Francesco,

Pólya frequency sequences and Koszulity

Vic Reiner

89^{th} Séminaire Lotharingien de Combinatoire
Brenti Fest, Bertinoro March 28, 2023

- Francesco and me

- Polya frequency sequences

- Koszul algebras

connections / analogies that we don't yet fully understand...

- Stirling numbers & Koszul algebras
  (joint with Ayah Almousa
  and Sheila Sundaram)

- Francesco and me

Stanley numbers:

	1986	1986	1986	1988	1990	1990	1990	1990	
…	9	10	11	12	13	14	15	16	…
	Lynne Butler	Karen Collins	Sheila Sundaram	Francesco Brenti	Mark Purtill	Vic Reiner	Dave Wagner	Julian West	

- MIT Math 18.318 Spring 1987
  Topics in Combinatorics : symmetric functions
  with Profs. Stanley & Brenti

- β-testing of Björner & Brenti's
  "Combinatorics of Coxeter groups"
  in Minnesota Topics in Combinatorics Fall 1998

- Minnesota colloquium by Brenti in Feb 2001,
  discussing KL combinatorial invariance conjecture,
  before his proof in March 2002 via special matchings

- Collaboration with Brenti & Roichman in 2005, 2006
  on alternating subgroups of Coxeter groups

• Polya frequency sequences

UNIMODAL, LOG-CONCAVE AND PÓLYA FREQUENCY SEQUENCES IN COMBINATORICS

by

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June, 1988

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Certified by Signature redacted

Professor Richard P. Stanley
Thesis Supervisor

Accepted by Signature redacted

Professor Sigurdur Helgason, Chairman
Depatmental Graduate Committee
Department of Mathematics

DEFINITION (Brenti thesis §2.2)

Given a real sequence $(a_i)_{i=0,1,2,\ldots} = (\underset{\substack{\| \\ a_0}}{1}, a_1, a_2, a_3, \ldots)$

consider its generating function $A(t) := \sum_{i=0}^{\infty} a_i t^i \in \mathbb{R}[[t]]$

and the (infinite) Toeplitz matrix

$$\begin{array}{c|cccccc} & 1 & t & t^2 & t^3 & t^4 & \cdots \\ \hline 1 & 1 & a_1 & a_2 & a_3 & \cdots & \\ t & 0 & 1 & a_1 & a_2 & a_3 & \cdots \\ t^2 & 0 & 0 & 1 & a_1 & a_2 & a_3 \cdots \\ t^3 & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\ \vdots & & & & & & \end{array}$$

whose transpose expresses multiplication by $A(t)$

$$\mathbb{R}[[t]] \xrightarrow{\cdot A(t)} \mathbb{R}[[t]]$$

in the $\mathbb{R}$-basis $\{1, t, t^2, \ldots\}$

Say (a_i) is a Pólya frequency (PF) sequence if every square subdeterminant of the Toeplitz matrix is ≥ 0.

(a_i) PF $\Longrightarrow$ PF_1 i.e. 1×1 minors ≥ 0

$a_i \geq 0 \ \forall i$

PF $\Longrightarrow$ PF_2 i.e. 2×2 minors ≥ 0

$$\det \begin{bmatrix} a_i & a_{i+r} \\ a_{j-r} & a_j \end{bmatrix} \geq 0$$

i.e. $a_i a_j \geq a_{i+r} a_{j-r}$

$\Downarrow$ $i=j$, $r=1$

$a_i^2 \geq a_{i+1} a_{i-1}$ log-concave

PF_2 also implies a_i has no internal zeroes (Brenti thesis §2.5)

$\swarrow$

$1 \leq a_1 \leq a_2 \leq \dots \leq a_m \geq a_{m+1} \geq \dots$ unimodal

(a_i) PF also generalizes a polynomial $A(t) = \sum_{i=0}^{d} a_i t^i$ in $\mathbb{R}_{\geq 0}[t]$ having only (nonpositive) real roots :

THEOREM:
(Aissen-Edrei-Schoenberg-Whitney) 1951

[Brenti thesis §4.5]

(a_i) is PF $\Longleftrightarrow$

$$A(t) = e^{\gamma t} \cdot \frac{\prod_{i=1}^{\infty}(1+\alpha_i t)}{\prod_{j=1}^{\infty}(1-\beta_j t)}$$

with
$$\begin{cases} \gamma, \alpha_i, \beta_j \geq 0 \\ \sum_i \alpha_i < \infty \\ \sum_j \beta_j < \infty \end{cases}$$

COROLLARY: PF sequences come in dual pairs:

$$(a_i)_{i=0,1,2,\ldots} \longleftrightarrow A(t) = \sum_{i=0}^{\infty} a_i t^i = e^{\gamma t} \cdot \frac{\prod_i (1+\alpha_i t)}{\prod_j (1-\beta_j t)}$$

$$(a^!_i)_{i=0,1,2,\ldots} \longleftrightarrow A^!(t) = \sum_{i=0}^{\infty} a^!_i t^i = e^{\gamma t} \cdot \frac{\prod_j (1+\beta_j t)}{\prod_i (1-\alpha_i t)}$$

$$:= \frac{1}{A(-t)}$$

DEFINITION of $(a^!_i)$:

$$\boxed{A(-t) \cdot A^!(t) = 1}$$

[Brenti thesis §7.5]

EXAMPLE

$$(a_i) = \binom{n}{i}_{i=0,1,2,\dots,n} \longleftrightarrow A(t) = \sum_{i=0}^{n} \binom{n}{i} t^i = (1+t)^n$$

$$(a^!_i) = \left(\!\!\binom{n}{i}\!\!\right)_{i=0,1,2,\dots} \longleftrightarrow A^!(t) = \sum_{i=0}^{\infty} \binom{n+i-1}{i} t^i = \frac{1}{(1-t)^n}$$

$$= \binom{n+i-1}{i}$$

EXAMPLE

$$(a_i) = \big(c(n, n-1-i)\big)_{i=0,1,2,\dots,n-1} \longleftrightarrow A(t) = (1+t)(1+2t)(1+3t)\cdots(1+(n-1)t)$$

$c(n,k)$ = (signless) Stirling number of 1^{st} kind

:= # permutations of $\{1,2,\dots,n\}$ with k cycles

$$(a^!_i) = \big(S(n-1+i, n-1)\big)_{i=0,1,2,\dots} \longleftrightarrow A^!(t) = \frac{1}{(1-t)(1-2t)(1-3t)\cdots(1-(n-1)t)}$$

$S(n,k)$ = Stirling number of 2^{nd} kind

:= # partitions of $\{1,2,\dots,n\}$ with k blocks

[Brenti thesis §6.5]

EXAMPLE The previous examples are subsumed by

$$(a_i)_{i=0,1,2,\dots} \longleftrightarrow A(t) = \sum_{i=0}^{n} e_i(\underline{x})\, t^i = \prod_{j=1}^{n} (1 + x_j t)$$

with $a_i = e_i(x_1, x_2, \dots, x_n)$ elementary symmetric function in $x_1, x_2, \dots, x_n \in \mathbb{R}_{\geq 0}$

$$(a_i^!)_{i=0,1,2,\dots} \longleftrightarrow A^!(t) = \sum_{i=0}^{n} h_i(\underline{x})\, t^i = \prod_{j=1}^{n} \frac{1}{1 - x_j t}$$

with $a_i^! = h_i(x_1, x_2, \dots, x_n)$ complete homogeneous symmetric function in $x_1, x_2, \dots, x_n \in \mathbb{R}_{\geq 0}$

[Brenti thesis §7.4]

Koszul algebras

DEFINITION: A standard graded $\mathbb{k}$-algebra is a ring

$$A = \underbrace{\mathbb{k}\langle x_1, x_2, \ldots, x_n\rangle}_{\text{free associative algebra on } n \text{ letters over } \mathbb{k}} \Big/ I$$

I ← 2-sided homogeneous ideal, i.e.

$$\begin{array}{ccc} I = & \bigoplus_{d=0}^{\infty} I_d & \text{where } I_d = I \cap \mathbb{k}\langle \underline{x}\rangle \\ \cap & \cap & \\ \mathbb{k}\langle \underline{x}\rangle = & \bigoplus_{d=0}^{\infty} \underbrace{\mathbb{k}\langle \underline{x}\rangle_d}_{\text{span of monomials of degree } d} & \end{array}$$

DEFINITION: Hilbert function $(a_i)_{i=0,1,2,\ldots}$ where $a_i = \dim_{\mathbb{k}}(A_i)$

Hilbert series $A(t) = \sum_{i=0}^{\infty} a_i t^i$

EXAMPLES

$$\underbrace{\mathbb{k}[x_1,\dots,x_n]}_{\text{commutative polynomial ring over } \mathbb{k}} = \mathbb{k}\langle x_1,\dots,x_n\rangle \Big/ (x_i x_j - x_j x_i)_{1\le i<j\le n}$$

$$A(t) = \sum_{i=0}^{\infty} \left(\!\binom{n}{i}\!\right) t^i = \sum \binom{n+i-1}{i} t^i = \frac{1}{(1-t)^n}$$

$$\underbrace{\Lambda_{\mathbb{k}}\langle y_1,\dots,y_n\rangle}_{\text{exterior/skew-commutative polynomial ring over } \mathbb{k}} = \mathbb{k}\langle y_1,\dots,y_n\rangle \Big/ (y_i y_j + y_j y_i)_{1\le i<j\le n} + (y_i^2)_{1\le i\le n}$$

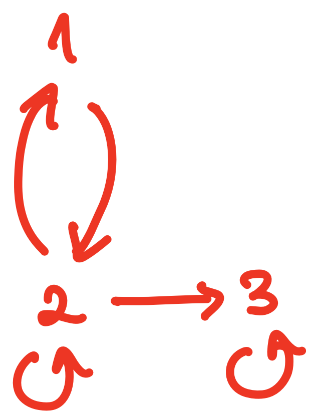
$$A^!(t) = \sum_{i=0}^{n} \binom{n}{i} t^i = (1+t)^n$$

EXAMPLE [related to Brenti's thesis Chap. 7 "PF digraphs"]

Given a directed graph (= digraph) D on vertices $\{1,2,\ldots,n\}$

let $A_D := \mathbb{k}\langle x_1,\ldots,x_n\rangle \big/ \big(\underbrace{x_i x_j}_{\substack{i \to j \\ \text{not in } D}} \;,\; \underbrace{x_i^2}_{\substack{i \circlearrowleft \\ \text{not in } D}} \big)$

EXAMPLE

$D =$ 

$$A_D = \mathbb{k}\langle x_1,x_2,x_3\rangle \big/ (x_1x_3,\, x_3x_1,\, x_3x_2,\, x_1^2)$$

$$= \mathrm{span}_{\mathbb{k}}\{1,\ \big|\ x_1, x_2, x_3,\ \big|\ x_1x_2, x_2x_1, x_2x_3, x_2^2, x_3^2,\ \big|\ x_1x_2^2, x_2^2x_1, x_1x_2x_3, x_2^2x_3, x_2x_1x_2, x_3^3, x_2x_3^2\ \big|\ \ldots\}$$

$$(a_D(i))_{i=0,1,2,\ldots} = (1, 3, 5, 7, \ldots)$$

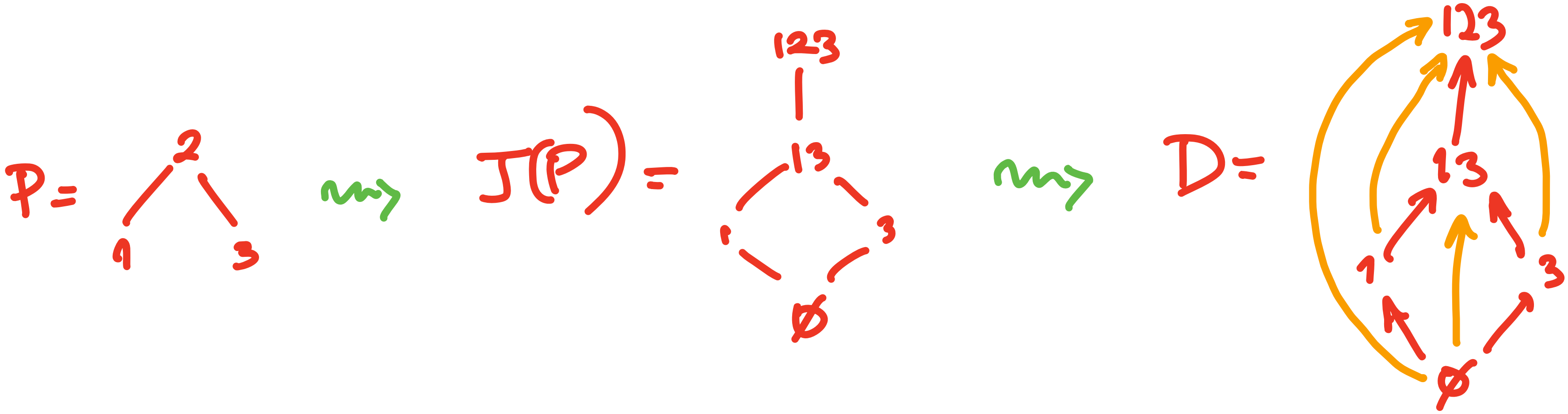
$a_D(i)$ = # of walks $(v_1, v_2, \ldots, v_i)$ of length i in D, that is, all steps v_j, v_{j+1} have $v_j \longrightarrow v_{j+1}$ in D.

QUESTION: Which digraphs D have $(a_D(i))_{i=0,1,2,\ldots}$ PF ?

(Brenti thesis Chapter 7)

Call them Pólya Frequency digraphs.

MOTIVATION: Brenti (§7.2) shows the Neggers-Stanley Poset Conjecture for a naturally labeled poset P is equivalent to the distributive lattice $D = J(P)$ giving a PF digraph.



REMARK: The Neggers-Stanley Conjecture was

resolved negatively

by Braenden 2004 ← labeled poset counterexamples

& Stembridge 2007 ← naturally labeled poset counterexamples

but led to other interesting open questions.

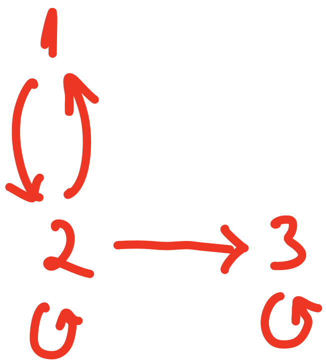
THEOREM [Brenti §7.5]

PF digraphs come in dual pairs $D, \bar{D}$ where $\bar{D}$ = complementary digraph of D,

that is, $$A_{\bar{D}}(t) = \frac{1}{A_D(-t)}$$

(and both have simple rational expressions in $\mathbb{Q}(t)$ using Transfer-Matrix Method)

EXAMPLE

$D =$ 

$$A_D = \mathbb{k}\langle x_1, x_2, x_3 \rangle / (x_1x_3, x_3x_1, x_3x_2, x_1^2)$$

$\bar{D} =$ 

$$A_{\bar{D}} = \mathbb{k}\langle y_1, y_2, y_3 \rangle / (y_1y_2, y_2y_1, y_2y_3, y_2^2, y_3^2)$$

DEFINITION: The standard graded $\Bbbk$-algebra A is **Koszul** if the field $\Bbbk = A/(x_1,\dots,x_n)$ has an A-free resolution that is **linear**:

$$0 \leftarrow \Bbbk \leftarrow A \xleftarrow{d_1} A(-1)^n \xleftarrow{d_2} A(-2)^{\beta_2} \xleftarrow{d_3} A(-3)^{\beta_3} \xleftarrow{d_4} A(-4)^{\beta_4} \leftarrow \cdots$$

$$\begin{array}{ccc} 0 \leftarrow\!\shortmid & x_1 \\ & \vdots \\ 0 \leftarrow\!\shortmid & x_n \end{array} \qquad [x_1 \cdots x_n] \qquad \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ - & - & & \cdot \end{bmatrix} \qquad \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ - & \cdot & \cdot \end{bmatrix} \qquad \begin{bmatrix} \cdot & - & \cdot & \cdot \\ \cdot & \cdot & - & \cdot \end{bmatrix}$$

all matrix entries in d_i are **linear**, i.e. lie in A_1

MOTIVATING EXAMPLE:

The Koszul complex resolving $\Bbbk$ over $A = \Bbbk[x_1, \dots, x_n]$:

$$0 \leftarrow \Bbbk \leftarrow A \xleftarrow{d_1} A(-1)^n \xleftarrow{d_2} A(-2)^{\binom{n}{2}} \xleftarrow{d_3} A(-3)^{\binom{n}{3}} \leftarrow \cdots$$

$$A(-1)^n = A \otimes \Lambda^1 \langle y_1, \dots, y_n \rangle, \qquad A(-2)^{\binom{n}{2}} = A \otimes \Lambda^2_{\Bbbk} \langle y_1, \dots, y_n \rangle, \qquad A(-3)^{\binom{n}{3}} = A \otimes \Lambda^3_{\Bbbk} \langle y_1, \dots, y_n \rangle$$

$$0 \leftarrow\!\shortmid x_1, \quad \vdots \quad 0 \leftarrow\!\shortmid x_n$$

$$x_1 \xleftarrow{d_1} y_1$$

$$\vdots$$

$$x_n \leftarrow\!\shortmid y_n$$

$$x_i y_j - x_j y_i \xleftarrow{d_2} y_i \wedge y_j$$

$$x_i y_j \wedge y_k - x_j y_i \wedge y_k + x_k y_i \wedge y_j \leftarrow\!\shortmid y_i \wedge y_j \wedge y_k$$

THEOREM (S. Priddy 1970) Koszul algebras come in pairs $A, A^!$ that are both quadratic algebras, and quadratic duals:

$$A = \Bbbk\langle x_1, \ldots, x_n\rangle / I \quad \text{where } I = (I_2) \qquad I_2 = I \cap \Bbbk\langle \underline{x}\rangle_2$$

$$A^! = \Bbbk\langle y_1, \ldots, y_n\rangle / J \quad \text{where } J = (J_2) \qquad J_2 = J \cap \Bbbk\langle \underline{y}\rangle_2$$

where $I_2 = J_2^{\perp}$ with respect to the bilinear pairing

$$\Bbbk\langle \underline{x}\rangle_2 \times \Bbbk\langle \underline{y}\rangle_2 \longrightarrow \Bbbk$$

that makes $\{x_i x_j\}_{1\le i,j\le n}$ and $\{y_i y_j\}_{1\le i,j\le n}$ dual bases.

Furthermore, $A^!(t)\cdot A(-t) = 1$ i.e. $A^!(t) = \dfrac{1}{A(-t)}$

because Priddy's complex on $A\otimes A^!$

canonically resolves $\Bbbk$ over A:

$$0 \leftarrow \Bbbk \leftarrow A\otimes A^!_0 \xleftarrow{d_1} A\otimes A^!_1 \xleftarrow{d_2} A\otimes A^!_2 \longleftarrow \cdots$$

EXAMPLE $A = \Bbbk[x_1,\dots,x_n]$

$A^! = \Lambda_{\Bbbk}\langle y_1,\dots,y_n\rangle$

Here Priddy's complex = Koszul complex

$$\begin{tikzcd}
0 & \Bbbk \arrow[l] & A \arrow[l] & A(-1)^n \arrow[l, "d_1"'] \arrow[d, equal] & A(-2)^{\binom{n}{2}} \arrow[l, "d_2"'] \arrow[d, equal] & A(-3)^{\binom{n}{3}} \arrow[l, "d_3"'] \arrow[d, equal] & \cdots \arrow[l] \\
 & & & A\otimes\Lambda^1\langle y_1,\dots,y_n\rangle & A\otimes\Lambda^2_{\Bbbk}\langle y_1,\dots,y_n\rangle & A\otimes\Lambda^3_{\Bbbk}\langle y_1,\dots,y_n\rangle &
\end{tikzcd}$$

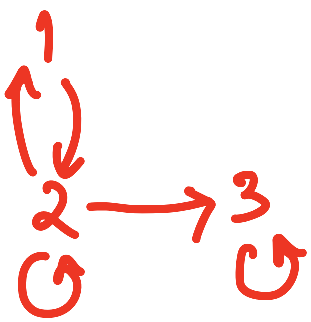
$0 \leftarrow\!\shortmid\ x_1$

$\vdots$

$0 \leftarrow\!\shortmid\ x_n$

THEOREM (Fröberg 1975, Kobayashi 1992, Bruns-Herzog-Vetter 1994) For any digraph D, A_D is a Koszul algebra with $A_{\bar{D}} = A_D^!$ its quadratic dual.

EXAMPLE

$D =$ 

$$A_D = \Bbbk\langle x_1, x_2, x_3\rangle \big/ (\underbrace{x_1x_3,\ x_3x_1,\ x_3x_2,\ x_1^2}_{I_2})$$

$\bar{D} =$ 

$$A_{\bar{D}} = \Bbbk\langle y_1, y_2, y_3\rangle \big/ (\underbrace{y_1y_2,\ y_2y_1,\ y_2y_3,\ y_2^2,\ y_3^2}_{J_2 = I_2^\perp})$$

$$\cong A_D^!$$

QUESTION (R.-Welker 2005) Which Koszul algebras A have their Hilbert functions $(a_i)_{i=0,1,2,\ldots}$ PF ?

We were motivated partly by the case where $A = \Bbbk\langle x_1, \ldots, x_n \rangle / I$ with I generated by monomials $x_i x_j$, x_k^2

i.e. Brenti's question [thesis Chap. 7]:

Which A_D have $(a_D(i))_{i=0,1,2,\ldots}$ PF ?

FURTHER MOTIVATION: Parallel constructions

... preserving PF	... preserving Koszulity
• $(a_i),(b_i) \mapsto (c_i)$ $\quad c_i = \sum_{j+k=i} a_j b_k$	• $A, B \longmapsto A \otimes B$ tensor product
• $(a_i) \mapsto (a_{di}) = (a_0, a_d, a_{2d}, a_{3d}, \dots)$	• $A \longmapsto A^{(d)} := \bigoplus_{i=0}^{\infty} A_{di}$ d^{th} Veronese subalgebra
• $(a_i),(b_i)$ both finite or polynomial in i $\mapsto (a_i b_i)$ Hadamard product	• $A, B \longmapsto A * B := \bigoplus_{i=0}^{\infty} A_i \otimes B_i$ Segre subalgebra
• (a_i) defining $A(t)$ $\mapsto$ (b_i) defining $(1-t)A(t)$	• $A \longmapsto A/(\ell(x))$ non-zero-divisor of deg 1

ONE MORE ANALOGY:

PROPOSITION: For a Koszul algebra A with Hilbert function (a_i), certain Toeplitz minors are always ≥ 0: those corresponding to ribbon skew Schur functions

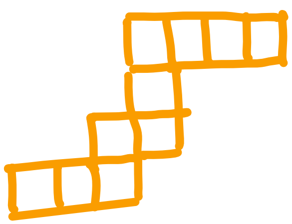
(K. Van de Bogert) 2023

[cf. Brenti thesis § 7.4]

$$s_{\lambda/\mu} = \det\left(a_{\lambda_i - \mu_j - i + j} \right)$$

EXAMPLE:

$\lambda/\mu =$



ribbon

row sizes
← 4
← 1
← 2
← 3

$\longleftrightarrow$

$$\det \begin{bmatrix} a_3 & a_{3+2} & a_{3+2+1} & a_{3+2+1+4} \\ 1 & a_2 & a_{2+1} & a_{2+1+4} \\ 0 & 1 & a_1 & a_{1+4} \\ 0 & 0 & 1 & a_4 \end{bmatrix} \geq 0$$

Why?

$$\det\begin{bmatrix} a_3 & a_{3+2} & a_{3+2+1} & a_{3+2+1+4} \\ 1 & a_2 & a_{2+1} & a_{2+1+4} \\ 0 & 1 & a_1 & a_{1+4} \\ 0 & 0 & 1 & a_4 \end{bmatrix}$$

$$= a_3a_2a_1a_4 - \begin{pmatrix} a_5a_1a_4 \\ + \\ a_3a_3a_4 \\ + \\ a_3a_2a_5 \end{pmatrix} + \begin{pmatrix} a_6a_4 \\ + \\ a_5a_5 \\ + \\ a_3a_7 \end{pmatrix} - a_{10}$$

interprets $\ker(d)$ in this exact sequence:

$$0 \to \ker(d) \to A_3\otimes A_2\otimes A_1\otimes A_4 \xrightarrow{d} \begin{matrix} A_5\otimes A_1\otimes A_4 \\ \oplus \\ A_3\otimes A_3\otimes A_4 \\ \oplus \\ A_3\otimes A_2\otimes A_5 \end{matrix} \to \begin{matrix} A_6\otimes A_4 \\ \oplus \\ A_5\otimes A_5 \\ \oplus \\ A_3\otimes A_7 \end{matrix} \to A_{10} \to 0$$

- Stirling numbers & Koszul algebras

(joint with Ayah Almousa and Sheila Sundaram)

Recall our Stirling number dual pair of PF sequences:

$$(a_i) = \big(c(n, n-1-i)\big)_{i=0,1,2,\ldots,n-1} \longleftrightarrow A(t) = (1+t)(1+2t)(1+3t)\cdots(1+(n-1)t)$$

Stirling #'s of 1st kind

$$(a^!_i) = \big(S(n-1+i, n-1)\big)_{i=0,1,2,\ldots} \longleftrightarrow A^!(t) = \frac{1}{(1-t)(1-2t)(1-3t)\cdots(1-(n-1)t)}$$

Stirling #'s of 2nd kind

There are important Koszul algebras $A(n)$ interpreting

$$A(t) = (1+t)(1+2t)(1+3t)\cdots(1+(n-1)t) \quad \text{or} \quad c(n,k):$$

(ordinary) cohomology

configuration space of labeled distinct points $p_1, p_2, \ldots, p_n$ in $\mathbb{R}^d$

configuration space of n label $p_1, \ldots, p_n$

$$A(n) = H^{\bullet}\,\mathrm{Conf}(n, \mathbb{R}^d)$$

$$\cong \begin{cases} \Lambda_{\Bbbk}\langle x_{ij} \rangle_{1\le i<j\le n} \Big/ \left(x_{ij}x_{ik} - x_{ij}x_{jk} + x_{ik}x_{jk} \right)_{1 \le i<j<k\le n} & \text{if } d \text{ even} \\ \Bbbk[x_{ij}] \Big/ \left(x_{ij}^2,\ x_{ij}x_{ik} - x_{ij}x_{jk} + x_{ik}x_{jk} \right) & \text{if } d \text{ odd} \end{cases}$$

F. Cohen 1972

type A Orlik-Solomon algebra $A_{OS}(n)$

type A graded Varchenko-Gelfand algebra $A_{VG}(n)$

QUESTION: What about the Koszul dual algebras $A^!(n)$ and $S(n,k)$?

As $\mathfrak{S}_n$-representations, both $A(n) = A_{OS}(n), A_{VG}(n)$ are well-studied, but not completely understood.

$n=4$	1 +	$6t$ +	$11t^2$ +	$6t^3$	total rep'n (ungraded)
	A_0	A_1	A_2	A_3	
$A_{VG}(4)$	$S_{(4)}$	$S_{(3,1)}$ $S_{(2,1,1)}$	$S_{(3,1)}$ $S_{(2,1,1)}$ $S_{(2,2)}$ $S_{(2,2)}$ $S_{(1,1,1,1)}$	$S_{(3,1)}$ $S_{(2,1,1)}$	$\Bbbk[\mathfrak{S}_4]$ = regular rep.
$A_{OS}(4)$	$S_{(4)}$	$S_{(3,1)}$ $S_{(2,2)}$	$S_{(3,1)}$ $S_{(2,1,1)}$		2 copies of $\Bbbk[\mathfrak{S}_4 / \mathfrak{S}_2 \times \mathfrak{S}_1 \times \mathfrak{S}_1]$
		$S_{(4)}$	$S_{(3,1)}$ $S_{(2,2)}$	$S_{(3,1)}$ $S_{(2,1,1)}$	

THEOREM (Sundaram-Welker 1997) As $\mathfrak{S}_n$-representations,

$$\sum_{n=0}^{\infty} \sum_{k=1}^{n} \operatorname{ch} A(n)_{n-k}\, t^k =$$

$$\begin{cases} \displaystyle\sum_{\lambda=1^{m_1}2^{m_2}\cdots} t^{\ell(\lambda)} \cdot \prod_{j=1}^{\infty} h_{m_j}[\mathrm{Lie}_j] = \prod_{m=1}^{\infty} (1-p_m)^{-a_m(t)} & \text{for } A_{VG}(n) \\ \displaystyle\sum_{\lambda=1^{m_1}2^{m_2}\cdots} t^{\ell(\lambda)} \prod_{j \text{ odd}} h_{m_j}[\pi_j] \cdot \prod_{j \text{ even}} e_{m_j}[\pi_j] = \prod_{m=1}^{\infty} (1+(-1)^m p_m)^{a_m(-t)} & \text{for } A_{OS}(n) \end{cases}$$

plethysm formulas

product generating functions

where $a_m(t) = \frac{1}{m} \sum_{d|m} \mu(d)\, t^{m/d}$

PROBLEM: Find such formulas for the Koszul duals $A^!(n)$.

Stirling triangle recursions....

$c(n,k)$

$n=$ \ $k=$	1	2	3	4	5
1	1				
2	1	1			
3	2	3	1		
4	6	11	6	1	
5	24	50	35	10	1

$$c(n,k) = (n-1)\cdot c(n-1,k) + c(n-1,k-1)$$

$S(n,k)$

$n=$ \ $k=$	1	2	3	4	5
1	1				
2	1	1			
3	1	3	1		
4	1	7	6	1	
5	1	15	25	10	1

$$S(n,k) = k\cdot S(n-1,k) + S(n-1,k-1)$$

... lift to branching rules :

THEOREM : Both $A(n) = A_{OS}(n), A_{VG}(n)$ satisfy

(a) $$A(n)_i \Big\downarrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} \cong \chi^{(n-1)}_{def} \otimes A(n-1)_{i-1} \oplus A(n-1)_i$$

(Sundaram 1994, 2020)

$$c(n,k) = (n-1) \cdot c(n-1,k) + c(n-1,k-1)$$

(b) $$A(n)^{!}_i \Big\downarrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} \cong \chi^{(n-1)}_{def} \otimes A(n)^{!}_{i-1} \Big\downarrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} \oplus A(n-1)^{!}_i$$

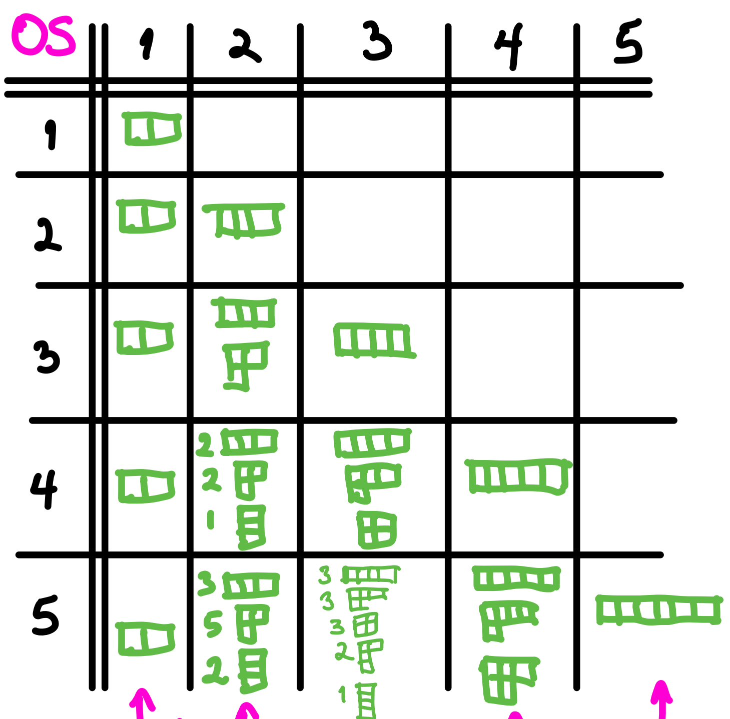
(Almonsa-R.-Sundaram 2023)

$$S(n,k) = k \cdot S(n-1,k) + S(n-1,k-1)$$

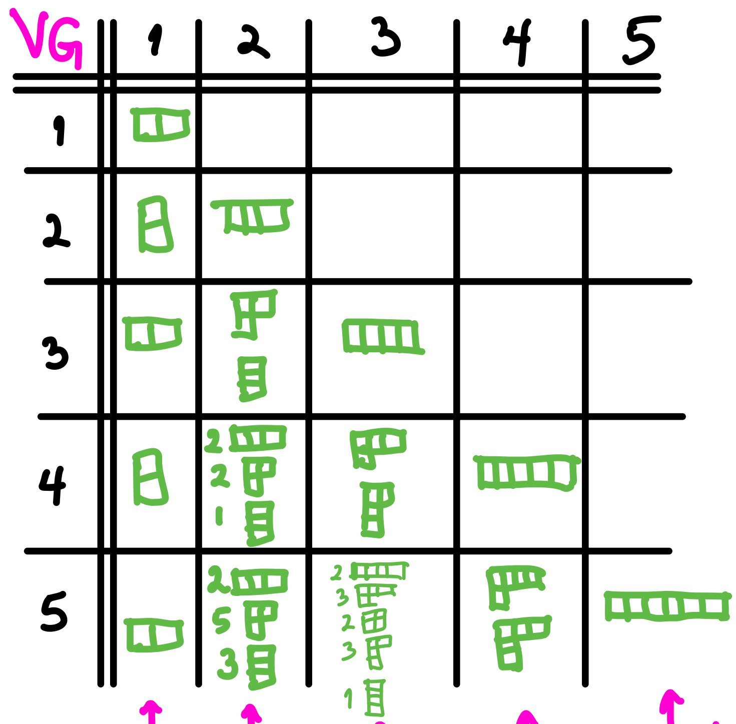
where $\chi^{(n-1)}_{def}$:= defining $\mathfrak{S}_{n-1}$-rep as permutation matrices

$S(n,k)$

n= \ k=	1	2	3	4	5
1	1				
2	1	1			
3	1	3	1		
4	1	7	6	1	
5	1	15	25	10	1

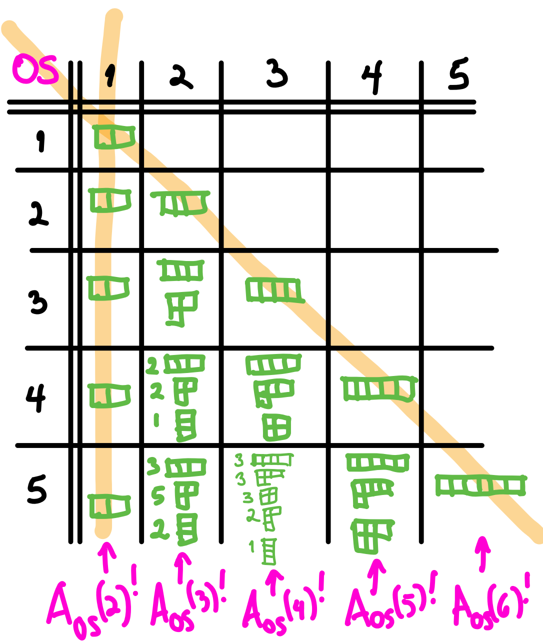


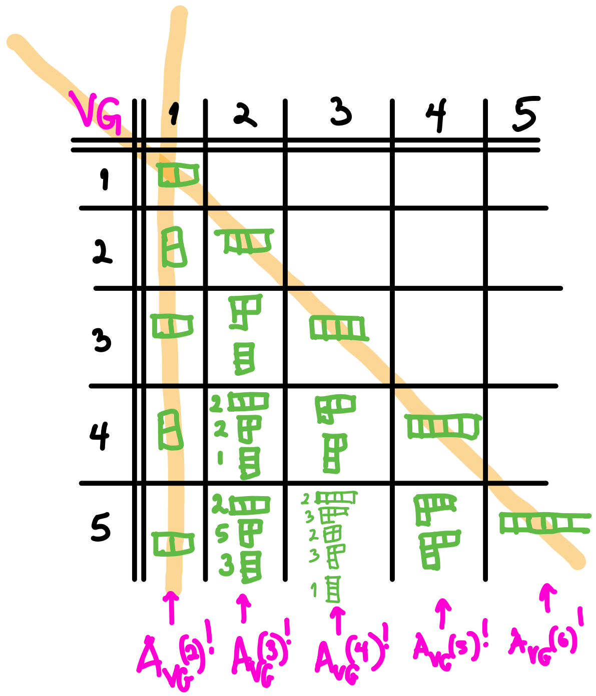
$A_{OS}(2)!$ $A_{OS}(3)!$ $A_{OS}(4)!$ $A_{OS}(5)!$ $A_{OS}(6)!$



$A_{VG}(2)!$ $A_{VG}(3)!$ $A_{VG}(4)!$ $A_{VG}(5)!$ $A_{VG}(6)!$

- Boundary cases of $A(n)^!$

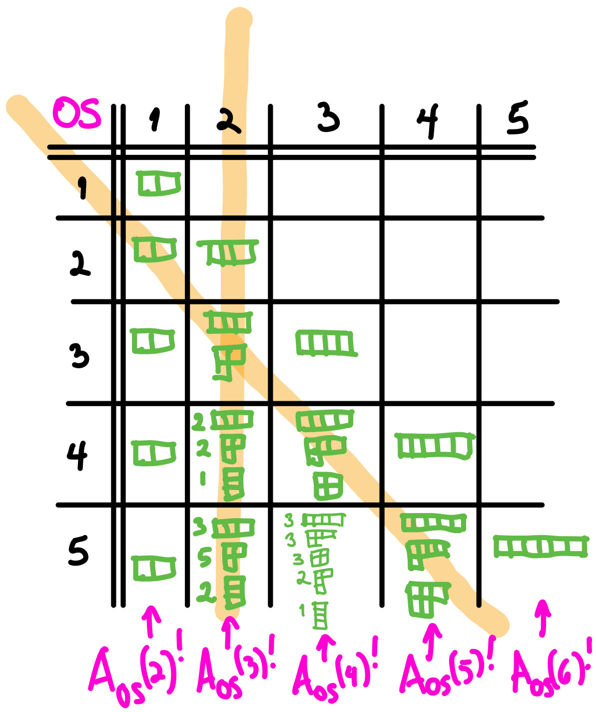


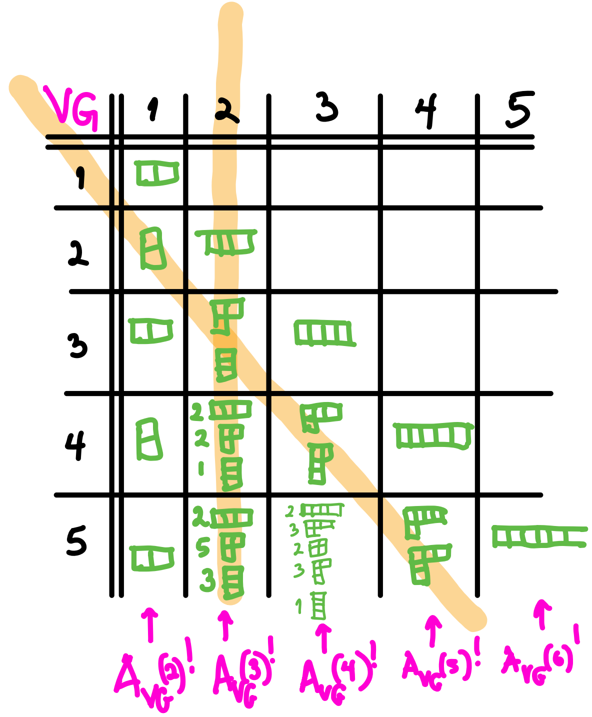


$$S(n-1, n-1) = 1 \rightsquigarrow \mathrm{ch}\, A(n)^!_{n-1} = s_{(n-1)}$$

$$S(m, 1) = 1 \rightsquigarrow \mathrm{ch}\, A_{OS}(2)^!_i = s_{(2)}$$

$$\mathrm{ch}\, A_{VG}(2)^!_i = \begin{cases} s_{(2)} & i \text{ even} \\ s_{(1,1)} & i \text{ odd} \end{cases}$$





$$S(n,n-1)=\binom{n}{2} \rightsquigarrow \mathrm{ch}\, A(n)^!_{n-1} = \begin{cases} s_{\square\square}\; s_{\underbrace{\square\square\square\square\square}_{n-2}} & \text{for OS} \\ s_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}\; s_{\underbrace{\square\square\square\square\square}_{n-2}} & \text{for VG} \end{cases}$$

$$S(n,2)=2^{n-1}-1 \rightsquigarrow A_{OS}(3)^!_i = 1 + \chi^{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} + \chi^{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}\otimes\chi^{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} + \chi^{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}\otimes\chi^{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}\otimes\chi^{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} + \dots + \left(\chi^{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}\right)^{\otimes i}$$

$$= 1+2+2^2+\dots+2^{n-2}$$

The $\mathfrak{S}_n$-rep'ns $\{A(n)_i\}_{n=1,2,\ldots}$ for fixed i are known to be representation stable in the sense of Church & Farb 2005:

$$A(n)_i \underset{\substack{\text{for}\\ n \gg 0}}{\cong} \bigoplus_{j=1}^{t} \left(\mathrm{Specht}^{\overbrace{\qquad}^{n-|\lambda^{(j)}|}\lambda^{(j)}} \right)^{\oplus c_j}$$

for certain $\lambda^{(1)}, \ldots, \lambda^{(t)}$, $c_1, \ldots, c_t$

COROLLARY: $\{A(n)^!_i\}_{n=1,2,\ldots}$ for fixed i are also representation stable.

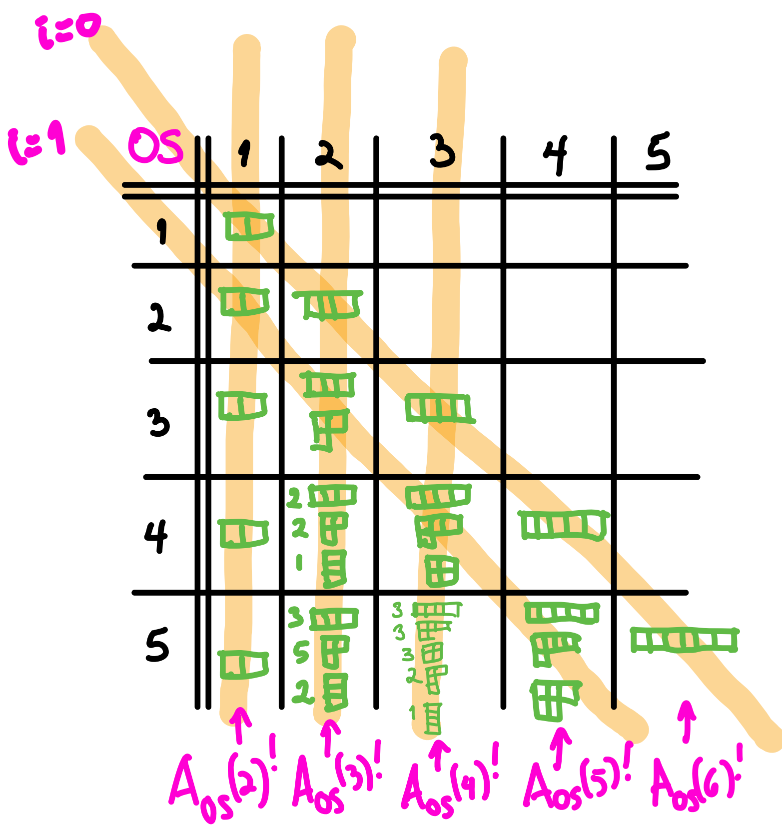
---

proof idea: induct, using virtual expression

$$A(n)^!_i = \sum_{j=1}^{i} A(n)_j \otimes A(n)^!_{i-j}$$

Most mysterious ...

CONJECTURE: $A_{OS}(n)^!_i$ is always an $\mathfrak{S}_n$-permutation representation (!)



Verified for $n = 2, 3, 4$
$i = 0, 1$

(The $\mathfrak{S}_n$-orbit stabilizers are **not** all parabolic subgroups $\mathfrak{S}_{\alpha_1} \times \ldots \times \mathfrak{S}_{\alpha_\ell}$)

Thanks to SLC

and

Happy Birthday,

Francesco!