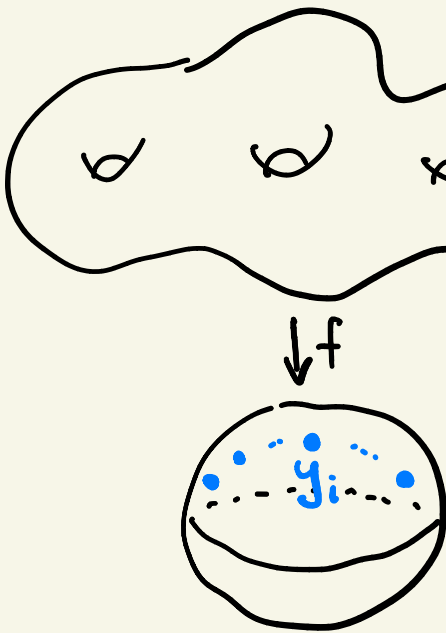


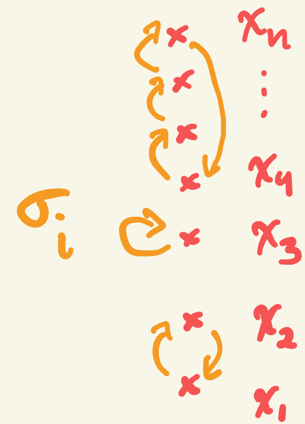
Equivalence classes parametrized by choice of

monodromy permutations $(\sigma_1, \sigma_2, \dots, \sigma_k)$ above $(y_1, y_2, \dots, y_k)$

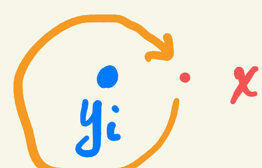
branch points

satisfying $\sigma_1 \sigma_2 \cdots \sigma_{k-1} \sigma_k = 1$ in $\mathfrak{S}_n$



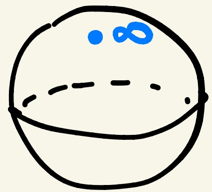


$f^{-1}(x) = \{x_1, \dots, x_n\}$



[See Lando & Zvonkin Chap. 1]

SPECIAL CASE:



$\Sigma = S^2 = \mathbb{C} \cup \{\infty\}$

$f(z) = z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$

generic polynomial of degree n

$S^2 = \mathbb{C} \cup \{\infty\}$

Here f generically has $n-1$ non-∞ branch points

$y_1, \dots, y_{n-1}$ (= zeroes of $f'(z)$)

Each y_i has monodromy permutation a transposition t_k (= simple branching)

$y_n = \infty$ has monodromy permutation an n-cycle $c^{-1} = (n, n-1, \dots, 2, 1)$

$t_1 t_2 \cdots t_{n-1} c^{-1} = 1$ means $c = t_1 t_2 \cdots t_{n-1}$

- Hurwitz's count & why
- Proof(s)
- q-Deformation 1 and a CSP
- q-Deformation 2 and $GL_n(\mathbb{F}_q)$

How did Hurwitz prove it?

- Roughly, generating function-ology.
- He solves a more general problem, replacing c by any $\sigma \in \mathfrak{S}_n$.
- He finds a recurrence based on cycle sizes of σ.
- He shows how the recurrence leads to a functional equation for a generating function.
- He sketches how to solve it [Strehl 1996 completes the sketch]

How didn't Hurwitz prove it?

[See Lando & Zvonkin §5.1]

V.I. Arnold's method: Set up two parameter spaces

- $\mathbb{C}^{n-1}$ for degree n polynomials $f(z) = z^n + a_{n-2}z^{n-2} + \dots + a_1 z + a_0$
- $\mathbb{C}^{n-1}$ for unordered sets $\{y_1, \dots, y_{n-1}\} \subset \mathbb{C}$

"LL" stands for "Lyashko-Looijenga"

and a homogeneous polynomial map $\mathbb{C}^{n-1} \xrightarrow{LL} \mathbb{C}^{n-1}$

$$f(z) \longmapsto \text{zeroes } \{y_1, \dots, y_{n-1}\} \text{ of } f'(z)$$

$=$ branch points for f

so the LL map has generic fiber size giving Hurwitz's count.

Then do a degree calculation showing $\deg(LL) = \dfrac{2n \cdot 3n \cdots (n-1)n}{2 \cdot 3 \cdots (n-1)} = n^{n-2}$

How else didn't Hurwitz prove it?

The beautiful combinatorial proof of Dénes (1959) using...

THEOREM (Borchardt 1880 / Cayley 1889) $n^{n-2} = \#\{\text{trees on vertices } \{1,2,3,\dots,n\}\}$

EXAMPLES

$n=3$

1 — 2 — 3
2 — 1 — 3
2 — 3 — 1

3^1

$n=4$

1 — 2, 3, 4
2 — 1, 3, 4
3 — 1, 2, 4
4 — 1, 2, 3

4

1 — 2 — 3 — 4
1 — 2 — 4 — 3
1 — 3 — 2 — 4
1 — 3 — 4 — 2
1 — 4 — 2 — 3
1 — 4 — 3 — 2
2 — 1 — 3 — 4
2 — 1 — 4 — 3
2 — 3 — 1 — 4
2 — 4 — 1 — 3
3 — 1 — 2 — 4
3 — 2 — 1 — 4

12

$4^{4-2} = 4 + 12$

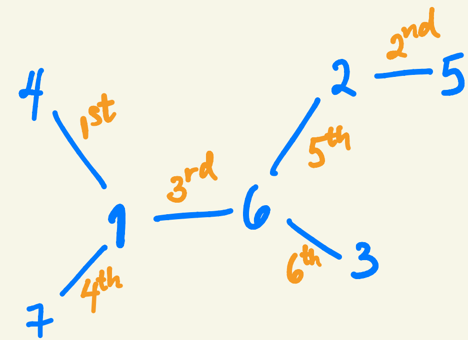
To show $\#\left\{\begin{array}{l}\text{factorizations}\\ c=(1,2,\ldots,n)=t_1t_2\cdots t_{n-1}\end{array}\right\} = n^{n-2} = \#\{\text{trees on } \{1,2,\ldots,n\}\}$,

Dénes shows instead

$$(n-1)!\ \#\left\{\begin{array}{l}\text{factorizations}\\ c=(1,2,\ldots,n)=t_1t_2\cdots t_{n-1}\end{array}\right\} = (n-1)!\ \#\{\text{trees on }\{1,2,\ldots,n\}\}$$

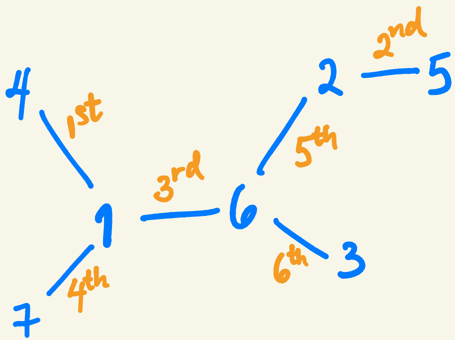
$$\| \qquad\qquad\qquad\qquad \|$$

$\#\left\{\begin{array}{l}\text{factorizations of}\\ \text{all } n\text{-cycles } \sigma = t_1t_2\cdots t_{n-1}\end{array}\right\}$ $\longleftrightarrow$ via a bijection $\#\left\{\begin{array}{l}\text{edge-ordered trees}\\ \text{on } \{1,2,\ldots,n\}\end{array}\right\}$



But Dénes's bijection is now easy:

$\{$edge-ordered trees on $\{1,2,\dots,n\}\}$ $\longleftrightarrow$ $\{$factorizations of all n-cycles $\sigma = t_1 t_2 \cdots t_n\}$



$\longleftrightarrow$

$$t_1 \; t_2 \; t_3 \; t_4 \; t_5 \; t_6$$
$$(14)(25)(16)(17)(26)(36)$$
$$= (1763524) = \sigma$$

$t_k = (ij)$

$\Longleftrightarrow$

k^{th} edge labeled $i - j$

KEY POINT: $\sigma \cdot (ij)$ has either 1 fewer or 1 more cycle depending on whether $\{i,j\}$ are in different or same cycle of σ

- Hurwitz's count & why
- Proof(s)
- q-Deformation 1 and a CSP
- q-Deformation 2 and $GL_n(\mathbb{F}_q)$

Let's introduce some q-analogues of...

- positive integers $n \xleftarrow{q=1} [n]_q := 1+q+q^2+\dots+q^{n-1} = \frac{1-q^n}{1-q}$

- our count $n^{n-2} \xleftarrow{q=1} \dfrac{[2n]_q [3n]_q \cdots [(n-1)n]_q}{[2]_q \; [3]_q \cdots [n-1]_q}$

$$= [n]_{q^2} [n]_{q^3} \cdots [n]_{q^{n-1}}$$

EXAMPLES: $n=3$: $[3]_{q^2} = 1+q^2+q^4$ $\xrightarrow{q=1}$ $3^1 = 3$

$n=4$: $[4]_{q^2} \cdot [4]_{q^3} = (1+q^2+q^4+q^6)(1+q^3+q^6+q^9)$ $\xrightarrow{q=1}$ $4^2 = 16$

Why introduce these q-analogues?

They miraculously **predict orbit structure** for this **cyclic action** Ψ on $\{\text{factorizations } c = t_1 t_2 \cdots t_{n-2} t_{n-1}\}$:

$$(t_1, t_2, \ldots, t_{n-2}, t_{n-1}) \xmapsto{\ \Psi\ } \Big(c\, t_{n-1} c^{-1}, \underbrace{t_1, t_2, \ldots, t_{n-2}}_{\substack{\text{their product is} \\ t_1 t_2 \cdots t_{n-2} = c\, t_{n-1}^{-1}}} \Big)$$

EXAMPLE:
$n=3$

$$\begin{tikzcd}[column sep=small]
 & c = (1,2,3) = (12)(23) \arrow[dr, bend left, "\Psi"] & \\
(23)(13) \arrow[ur, bend left, "\Psi"] & & (13)(12) \arrow[ll, bend left, "\Psi"]
\end{tikzcd}$$

EXAMPLE

$n=4$ $\qquad |\langle\Psi\rangle| = 3\cdot 4 = 12$

$c=(1,2,3,4)=(12)(23)(34)$

$$\begin{tikzcd}
(12)(23)(34) \arrow[r, "\Psi"] & (14)(12)(23) \arrow[d] \\
(23)(34)(14) \arrow[u] & (34)(14)(12) \arrow[l]
\end{tikzcd}$$

a non-free $\langle\Psi\rangle$-orbit

$$\begin{tikzcd}
(23)(13)(34) \arrow[d, "\Psi"] \\
(14)(23)(13) \arrow[d] \\
(24)(14)(23) \arrow[d] \\
(34)(24)(14) \arrow[d] \\
(12)(34)(24) \arrow[d] \\
(13)(12)(34) \arrow[d] \\
(14)(13)(12) \arrow[d] \\
(23)(14)(13) \arrow[d] \\
(24)(23)(14) \arrow[d] \\
(12)(24)(23) \arrow[d] \\
(34)(12)(24) \arrow[d] \\
(13)(34)(12) \arrow[uuuuuuuuuuu, bend left]
\end{tikzcd}$$

a free $\langle\Psi\rangle$-orbit

In general Ψ generates a cyclic group $\langle\Psi\rangle$ of order $(n-1)n$,

because $(t_1,\dots,t_{n-1}) \xmapsto{\Psi^{n-1}} (ct_1c^{-1},\dots,ct_{n-1}c^{-1})$

The q-analogue predicts the $\langle\Psi\rangle$-orbit structure as follows.

THEOREM (T. Douvropoulos 2018, CONJ. by N. Williams 2013)

For any $\Psi^d \in \langle \Psi \rangle$, letting $\zeta := e^{2\pi i/(n-1)n}$

$$\#\{\text{factorizations } (t_1,\ldots,t_{n-1}) \text{ fixed by } \Psi^d\} = [n]_{q^2}[n]_{q^3}\cdots[n]_{q^{n-1}}\Big|_{q=\zeta^d}$$

EXAMPLE: $n=3$

$$\begin{tikzcd}
 & (12)(23) \arrow[dr, bend left, "\Psi"] & \\
(23)(13) \arrow[ur, bend left, "\Psi"] & & (13)(12) \arrow[ll, bend left, "\Psi"]
\end{tikzcd}$$

$\zeta = e^{2\pi i/6}$

$$[3]_{q^2} = 1+q^2+q^4\Big|_{q=\zeta^d} = \begin{cases} 3 & \text{if } d=0 \\ 3 & \text{if } d=3 \\ 0 & \text{if } d=2,4 \\ 0 & \text{if } d=1,5 \end{cases}$$

EXAMPLE

$n=4$

$$\begin{tikzcd}
 & (12)(23)(34) \arrow[dr, "\psi"] & \\
(23)(34)(14) \arrow[ur] & & (14)(12)(23) \arrow[dl] \\
 & (34)(14)(12) \arrow[ul] &
\end{tikzcd}$$

$$\begin{tikzcd}
(23)(13)(34) \arrow[d, "\psi"] \\
(14)(23)(13) \arrow[d] \\
(24)(14)(23) \arrow[d] \\
(34)(24)(14) \arrow[d] \\
(12)(34)(24) \arrow[d] \\
(13)(12)(34) \arrow[d] \\
(14)(13)(12) \arrow[d] \\
(23)(14)(13) \arrow[d] \\
(24)(23)(14) \arrow[d] \\
(12)(24)(23) \arrow[d] \\
(34)(12)(24) \arrow[d] \\
(13)(34)(12) \arrow[uuuuuuuuuuu, bend left]
\end{tikzcd}$$

$$\zeta = e^{2\pi i/12}$$

$$[4]_{q^2}[4]_{q^3} = (1+q^2+q^4+q^8)(1+q^3+q^6+q^9)$$

$$\xrightarrow[q=\zeta^d]{} \begin{cases} 16 & \text{if } d=0 \\ 4 & \text{if } d=4,8 \\ 0 & \text{if } d=3,9 \\ 0 & \text{if } d=6 \\ 0 & \text{if } d=1,5,7,11 \end{cases}$$

EXERCISE: These numbers recover the $\langle \psi \rangle$-orbit sizes

ASIDE: D. Stanton, D. White and I called this situation

a cyclic sieving phenomenon (CSP):

- $\langle \Psi \rangle$ permutes a finite set X
- $X(q)$ is a q-analogue of $\#X$ meaning $\#X = X(q)\big|_{q=1}$

and more generally, if $\zeta := e^{2\pi i / \#\langle \Psi \rangle}$

then any ψ^d has $\#\{x \in X : \psi^d(x) = x\} = X(q)\Big|_{q=\zeta^d}$

It happens a lot!

Douvropoulos's proof ?

- Applies to a more general conjecture by N. Williams
  not just factoring n-cycles in $\mathfrak{S}_n$ into transpositions,
  but factoring Coxeter elements in reflection groups into reflections.

- Uses Lyashko-Looijenga generalization of Arnold's $\mathbb{C}^{n-1} \xrightarrow{LL} \mathbb{C}^{n-1}$
  via invariant theory of reflection groups

- Uses D. Bessis's beautiful analysis of fibers of LL
  via factorizations into reflections

- Hurwitz's count & why
- Proof(s)
- q-Deformation 1 and a CSP
- q-Deformation 2 and $GL_n(\mathbb{F}_q)$

A different well-established avenue of q-deformation ...

$\mathfrak{S}_n$ = symmetric group $= \left\{ \begin{array}{c} \text{invertible maps} \\ \{1,2,\ldots,n\} \xrightarrow{\sigma} \{1,2,\ldots,n\} \end{array} \right\}$ $\overset{\text{"}q=1\text{"}}{\leftsquigarrow}$ $GL_n(\mathbb{F}_q)$ = general linear group $= \left\{ \begin{array}{c} \text{invertible } \mathbb{F}_q\text{-linear maps} \\ \mathbb{F}_q^n \xrightarrow{\sigma} \mathbb{F}_q^n \end{array} \right\}$

{transpositions $t_k = (ij)$ in $\mathfrak{S}_n$} $\leftsquigarrow$ {reflections t in $GL_n(\mathbb{F}_q)$}

i.e. fixed space $(\mathbb{F}_q^n)^t = \{v \in \mathbb{F}_q^n : t(v) = v\}$ is $(n-1)$-dimensional

---

{n-cycles $(i_1 i_2 \cdots i_n)$ in $\mathfrak{S}_n$} $\leftsquigarrow$ {Singer cycles c in $GL_n(\mathbb{F}_q)$}

i.e. generators for $\langle c \rangle = \mathbb{F}_{q^n}^{\times} \hookrightarrow GL_n(\mathbb{F}_q)$

---

n = order of an n-cycle $\leftsquigarrow$ $q^n - 1$ = order of a Singer cycle

REMARK: **Singer cycles** c in $GL_n(\mathbb{F}_q)$ are the conjugacy classes of the **companion matrices** for a **primitive irreducible** polynomial

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \text{ in } \mathbb{F}_q[x]$$

$$\left[\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & & \vdots & -a_2 \\ \vdots & \vdots & \ddots & 0 & \vdots \\ & & & 1 & -a_{n-1} \end{array}\right]$$

EXAMPLE In $GL_4(\mathbb{F}_2)$, Singer cycles are conjugates of the companion matrices for x^4+x+1, x^4+x^3+1 in $\mathbb{F}_2[x]$

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

In $GL_2(\mathbb{F}_3)$, Singer cycles are conjugates of the companion matrices for x^2+x-1, x^2-x+1 in $\mathbb{F}_3[x]$

$$\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \qquad \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

Bearing that analogy in mind...

THEOREM (Lewis-R.-Stanton 2014)

Inside the general linear group $GL_n(\mathbb{F}_q)$, a fixed Singer cycle c has exactly $(q^n-1)^{n-1}$ factorizations $c = t_1 \cdot t_2 \cdots t_n$ into n reflections t_k

"$q=1$"

THEOREM (Hurwitz 1891)

Inside the symmetric group $\mathfrak{S}_n$, a fixed n-cycle c has exactly n^{n-2} factorizations $c = t_1 \cdot t_2 \cdots t_{n-1}$ into $n-1$ transpositions t_k

How didn't we prove it (but wish we could)?

Via a simple, combinatorial Dénes-style overcounting proof, $GL_n(\mathbb{F}_q)$-analogized.

PROBLEM:

Find such a proof!

So how did we prove it?

A tried-and-true method of Frobenius (1896) [Lando & Zvonkin Appendix A.1]

lets one count for g in G any finite group (!)

and any choice of G-conjugacy stable subsets $C_1, C_2, \ldots, C_\ell \subset G$

$$\#\{\text{factorizations } g = t_1 t_2 \cdots t_\ell \text{ with } t_k \text{ in } C_k \text{ for } k = 1, 2, \ldots, \ell\}$$

via a sum over irreducible complex G-characters

It was a slog, but it worked for $G = GL_n(\mathbb{F}_q)$

g = Singer cycle

C_k = reflections

PROBLEM: Formulate and prove a Douvropoulos/Williams CONJECTURE for the $\langle\psi\rangle$-orbits

$$c = t_1 t_2 \cdots t_{n-1} t_n \overset{\psi}{\longmapsto} c t_n c^{-1} \cdot t_1 t_2 \cdots t_{n-1}$$

when c is a Singer cycle in $GL_n(\mathbb{F}_q)$.

What is the appropriate t-analogue of $(q^n-1)^{n-1}$ to plug in $t=\zeta^d$ for $\zeta = e^{2\pi i / n(q^n-1)}$?

Thanks for your attention!

References

- Dénes - Publ. Math. Inst. Hung. Acad. Sci. 4 (1959)
- Douvropoulos - Séminaire Lotharingien de Combinatoire 80B (2018)
- Hurwitz - Mathematische Annalen 39 (1891)
- Lando & Zvonkin - "Graphs on Surfaces" Springer (2004)
- Lewis-R.-Stanton - J. Algebraic Combinatorics 40 (2014)
- Strehl - Séminaire Lotharingien de Combinatoire 37 (1996)