Reflection group invariant theory and generatingfunctionology

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PART I

- Four combinatorial product formulas
- How they generalize to reflection groups via invariant theory

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PART II

- Some proofs

Four combinatorial product formulas

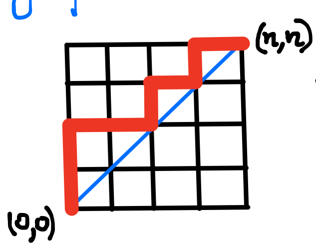
$$\sum_{\substack{\text{permutations}\\ w\in\mathfrak{S}_n}} q^{\#\text{inversions}(w)} = [1]_q[2]_q[3]_q\cdots[n]_q$$

where $[m]_q := 1+q+q^2+\cdots+q^{m-1}$

$$\sum_{\substack{\text{permutations}\\ w\in\mathfrak{S}_n}} q^{\#\text{cycles}(w)} = q(q+1)(q+2)\cdots(q+(n-1))$$

$$\sum_{\substack{\text{set partitions}\\ \pi \text{ of } \{1,2,\ldots,n\}}} \mu(\hat{0},\pi)\, q^{\#\text{blocks}(\pi)} = q(q-1)(q-2)\cdots(q-(n-1))$$

$$\#\left\{\text{Dyck paths } (0,0)\to(n,n)\right\} = \underbrace{\frac{1}{n+1}\binom{2n}{n}}_{\text{Catalan number}} = \frac{(n+2)(n+3)\cdots(2n)}{2\cdot 3\cdots n}$$

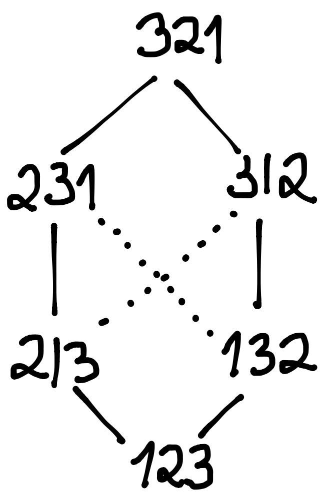


$$\sum_{\substack{\text{permutations} \\ w \in \mathfrak{S}_n}} q^{\#\text{inversions}(w)} = [1]_q [2]_q [3]_q \cdots [n]_q$$

where $[m]_q := 1+q+q^2+\ldots+q^{m-1}$

$$\text{inversions}(w) := \{(i,j) : 1 \le i < j \le n,\ w(i) > w(j)\}$$

$\boxed{n=3}$

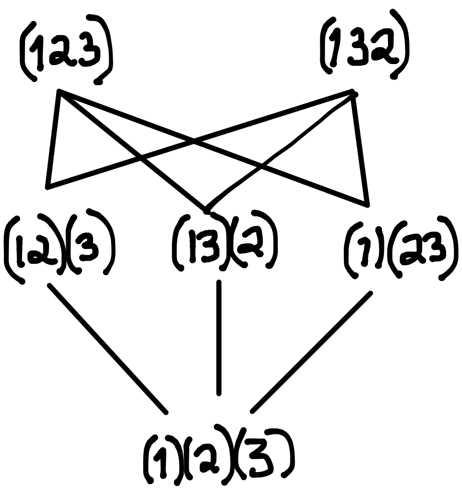


#inversions(w)	
3	q^3
2	$+2q^2$
1	$+2q^1$
0	$+q^0$

$$1+2q+2q^2+q^3$$
$$=(1+q)(1+q+q^2)$$
$$=[1]_q[2]_q[3]_q$$

$$\sum_{\substack{\text{permutations}\\ w\in \mathfrak{S}_n}} q^{\#\text{cycles}(w)} = q(q+1)(q+2)\cdots(q+(n-1))$$

$\boxed{n=3}$



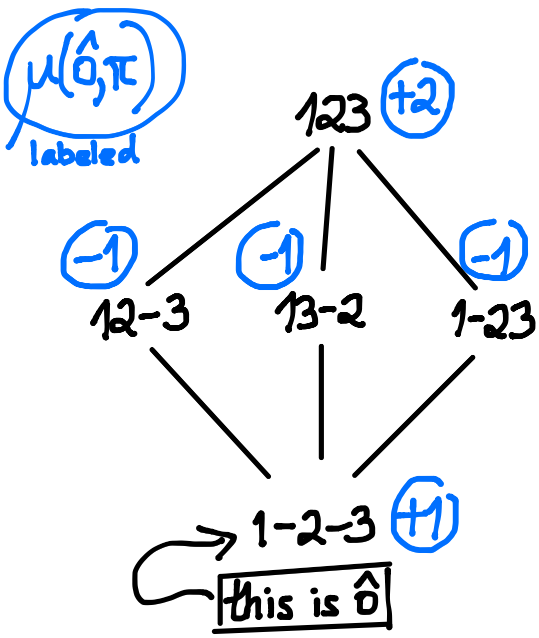
	#cycles(w)	
(123), (132)	1	$2q^1$
(12)(3), (13)(2), (1)(23)	2	$3q^2$
(1)(2)(3)	3	q^3

$$q^3+3q^2+2q$$
$$= q(q+1)(q+2)$$

$$\sum_{\substack{\text{set partitions}\\ \pi \text{ of } \{1,2,\dots,n\}}} \mu(\hat{0},\pi)\, q^{\#\text{blocks}(\pi)} = q(q-1)(q-2)\cdots(q-(n-1))$$

$$\mu(\hat{0},\pi) := \begin{cases} +1 & \text{if } \pi=\hat{0} \\ -\sum\limits_{\sigma<\pi} \mu(\hat{0},\sigma) & \end{cases}$$

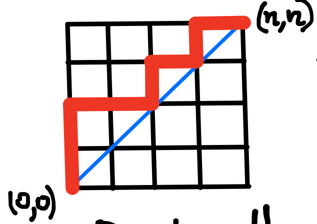
$\boxed{n=3}$



$\#\text{blocks}(\omega)$	
1	$+2q^1$
2	$-3q^2$
3	$+q^3$

$$q^3-3q^2+2q = q(q-1)(q-2)$$

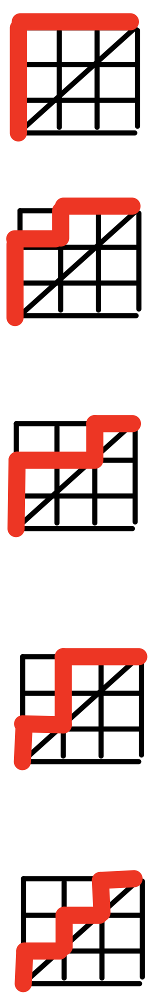
$$\#\left\{\text{Dyck paths } (0,0)\to(n,n)\right\} = \frac{1}{n+1}\binom{2n}{n} = \frac{(n+2)(n+3)\cdots(2n)}{2\cdot 3\cdots n}$$



Catalan number

Dyck paths $(0,0)\to(n,n)$ take unit steps north, east and stay weakly above $y=x$

$\boxed{n=3}$



$$5 = \frac{1}{3+1}\binom{6}{3}$$

$$= \frac{5\cdot 6}{2\cdot 3} = \frac{(3+2)(3+3)}{2\cdot 3}$$

How they generalize to reflection groups W

$$\sum_{\substack{\text{permutations}\\ w\in \mathfrak{S}_n}} q^{\#\text{inversions}(w)} = [1]_q[2]_q[3]_q\cdots[n]_q$$

where $[m]_q := 1+q+q^2+\dots+q^m$

$d_1,\dots,d_n$ = degrees

S = simple reflections

$$\rightsquigarrow \sum_{w\in W} q^{\ell_S(w)} \underset{W\text{ real}}{=} [d_1]_q[d_2]_q\cdots[d_n]_q$$

$$\sum_{\substack{\text{permutations}\\ w\in \mathfrak{S}_n}} q^{\#\text{cycles}(w)} = q(q+1)(q+2)\cdots(q+(n-1))$$

$e_1,\dots,e_n$ = exponents

T = all reflections

$$\rightsquigarrow \sum_{w\in W} q^{n-\ell_T(w)} \underset{W\text{ real}}{=} \sum_{w\in W} q^{\dim V^w} = (q+e_1)(q+e_2)\cdots(q+e_n)$$

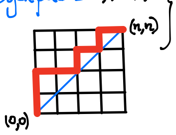
$$\sum_{\substack{\text{set partitions}\\ \pi \text{ of } \{1,2,\dots,n\}}} \mu(\hat{0},\pi)\, q^{\#\text{blocks}(\pi)} = q(q-1)(q-2)\cdots(q-(n-1))$$

$e_1^*,\dots,e_n^*$ = coexponents

$\mathcal{L}_W$ = hyperplane intersection poset

$$\rightsquigarrow \sum_{X\in\mathcal{L}_W} \mu(\hat{0},X)\, q^{\dim X} = \sum_{w\in W} \det(w)\, q^{\dim V^w} = (q-e_1^*)(q-e_2^*)\cdots(q-e_n^*)$$

$$\#\left\{\text{Dyck paths } (0,0)\to(n,n)\right\} = \frac{1}{n+1}\binom{2n}{n} = \frac{(n+2)(n+3)\cdots(2n)}{2\cdot 3\cdots n}$$



Catalan number

Q = root lattice

$h = \max\{d_1,\dots,d_n\}$

$$\rightsquigarrow \#\left\{\begin{matrix}W\text{-orbits}\\ \text{on}\\ Q/(h+1)Q\end{matrix}\right\} = \frac{1}{\#W}\sum_{w\in W}(h+1)^{\dim V^w} = \frac{(h+d_1)(h+d_2)\cdots(h+d_n)}{d_1 d_2\cdots d_n}$$

W crystallographic irreducible

$$=: \mathrm{Cat}(W)$$

DEFINITION:

A (complex / unitary) reflection group is a finite subgroup $W \subset GL(V)$, where $V = \mathbb{C}^n$, generated by its subset of (complex / unitary / pseudo-) reflections $t \in W$, that is, t having $V^t := \{v \in V : t(v) = v\}$
$$:= \ker(1_V - t)$$
a hyperplane (= a codim 1 linear subspace)

Alternatively, a reflection t diagonalizes to

$$\begin{bmatrix} \zeta & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots \; 1 \end{bmatrix}$$

where $\zeta = \det(t) \neq 1$ is a root-of-unity in $\mathbb{C}$

REMARK Why unitary?

Every finite subgroup $W \subset GL(V)$
$V = \mathbb{C}^n$

is conjugate to a subgroup of the unitary group U_n,

by averageing any positive definite Hermitian form $\langle \cdot , \cdot \rangle$ on V

to make it W-invariant:

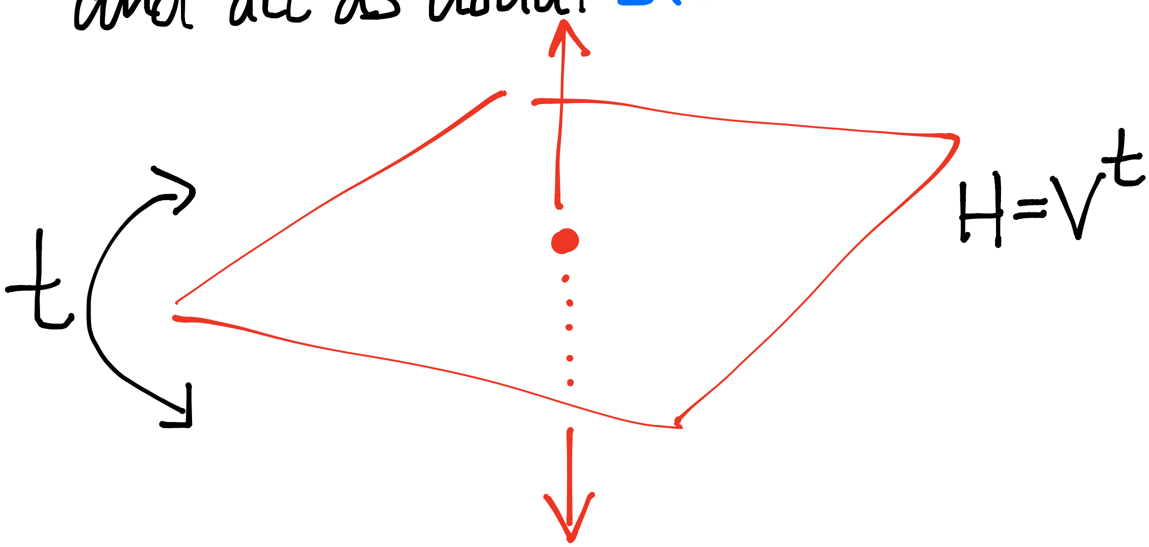
$$\langle u, v \rangle_W := \frac{1}{\# W} \sum_{w \in W} \langle w(u), w(v) \rangle$$

EXAMPLES

(1) When $W \subset GL(V)$ for $V = \mathbb{R}^n$,
W is a finite real reflection group
(and $W \subset O_n$ = orthogonal group WLOG)
so its reflections $t \in W$ diagonalize to

$$\begin{bmatrix} -1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}$$

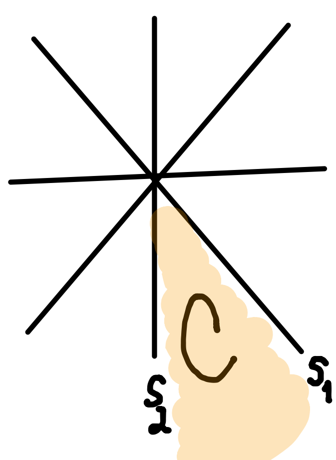
and act as usual Euclidean reflections

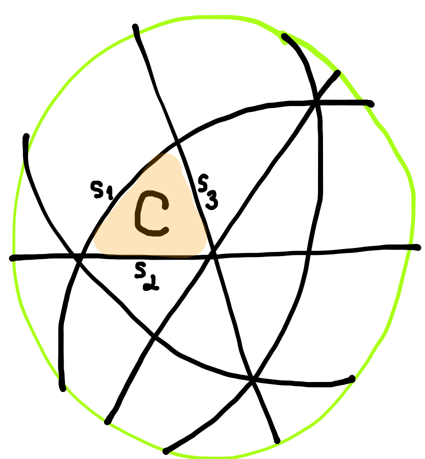


These real reflection groups W are exactly the finite groups having Coxeter presentations (W, S).

The simple reflections $S = \{s_1, s_2, \ldots\}$ can be chosen as reflections through the walls of a chamber C

a connected component of $V \setminus \bigcup_{\substack{\text{reflections} \\ t \text{ in } W}} V^t$



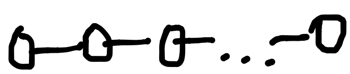


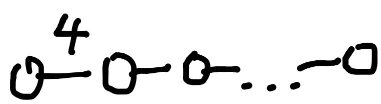
② Shephard & Todd's infinite family of monomial groups

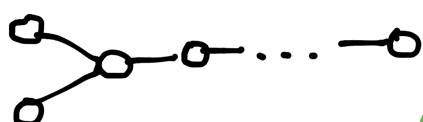
$$G(de,e,n) := \left\{ \begin{array}{c} n\times n \text{ monomial matrices} \\ \begin{bmatrix} 0&0&0&0&*\\ 0&*&0&0&0\\ *&0&0&0&0\\ 0&0&*&0&0\\ 0&0&0&*&0 \end{bmatrix} \\ \text{with nonzero entries} \\ * \text{ in } \sqrt[de]{1} \\ \text{and their product in } \sqrt[d]{1} \end{array} \right\}$$

for $d,e,n \geq 1$

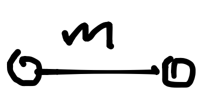
captures all of the real infinite families

$\mathfrak{S}_n = A_{n-1} =$  $= G(1,1,n)$

$B_n = C_n =$  $= G(2,1,n)$

$D_n =$  $= G(2,2,n)$

EXERCISE!

$I_2(m) =$  $= G(m,m,2)$

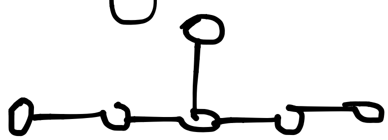
(3) Shephard & Todd (1954) showed that the irreducible complex reflection groups are either

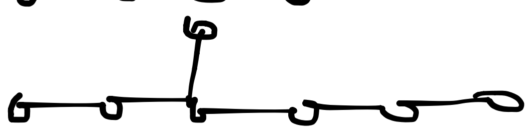
- in the $G(de,e,n)$ family

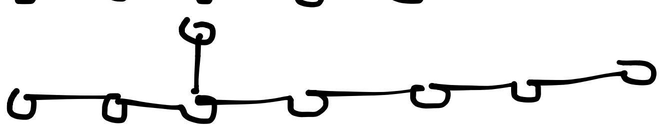
or

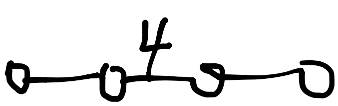
- among 34 exceptional groups

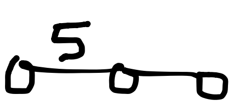
the latter containing the real exceptional groups

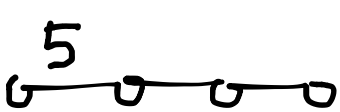
E_6 

E_7 

E_8 

F_4 

H_3 

H_4 

REMARK

A complex reflection group W acting on $V = \mathbb{C}^n$ has no natural candidate for a set of simple reflections $S = \{s_1, s_2, \ldots, s_n\}$, just its set of all reflections $T \subset W$.

In fact, sometimes W needs $n+1$ reflections to generate it, e.g. $W = G(de, e, n)$ with $d, e, n \geq 2$ (EXERCISE)

$$G(4,2,2) = \left\langle \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}}_{\substack{\text{these two only} \\ \text{generate } G(4,4,2)}}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right\rangle$$

Where to find degrees, exponents, coexponents?

Invariant theory!

$GL(V)$ acts on V

on V^* with $\mathbb{C}$-basis $x_1, x_2, \ldots, x_n$

on $S := \mathrm{Sym}(V^*) \cong \mathbb{C}[x_1, x_2, \ldots, x_n]$

via linear substitutions of variables

$$f = f(\underline{x}) = f(x_1, \ldots, x_n)$$

$$\stackrel{g}{\longmapsto} g(f) = f \circ g^{-1} = f(g^{-1}\underline{x})$$

Same for any subgroup W of $GL(V)$

DEFINITION

$$S^W := \{ f \in S : g(f) = f \ \ \forall w \in W \}$$

$=$ the W-invariant subalgebra of S

The numerology comes from considering

$$S = \mathrm{Sym}(V^*) = \mathbb{C}[x_1, \dots, x_n]$$

as a standard graded ring with $\deg(x_i) = 1$

$$S = \bigoplus_{d \geq 0} S_d = \bigoplus_{d \geq 0} \mathrm{Sym}^d(V^*)$$

$$S_i S_j \subset S_{i+j}$$

Since $W \subset GL(V)$ preserves degree,

$$S^W = \bigoplus_{d \geq 0} S^W_d$$

where $S^W_d := S^W \cap S_d$

is also a graded ring.

The first crucial result comes in 2 parts...

THEOREM (Shepard & Todd 1954, Chevalley 1955)

(a) For a finite subgroup W of $GL(V)$, $V = \mathbb{C}^n$,
W is a reflection group
$\iff S^W = \mathbb{C}[f_1, \ldots, f_n]$ is polynomial,
that is generated by
algebraically independent $f_1, \ldots, f_n$

(b) the graded quotient ring $S/(S^W_+)$
where $S^W_+ := \bigoplus_{d \geq 1} S^W_d$, called the
coinvariant algebra, carries the
regular representation $\mathbb{C}W$
(but a graded version of it!)

Both parts (a) and (b) of the theorem lead to numerology.

Since S^W is graded, when writing $S^W = \mathbb{C}[f_1, \dots, f_n]$, one can pick each of $f_1, f_2, \dots, f_n$ homogeneous.

The choice of f_i's is not unique.

But their multiset of degrees $d_1 \leq d_2 \leq \dots \leq d_n$ (after re-indexing) are unique.

This follows, for example, from a Hilbert series calculation

DEFINITION
For a graded $\mathbb{C}$-vector space $U = \bigoplus_{d \geq 0} U_d$

$$\mathrm{Hilb}(U, q) := \sum_{d \geq 0} \dim_{\mathbb{C}} U_d \, q^d \quad \in \mathbb{Z}[[q]]$$

Since $S^W = \mathbb{C}[f_1, f_2, \ldots, f_n]$,
S^W has a $\mathbb{C}$-basis of monomials $\{ f_1^{a_1} f_2^{a_2} \cdots f_n^{a_n} \}$.

$$\mathrm{Hilb}(S^W, q) = (1 + q^{d_1} + q^{2d_1} + \ldots)(1 + q^{d_2} + q^{2d_2} + \ldots) \cdots (1 + q^{d_n} + q^{2d_n} + \ldots)$$

$$= \frac{1}{1-q^{d_1}} \frac{1}{1-q^{d_2}} \cdots \frac{1}{1-q^{d_n}}$$

$$= \frac{1}{\prod_{i=1} (1-q^{d_i})}$$

EXERCISE

Show that the expression

$$\frac{1}{\mathrm{Hilb}(S^W, q)} = \prod_{i=1}^{n} (1-q^{d_i}),$$

together with unique factorization in $\mathbb{Z}[q]$ implies uniqueness of the degrees $d_1 \le \ldots \le d_n$.

(HINT: $1-q^{d_i}$ is not irreducible in $\mathbb{Z}[q]$, but one knows exactly its irreducible cyclotomic factors.)

EXAMPLE

$W = \mathfrak{S}_n$ permutes variables in

$$S = \mathbb{C}[x_1, x_2, \ldots, x_n]$$

$$S^W = \mathbb{C}[f_1, f_2, \ldots, f_n]$$

degrees $\underset{d_1}{\overset{\shortparallel}{1}} \le \underset{d_2}{\overset{\shortparallel}{2}} \le \ldots \le \underset{d_n}{\overset{\shortparallel}{n}}$

$$\Rightarrow \quad \mathrm{Hilb}(S^W, q) = \frac{1}{(1-q^1)(1-q^2)\cdots(1-q^n)}$$

Some choices for $f_1, f_2, \ldots, f_n$:

$f_1 = x_1 + x_2 + \ldots + x_n$
$f_2 = x_1x_2 + x_1x_3 + \ldots + x_{n-1}x_n$
$\vdots$
$f_n = x_1x_2 \cdots x_n$

elementary symmetric polynomials

OR

$f_1 = x_1 + x_2 + \ldots + x_n$
$f_2 = x_1^2 + x_2^2 + \ldots + x_n^2$
$\vdots$
$f_n = x_1^n + x_2^n + \ldots + x_n^n$

power sum symmetric polynomials

Exponents & coexponents will come from part (b) of Shephard-Todd-Chevalley...

Quick representation theory review

- Finite groups W have finitely many irreducible complex representations $\{U_i\}$

- Any W-representation $\mathcal{U}$ decomposes as $\mathcal{U} = \bigoplus_i U_i^{\oplus m_i}$

- The multiplicity $m_i = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}W}(U_i, \mathcal{U})$

- Alternatively, $m_i = \langle \chi_{U_i}, \chi_{\mathcal{U}} \rangle_W$
$$= \frac{1}{\# W} \sum_{w \in W} \chi_{U_i}(w^{-1}) \chi_{\mathcal{U}}(w)$$
where $\chi_{\mathcal{U}}(w) = \mathrm{Trace}(\mathcal{U} \xrightarrow{w} \mathcal{U})$

- $\langle \mathcal{U}, \mathbb{C}W \rangle_W = \dim_{\mathbb{C}} \mathcal{U}$

Since Shephard-Todd-Chevalley (b) says

$$S/(S_+^W) \cong \mathbb{C}W$$

every W-representation U has

- **U-exponents** $e_1(U) \leq e_2(U) \leq \cdots \leq e_{\dim U}(U)$
- **U-fake degree polynomial** $f^U(q) := \sum_{i=1}^{\dim U} q^{e_i(U)}$

defined by

$$f^U(q) = \sum_{d \geq 0} \langle \chi_U, \chi_{(S/(S_+^W))_d} \rangle_W \, q^d$$

DEFINITION

- **exponents** $e_1 \leq \cdots \leq e_n$ of W are the V-exponents
- **coexponents** $e_1^* \leq \cdots \leq e_n^*$ of W are the V^*-exponents

"EXAMPLES"

- We will see later that for any complex reflection group W the degrees and exponents are related by
$$(e_1,\dots,e_n)=(d_1-1,\dots,d_n-1)$$

- For real reflection groups W, since $V\cong V^*$, one further has
$$(e_1^*,\dots,e_n^*)=(e_1,\dots,e_n)$$
$$\left(=(d_1-1,\dots,d_n-1)\right)$$

- Thus $W=\mathfrak{S}_n$ acting on $V=\mathbb{R}^n$ has $(d_1,\dots,d_n)=(1,2,\dots,n)$
$$(e_1,\dots,e_n)=(0,1,\dots,n-1)=(e_1^*,\dots,e_n^*)$$

How do the 4 product formulas look for $W=\mathfrak{S}_n$?

$d_1,\dots,d_n=$ degrees

$S=$ simple reflections

$$\sum_{\substack{\text{permutations}\\ w\in\mathfrak{S}_n}} q^{\#\text{inversions}(w)} = [1]_q[2]_q[3]_q\cdots[n]_q$$

where $[m]_q := 1+q+q^2+\dots+q^{m}$

$$\rightsquigarrow \quad \sum_{w\in W} q^{\ell_S(w)} \underset{W\text{ real}}{=} [d_1]_q[d_2]_q\cdots[d_n]_q$$

$e_1,\dots,e_n=$ exponents

$T=$ all reflections

$$\sum_{\substack{\text{permutations}\\ w\in\mathfrak{S}_n}} q^{\#\text{cycles}(w)} = q(q+1)(q+2)\cdots(q+(n-1))$$

$$\rightsquigarrow \quad \sum_{w\in W} q^{n-\ell_T(w)} \underset{W\text{ real}}{=} \sum_{w\in W} q^{\dim V^w} = (q+e_1)(q+e_2)\cdots(q+e_n)$$

$e_1^*,\dots,e_n^*=$ coexponents

$\mathcal{L}_W=$ hyperplane intersection poset

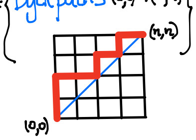
$$\sum_{\substack{\text{set partitions}\\ \pi \text{ of } \{1,2,\dots,n\}}} \mu(\hat{0},\pi)\, q^{\#\text{blocks}(\pi)} = q(q-1)(q-2)\cdots(q-(n-1))$$

$$\rightsquigarrow \quad \sum_{X\in\mathcal{L}_W} \mu(\hat{0},X)\, q^{\dim X} = \sum_{w\in W} \det(w)\, q^{\dim V^w} = (q-e_1^*)(q-e_2^*)\cdots(q-e_n^*)$$

$Q=$ root lattice

$h=\max\{d_1,\dots,d_n\}$

$$\#\left\{\begin{array}{c}\text{Dyck paths } (0,0)\to(n,n)\end{array}\right\} = \frac{1}{n+1}\binom{2n}{n} = \frac{(n+2)(n+3)\cdots(2n)}{2\cdot 3\cdots n}$$



Catalan number

$$\rightsquigarrow \quad \#\left\{\begin{array}{c}W\text{-orbits}\\ \text{on}\\ Q/(h+1)Q\end{array}\right\} \underset{\substack{W \text{ crystallographic}\\ \text{irreducible}}}{=} \frac{1}{\#W}\sum_{w\in W}(h+1)^{\dim V^w} = \frac{(h+d_1)(h+d_2)\cdots(h+d_n)}{d_1 d_2\cdots d_n} =: \mathrm{Cat}(W)$$

$$\sum_{\substack{\text{permutations} \\ w \in \mathfrak{S}_n}} q^{\#\text{inversions}(w)} = [1]_q [2]_q [3]_q \cdots [n]_q$$

where $[m]_q := 1+q+q^2+\ldots+q^m$

$d_1, \ldots, d_n =$ degrees

$S =$ simple reflections

$$\rightsquigarrow \sum_{w \in W} q^{\ell_S(w)} \underset{W \text{ real}}{=} [d_1]_q [d_2]_q \cdots [d_n]_q$$

This makes sense since

$$(W, S) = \Big(\mathfrak{S}_n, \{\underset{\substack{\| \\ (12)}}{s_1}, \underset{\substack{\| \\ (23)}}{s_2}, \ldots, \underset{\substack{\| \\ (n-1\, n)}}{s_{n-1}}\}\Big)$$

has $\ell_S(w) = \#\text{inversions}(w)$

e.g. $w = 4213$ has $\#\text{inversions}(w) = 4$

$= \ell_S(w)$

$$\left.\begin{array}{ll} 4213 & \\ 2413 & s_1 \\ 2143 & s_2 \\ 2134 & s_3 \\ (e=)\,1234 & s_1 \end{array}\right\} \ell_S(w) = 4$$

$$\sum_{\substack{\text{permutations} \\ w\in\mathfrak{S}_n}} q^{\#\text{cycles}(w)} = q(q+1)(q+2)\cdots(q+(n-1))$$

$e_1,\ldots,e_n$ = exponents
T = all reflections

$$\rightsquigarrow \sum_{w\in W} q^{n-\ell_T(w)} \underset{W \text{ real}}{=} \sum_{w\in W} q^{\dim V^w} = (q+e_1)(q+e_2)\cdots(q+e_n)$$

This makes sense since

$w\in\mathfrak{S}_n$ has $\dim V^w = \#\text{cycles}(w)$

e.g. $w = (1\,4\,7)(2\,8)(5\,9)(3)(6)$

acting on $V=\mathbb{R}^9$ has

$$V^w = \{x_1=x_4=x_7,\ x_2=x_8,\ x_5=x_9,\ x_3, x_6 \text{ arbitrary}\}$$

$$\cong \mathbb{R}^5 = \mathbb{R}^{\#\text{cycles}(w)}$$

$$\sum_{\substack{\text{set partitions}\\ \pi \text{ of } \{1,2,\ldots,n\}}} \mu(\hat{0},\pi)\, q^{\#\text{blocks}(\pi)} = q(q-1)(q-2)\cdots(q-(n-1))$$

$e_1^*, \ldots, e_n^*$ = coexponents

$\mathcal{L}_W$ = hyperplane intersection poset

$$\rightsquigarrow \sum_{X \in \mathcal{L}_W} \mu(\hat{0}, X)\, q^{\dim X} = \sum_{w \in W} \det(w)\, q^{\dim V^w} = (q-e_1^*)(q-e_2^*)\cdots(q-e_n^*)$$

This makes sense since for $\mathfrak{S}_n$,

$\mathcal{L}_W = \{$intersection subspaces of the hyperplanes $x_i = x_j\}$

$\updownarrow$

$\{$set partitions of $\{1,2,\ldots,n\}\}$

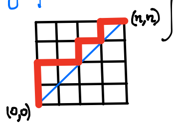
e.g. $X = \{x_1 = x_4\} \cap \{x_4 = x_7\} \cap \{x_2 = x_8\} \cap \{x_5 = x_9\}$

$= \{x_1 = x_4 = x_7,\ x_2 = x_8,\ x_5 = x_9,\ x_3, x_6 \text{ arbitrary}\}$

$\updownarrow$

$147 - 28 - 59 - 3 - 6$

$\#\left\{\text{Dyck paths } (0,0)\to(n,n)\right\} = \frac{1}{n+1}\binom{2n}{n} = \frac{(n+2)(n+3)\cdots(2n)}{2\cdot 3\cdots n}$



Catalan number

Q = root lattice
$h = \max\{d_1, \dots, d_n\}$

$$\rightsquigarrow \#\left\{\begin{matrix} W\text{-orbits} \\ \text{on} \\ Q/(h+1)Q \end{matrix}\right\} = \frac{1}{\#W}\sum_{w\in W}(h+1)^{\dim V^{w}} = \frac{(h+d_1)(h+d_2)\cdots(h+d_n)}{d_1\, d_2 \cdots d_n}$$

W crystallographic irreducible

$$=: Cat(W)$$

... is perhaps the least transparent!

Note $\mathfrak{S}_n$ acts reducibly on $\mathbb{R}^n$
but irreducibly on $V := \{\underline{x} \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 0\}$

Hence here $S = Sym(V^*)$

$$\cong \mathbb{C}[x_1, \dots, x_n]/(x_1 + \dots + x_n)$$

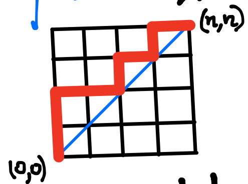
and $S^{\mathfrak{S}_n} = \mathbb{C}[e_2(\underline{x}), e_3(\underline{x}), \dots, e_n(\underline{x})] \hookrightarrow S$

with degrees $2, 3, \dots, n =: h$

$$\Rightarrow Cat(W) = \frac{(n+2)(n+3)\cdots(2n)}{2\cdot 3 \cdots n} = \frac{1}{n+1}\binom{2n}{n}$$

But how should one relate

$$\left\{ \text{Dyck paths } (0,0)\to(n,n) \right\} \text{ and } \left\{ \begin{matrix}W\text{-orbits}\\ \text{on}\\ Q/(h+1)Q\end{matrix} \right\}$$



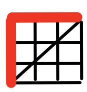
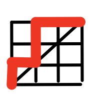
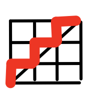
where the root lattice Q here is

$$Q = \{ \underline{x} \in \mathbb{Z}^n : \sum_{i=1}^{n} x_i = 0 \} \ ?$$

Start with $\left\{ \begin{matrix}\text{Dyck paths}\\ (0,0)\to(n,n)\end{matrix} \right\} \longleftrightarrow \left\{ \underline{x} \in \mathbb{Z}^n : \begin{matrix} x_1 \le \cdots \le x_n \\ 0 \le x_i \le i-1 \end{matrix} \right\}$

path $\longmapsto$ x coordinates $x_1 \le \ldots \le x_n$ of its north steps

$\boxed{n=3}$

		(x_1, x_2, x_3)
	$\longleftrightarrow$	0 0 0
	$\longleftrightarrow$	0 0 1
	$\longleftrightarrow$	0 0 2
	$\longleftrightarrow$	0 1 1
	$\longleftrightarrow$	0 1 2

$$\left\{\begin{array}{c}\text{Dyck paths}\\ (0,0)\to(n,n)\end{array}\right\} \longleftrightarrow \left\{\underline{x}\in\mathbb{Z}^n : \begin{array}{c} x_1\le\dots\le x_n\\ 0\le x_i\le i-1\end{array}\right\}$$

$$\|$$

$$\left\{\underline{x}\in\{0,1,2,\dots,n-1\}^n : \begin{array}{c} x_1\le\dots\le x_n \text{ and}\\ \#\{x_i\le j-1\}\ge j\ \forall j\end{array}\right\}$$

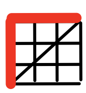
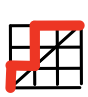
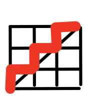
$$\|$$

$$\mathfrak{S}_n\text{-orbits on } \underbrace{\left\{\underline{x}\in\{0,1,2,\dots,n-1\}^n : \#\{x_i\le j-1\}\ge j\ \forall j\right\}}$$

called parking functions of length n, equipped with $\mathfrak{S}_n$-action on positions

Each row here is an $\mathfrak{S}_3$-orbit of parking functions:

$\boxed{n=3}$

	$\longleftrightarrow$	000	
	$\longleftrightarrow$	001	010 100
	$\longleftrightarrow$	002	020 200
	$\longleftrightarrow$	011	101 110
	$\longleftrightarrow$	012	021 102 120 201 210

Haiman observed that the parking functions (1993) of length n give coset reps for

$$\mathbb{Z}^n / \left((n+1)\mathbb{Z}^n + \mathbb{Z} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \right), \text{ of size } (n+1)^{n-1}$$

$\Rightarrow \exists$ $\mathfrak{S}_n$-equivariant bijection

$$\mathbb{Z}^n / \left((n+1)\mathbb{Z}^n + \mathbb{Z} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \right) \leftrightarrow \left\{ \begin{matrix} \text{parking functions} \\ \text{of length } n \end{matrix} \right\}$$

$\Longrightarrow \exists$ bijection

$$\left\{ \begin{matrix} \mathfrak{S}_n\text{-orbits on} \\ \mathbb{Z}^n / \left((n+1)\mathbb{Z}^n + \mathbb{Z} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \right) \end{matrix} \right\} \leftrightarrow \left\{ \begin{matrix} \mathfrak{S}_n\text{-orbits on} \\ \text{parking} \\ \text{functions of} \\ \text{length } n \end{matrix} \right\},$$

$\Big\updownarrow$ Pontryagin duality $\qquad\qquad\qquad\qquad \Big\updownarrow$

$$\left\{ \begin{matrix} \mathfrak{S}_n\text{-orbits on} \\ Q/(n+1)Q \end{matrix} \right\} \qquad\qquad \left\{ \begin{matrix} \text{Dyck paths} \\ (0,0) \to (n,n) \end{matrix} \right\}$$

where $Q = \{ \underline{x} \in \mathbb{Z}^n : \sum_{i=1}^{n} x_i = 0 \} = \mathfrak{S}_n$-root lattice

EXAMPLE (with W genuinely complex)

$$W = G(d,1,1) = \langle \underset{\substack{\| \\ e^{\frac{2\pi i}{d}}}}{\zeta} \rangle \subset GL_1(\mathbb{C})$$

acts on $S = \mathrm{Sym}(V^*) = \mathbb{C}[x]$

sending $x \xmapsto{\zeta} \zeta^{-1} x$

so $S^W = \mathbb{C}[\underset{\substack{\| \\ f_1}}{x^d}]$, and degree $d_1 = d$

$$S/(S^W_+) = \mathbb{C}[x]/(x^d)$$

$= \mathbb{C}$-span of $\{1, x, x^2, \ldots, x^{d-1}\}$

carries V^*	carries V
$\Rightarrow$ coexponent $e_1^* = 1$	$\Rightarrow$ exponent $e_1 = d-1$

How do the 4 product formulas look for $G(d,1,1)$?

$d_1,\dots,d_n$ = degrees

S = simple reflections

$$\sum_{\substack{\text{permutations}\\ w\in\mathfrak{S}_n}} q^{\#\text{inversions}(w)} = [1]_q[2]_q[3]_q\cdots[n]_q$$

where $[m]_q := 1+q+q^2+\dots+q^m$

$$\rightsquigarrow \quad \sum_{w\in W} q^{\ell_S(w)} \underset{W \text{ real}}{=} [d_1]_q[d_2]_q\cdots[d_n]_q$$

$e_1,\dots,e_n$ = exponents

T = all reflections

$$\sum_{\substack{\text{permutations}\\ w\in\mathfrak{S}_n}} q^{\#\text{cycles}(w)} = q(q+1)(q+2)\cdots(q+(n-1))$$

$$\rightsquigarrow \quad \sum_{w\in W} q^{n-\ell_T(w)} \underset{W \text{ real}}{=} \sum_{w\in W} q^{\dim V^w} = (q+e_1)(q+e_2)\cdots(q+e_n)$$

$e_1^*,\dots,e_n^*$ = coexponents

$\mathcal{L}_W$ = hyperplane intersection poset

$$\sum_{\substack{\text{set partitions}\\ \pi \text{ of } \{1,2,\dots,n\}}} \mu(\hat{0},\pi)\, q^{\#\text{blocks}(\pi)} = q(q-1)(q-2)\cdots(q-(n-1))$$

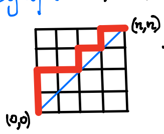
$$\rightsquigarrow \quad \sum_{X\in\mathcal{L}_W} \mu(\hat{0},X)\, q^{\dim X} = \sum_{w\in W} \det(w)\, q^{\dim V^w} = (q-e_1^*)(q-e_2^*)\cdots(q-e_n^*)$$

Q = root lattice

$h = \max\{d_1,\dots,d_n\}$

$$\#\left\{\text{Dyck paths } (0,0)\to(n,n)\right\} = \frac{1}{n+1}\binom{2n}{n} = \frac{(n+2)(n+3)\cdots(2n)}{2\cdot 3\cdots n}$$

Catalan number



$$\rightsquigarrow \quad \#\left\{\begin{matrix}W\text{-orbits}\\ \text{on}\\ Q/(h+1)Q\end{matrix}\right\} = \frac{1}{\#W}\sum_{w\in W}(h+1)^{\dim V^w} = \frac{(h+d_1)(h+d_2)\cdots(h+d_n)}{d_1 d_2\cdots d_n}$$

W crystallographic irreducible

$$=: \mathrm{Cat}(W)$$

$$\sum_{\substack{\text{permutations} \\ w \in \mathfrak{S}_n}} q^{\#\text{inversions}(w)} = [1]_q [2]_q [3]_q \cdots [n]_q$$

where $[m]_q := 1+q+q^2+\ldots+q^m$

$d_1, \ldots, d_n =$ degrees

$S =$ simple reflections

$$\rightsquigarrow \quad \sum_{w \in W} q^{\ell_S(w)} \underset{W \text{ real}}{=} [d_1]_q [d_2]_q \cdots [d_n]_q$$

We saw $W = G(d,1,1)$ has $S^W = \mathbb{C}[x^d]$

so its degree $d_1 = d$, and the right side

therefore equals $[d]_q = 1+q+q^2+\ldots+q^{d-1}$.

But $W = G(d,1,1)$ is a real reflection group

only when $d=2$,

i.e. $W = G(2,1,1) = \langle \underset{\overset{\shortparallel}{s}}{-1} \rangle = \{e, s\}$,

and then $\sum_{w \in W} q^{\ell(w)} = q^{\ell(e)} + q^{\ell(s)} = 1+q = [2]_q \; \left(\overset{\checkmark}{=} [d]_q \right)$

$$\sum_{\substack{\text{permutations}\\ w\in\mathfrak{S}_n}} q^{\#\text{cycles}(w)} = q(q+1)(q+2)\cdots(q+(n-1))$$

$e_1,\dots,e_n$ = exponents
T = all reflections

$$\rightsquigarrow \sum_{w\in W} q^{n-\ell_T(w)} \underset{W\text{ real}}{=} \sum_{w\in W} q^{\dim V^w} = (q+e_1)(q+e_2)\cdots(q+e_n)$$

We saw $W=G(d,1,1)=\langle\zeta\rangle$ has

$$S/(S_+^W) = \mathbb{C}[x]/(x^d)$$

$$=\mathbb{C}\text{-span of }\{1,x,x^2,\dots,x^{d-1}\}$$

carries V $\Rightarrow$ exponent $e_1=d-1$

Meanwhile, $W=\{1,\underbrace{\zeta,\zeta^2,\dots,\zeta^{d-1}}_{T=\text{all reflections}}\}$

$\dim V^w$	$1, 0, 0, \dots, 0$

$$\Rightarrow \sum_{w\in W} q^{\dim V^w} = q^1+q^0+q^0+\dots+q^0 = q+d-1$$

$$(= q+e_1)$$

$e_1^*, \ldots, e_n^*$ = coexponents

$\mathcal{L}_W$ = hyperplane intersection poset

$$\sum_{\substack{\text{set partitions} \\ \pi \text{ of } \{1,2,\ldots,n\}}} \mu(\hat{0},\pi)\, q^{\#\text{blocks}(\pi)} = q(q-1)(q-2)\cdots(q-(n-1))$$

$$\rightsquigarrow \sum_{X \in \mathcal{L}_W} \mu(\hat{0},X)\, q^{\dim X} = \sum_{w \in W} \det(w)\, q^{\dim V^w} = (q-e_1^*)(q-e_2^*)\cdots(q-e_n^*)$$

We saw $W = G(d,1,1) = \langle \zeta \rangle$ has

$$S/(S_+^W) = \mathbb{C}[x]/(x^d)$$

$$= \mathbb{C}\text{-span of } \{1, x, x^2, \ldots, x^{d-1}\}$$

carries $V^* \Rightarrow$ coexponent $e_1^* = 1$

$$\mathcal{L}_W = \boxed{\begin{array}{c} (-1)\ \{0\} \\ | \\ (+1)\ V = \mathbb{C}^1 \end{array}}$$

$\mu(\hat{0},\pi)$ labeled

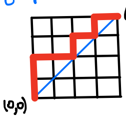
$$\sum_{X \in \mathcal{L}_W} \mu(\hat{0},X)\, q^{\dim X}$$

$$= +q^1 - q^0 = q-1$$

$$\left(\overset{\checkmark}{=} q - e_1^* \right)$$

$$\#\left\{\text{Dyck paths } (0,0)\to(n,n)\right\} = \frac{1}{n+1}\binom{2n}{n} = \frac{(n+2)(n+3)\cdots(2n)}{2\cdot 3\cdots n}$$

Catalan number



(n,n)

$(0,0)$

Q = root lattice
$h = \max\{d_1, \dots, d_n\}$

$$\rightsquigarrow \#\left\{\begin{matrix}W\text{-orbits}\\ \text{on}\\ Q/(h+1)Q\end{matrix}\right\} = \frac{1}{\#W}\sum_{w\in W}(h+1)^{\dim V^w} = \frac{(h+d_1)(h+d_2)\cdots(h+d_n)}{d_1\, d_2 \cdots d_n}$$

W crystallographic irreducible

$$=: \mathrm{Cat}(W)$$

$W = G(d,1,1)$ crystallographic $\Rightarrow d=2$

i.e. $W = G(2,1,1) = \langle \underset{\underset{s}{\shortparallel}}{-1} \rangle = \{e, s\}$,

$Q = \mathbb{Z}$, $h = 2$

$$Q/(h+1)Q = \mathbb{Z}/3\mathbb{Z} = \{-1, 0, +1\}$$

s s

$$\underset{\underset{2}{\shortparallel}}{\#\left\{\begin{matrix}W\text{-orbits}\\ \text{on}\\ Q/(h+1)Q\end{matrix}\right\}} = \underset{\underset{\frac{2+2}{2}}{\shortparallel}}{\frac{h+d_1}{d_1}}$$

END

Part 1