Reflection group invariant theory and generatingfunctionology

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PART I

- Four combinatorial product formulas
- How they generalize to reflection groups via invariant theory

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PART II

- Some proofs

Four combinatorial product formulas

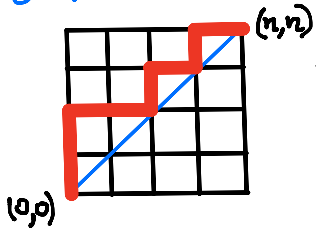
$$\sum_{\substack{\text{permutations} \\ w \in \mathfrak{S}_n}} q^{\#\text{inversions}(w)} = [1]_q [2]_q [3]_q \cdots [n]_q$$

where $[m]_q := 1+q+q^2+\ldots+q^{m-1}$

$$\sum_{\substack{\text{permutations} \\ w \in \mathfrak{S}_n}} q^{\#\text{cycles}(w)} = q(q+1)(q+2)\cdots(q+(n-1))$$

$$\sum_{\substack{\text{set partitions} \\ \pi \text{ of } \{1,2,\ldots,n\}}} \mu(\hat{0},\pi)\, q^{\#\text{blocks}(\pi)} = q(q-1)(q-2)\cdots(q-(n-1))$$

$$\#\left\{\text{Dyck paths } (0,0)\to(n,n)\right\} = \frac{1}{n+1}\binom{2n}{n} = \frac{(n+2)(n+3)\cdots(2n)}{2\cdot 3\cdots n}$$



Catalan number

How they generalize to reflection groups W

$d_1,\ldots,d_n =$ degrees

$S =$ simple reflections

$$\sum_{\substack{\text{permutations}\\ w\in\mathfrak{S}_n}} q^{\#\text{inversions}(w)} = [1]_q[2]_q[3]_q\cdots[n]_q$$

where $[m]_q := 1+q+q^2+\cdots+q^{m-1}$

$$\rightsquigarrow \quad \sum_{w\in W} q^{\ell_S(w)} \underset{W\text{ real}}{=} [d_1]_q[d_2]_q\cdots[d_n]_q$$

$e_1,\ldots,e_n =$ exponents

$T =$ all reflections

$$\sum_{\substack{\text{permutations}\\ w\in\mathfrak{S}_n}} q^{\#\text{cycles}(w)} = q(q+1)(q+2)\cdots(q+(n-1))$$

$$\rightsquigarrow \quad \sum_{w\in W} q^{n-\ell_T(w)} \underset{W\text{ real}}{=} \sum_{w\in W} q^{\dim V^w} = (q+e_1)(q+e_2)\cdots(q+e_n)$$

$e_1^*,\ldots,e_n^* =$ coexponents

$\mathcal{L}_W =$ hyperplane intersection poset

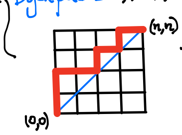
$$\sum_{\substack{\text{set partitions}\\ \pi\text{ of }\{1,2,\ldots,n\}}} \mu(\hat{0},\pi)\, q^{\#\text{blocks}(\pi)} = q(q-1)(q-2)\cdots(q-(n-1))$$

$$\rightsquigarrow \quad \sum_{X\in\mathcal{L}_W} \mu(\hat{0},X)\, q^{\dim X} = \sum_{w\in W} \det(w)\, q^{\dim V^w} = (q-e_1^*)(q-e_2^*)\cdots(q-e_n^*)$$

$Q =$ root lattice

$h = \max\{d_1,\ldots,d_n\}$

$$\#\left\{\text{Dyck paths } (0,0)\to(n,n)\right\} = \underbrace{\frac{1}{n+1}\binom{2n}{n}}_{\text{Catalan number}} = \frac{(n+2)(n+3)\cdots(2n)}{2\cdot 3\cdots n}$$



$$\rightsquigarrow \quad \#\left\{\begin{matrix}W\text{-orbits}\\ \text{on}\\ Q/(h+1)Q\end{matrix}\right\} \underset{\substack{W\text{ crystallographic}\\ \text{irreducible}}}{=} \frac{1}{\#W}\sum_{w\in W}(h+1)^{\dim V^w} = \frac{(h+d_1)(h+d_2)\cdots(h+d_n)}{d_1 d_2\cdots d_n} =: \mathrm{Cat}(W)$$

PART II Some proofs

THEOREM (Chevalley 1950, Solomon 1966)

$$\sum_{w \in W} q^{\ell_S(w)} \underset{W \text{ real}}{=} [d_1]_q [d_2]_q \cdots [d_n]_q$$

$d_1, \ldots, d_n =$ degrees

proof: Rephrase it as

$$\sum_{w \in W} q^{\ell_S(w)} = \prod_{i=1}^{n} \frac{1-q^{d_i}}{1-q}$$

$$= \frac{1}{(1-q)^n} \Big/ \frac{1}{\prod_{i=1}^{n}(1-q^{d_i})}$$

$$= \frac{\mathrm{Hilb}(\mathbb{C}[\underline{x}], q)}{\mathrm{Hilb}(\mathbb{C}[\underline{x}]^W, q)}$$

(where we are using $\mathbb{C}[\underline{x}]$ for $S = \mathbb{C}[x_1, \ldots, x_n]$ to avoid conflict with (W,S) and $\ell_S(w)$ in this proof!)

To show $\sum_{w \in W} q^{\ell_S(w)} = \frac{\mathrm{Hilb}(\mathbb{C}[\underline{x}], q)}{\mathrm{Hilb}(\mathbb{C}[\underline{x}]^W, q)}$

it will suffice to show that $\forall J \subseteq S$ both of these functions

$$f(J) := \begin{cases} \dfrac{1}{W_J(q)} & \text{where } W_J(q) := \sum_{w \in W_J} q^{\ell_S(w)} \\[2ex] \dfrac{\mathrm{Hilb}(\mathbb{C}[\underline{x}]^{W_J}, q)}{\mathrm{Hilb}(\mathbb{C}[\underline{x}], q)} \end{cases}$$

satisfy the identity

$$\boxed{\sum_{J \subseteq S} (-1)^{\#J} f(J) = q^{\ell_S(w_0)} f(S)}$$

Why? $$\boxed{\sum_{J \subseteq S} (-1)^{\#J} f(J) = q^{\ell_S(w_0)} f(S)}$$

can be rewritten as a recurrence

$$f(S) = \frac{\displaystyle\sum_{J \subsetneq S} (-1)^{\#J} f(J)}{q^{\ell_S(w_0)} - (-1)^{\#S}}$$

showing both functions $f(J)$ are equal by induction on $\#J$

Base case $J = \emptyset$ says $$\frac{1}{W_\emptyset(q)} \overset{\checkmark}{=} \frac{\mathrm{Hilb}(\mathbb{C}[x]^{W_\emptyset}, q)}{\mathrm{Hilb}(\mathbb{C}[x], q)}$$

$$\frac{1}{W_\emptyset(q)} = 1 = \frac{\mathrm{Hilb}(\mathbb{C}[x]^{W_\emptyset}, q)}{\mathrm{Hilb}(\mathbb{C}[x], q)}$$

Case $J = S$ says $$\frac{1}{W(q)} = \frac{\mathrm{Hilb}(\mathbb{C}[x]^{W}, q)}{\mathrm{Hilb}(\mathbb{C}[x], q)}$$
equivalent to the theorem.

Showing $\boxed{\sum_{J\subseteq S}(-1)^{\#J} f(J) = q^{\ell_S(w_0)} f(S)}$

for $f(J) = \frac{1}{W_J(q)}$ is equivalent to

$$\sum_{J\subseteq S}(-1)^{\#J}\frac{W(q)}{W_J(q)} \overset{?}{=} q^{\ell_S(w_0)}$$

$\Big\|$ $\left.\begin{array}{l} W = W^J \cdot W_J \\ W(q) = W^J(q)\, W_J(q) \end{array}\right\}$ length-additive parabolic coset decomposition

$$\sum_{J\subseteq S}(-1)^{\#J} W^J(q) \quad \text{where } W^J(q) = \sum_{\substack{w\in W:\\ \ell_S(ws)>\ell_S(w)\\ \forall s\in J}} q^{\ell_S(w)}$$

$\Big\|$ Inclusion-Exclusion

$$\sum_{\substack{w\in W:\\ \ell_S(ws)<\ell_S(w)\\ \forall s\in S}} q^{\ell_S(w)} = q^{\ell_S(w_0)} \quad \checkmark$$

Showing $$\boxed{\sum_{J \subseteq S} (-1)^{\#J} f(J) = q^{\ell_S(w_0)} f(S)}$$

for $$f(J) = \frac{\mathrm{Hilb}(\mathbb{C}[\underline{x}]^{W_J}, q)}{\mathrm{Hilb}(\mathbb{C}[\underline{x}], q)}$$

is equivalent, after clearing the denominator $\mathrm{Hilb}(\mathbb{C}[\underline{x}], q)$ everywhere, to showing

$$\boxed{\sum_{J \subseteq S} (-1)^{\#J} \mathrm{Hilb}(\mathbb{C}[\underline{x}]^{W_J}, q) = q^{\ell_S(w_0)} \mathrm{Hilb}(\mathbb{C}[\underline{x}]^{W}, q)}$$

$$\boxed{\sum_{J\subseteq S}(-1)^{\#J}\,\mathrm{Hilb}\left(\mathbb{C}[\underline{x}]^{W_J},q\right)=q^{\ell_S(w_0)}\,\mathrm{Hilb}\left(\mathbb{C}[\underline{x}]^{W},q\right)}$$

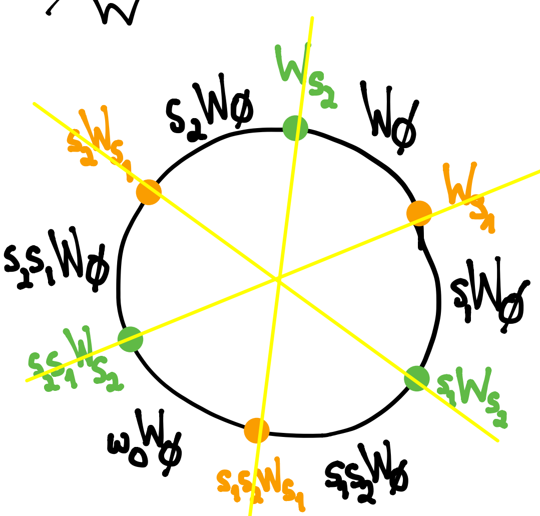
has left side equal to

$$\sum_{J\subseteq S}(-1)^{\#J}\sum_{d\geq 0}q^d\,\underbrace{\left\langle \mathbb{C}[\underline{x}]_d\big\downarrow^{W}_{W_J},\,1_{W_J}\right\rangle_{W_J}}_{\underset{\text{Frobenius reciprocity}}{=}\left\langle \mathbb{C}[\underline{x}]_d,\,1_{W_J}\uparrow^{W}\right\rangle_W}$$

$$=\sum_{d\geq 0}q^d\left\langle \mathbb{C}[\underline{x}]_d,\,\sum_{J\subseteq S}(-1)^{\#J}\,1_{W_J}\uparrow^{W}\right\rangle_W$$

Hopf-Lefschetz trace formula on Coxeter complex Δ

$$\sum_i(-1)^i\tilde{C}_i(\Delta)\cong\sum_i(-1)^i\tilde{H}_i(\Delta)$$



$$=\sum_{d\geq 0}q^d\left\langle \mathbb{C}[\underline{x}]_d,\,\det\right\rangle_W$$

$$=\mathrm{Hilb}\left(\mathbb{C}[\underline{x}]^{W,\det},q\right)$$

det-isotypic component of $\mathbb{C}[\underline{x}]$

Thus $\boxed{\sum_{J\subseteq S}(-1)^{\#J}\,\mathrm{Hilb}(\mathbb{C}[\underline{x}]^{W_J},q) = q^{\ell_S(w_0)}\,\mathrm{Hilb}(\mathbb{C}[\underline{x}]^{W},q)}$

follows if we can show that

$$\mathrm{Hilb}(\mathbb{C}[\underline{x}]^{W,\det},q) = q^{\ell_S(w_0)}\,\mathrm{Hilb}(\mathbb{C}[\underline{x}]^{W},q)$$

for real reflection groups (W,S).

In fact, let's analyze the structure of the χ-isotypic component

$$\mathbb{C}[\underline{x}]^{W,\chi} = \{f\in\mathbb{C}[\underline{x}] : w(f)=\chi(w)\cdot f\}$$

for any 1-dimensional character

$$\chi : W \longrightarrow \mathbb{C}^{\times}$$

of a complex reflection group W

PROPOSITION For any $\chi : W \to \mathbb{C}^\times$,

$$\mathbb{C}[\underline{x}]^{W,\chi} = f_\chi \cdot \mathbb{C}[\underline{x}]^W \quad \left(\text{a free } \mathbb{C}[\underline{x}]^W\text{-module of rank } 1\right)$$

where $f_\chi := \prod_{\substack{\text{reflection} \\ \text{hyperplanes } H}} \alpha_H^{d_H(\chi)}$

- α_H is linear in $\mathbb{C}[\underline{x}]$, vanishing on H
- $W_H = \langle t_H \rangle = \{w \in W : w(H) = H\} \cong \mathbb{Z}/d_H\mathbb{Z}$
- $\chi(t_H) = \zeta^{-d_H(\chi)}$ where $\zeta = e^{\frac{2\pi i}{d_H}}$

  and $0 \leq d_H(\chi) < d_H - 1$

NOTE that this would imply the needed

$$\mathrm{Hilb}\left(\mathbb{C}[\underline{x}]^{W,\det}, q\right) = q^{\ell_S(w_0)}\,\mathrm{Hilb}\left(\mathbb{C}[\underline{x}]^W, q\right)$$

for real reflection groups (W,S):

$\chi = \det : W \longrightarrow \mathbb{C}^\times$ has

$$\det(t_H) = -1 = (-1)^1 \text{ for all } H$$

$\Longrightarrow \quad d_H(\chi) = 1$ for all H

$\Longrightarrow \quad f_\chi = \prod_H \alpha_H^1$ has $\deg(f_\chi) = \#\left\{\begin{array}{c}\text{reflecting}\\ \text{hyperplanes}\\ H\end{array}\right\}$

$$= \ell_S(w_0)$$

$$\Longrightarrow \mathrm{Hilb}\left(\mathbb{C}[\underline{x}]^{W,\det}, q\right) = q^{\ell_S(w_0)}\,\mathrm{Hilb}\left(\mathbb{C}[\underline{x}]^W, q\right)$$

proof of PROPOSITION

Fixing H for the moment, pick dual bases

$$y_1, y_2, \ldots, y_n \text{ for } V$$

$$(\alpha_H =) x_1, x_2, \ldots, x_n \text{ for } V^*$$

so that $W_H = \langle t_H \rangle$ where

$t_H(y_1) = \zeta y_1$

t_H fixes $\underbrace{y_2, \ldots, y_n}_{\text{spanning } H}$

$t_H(x_1) = \zeta^{-1} x_1$

t_H fixes $x_2, \ldots, x_n$

$$\begin{array}{c|c} & \begin{array}{cccc} y_1 & y_2 & \cdots & y_n \end{array} \\ \begin{array}{c} y_1 \\ y_2 \\ \vdots \\ y_n \end{array} & \begin{bmatrix} \zeta & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots \\ & & & & 1 \end{bmatrix} \end{array}$$

t_H on V

$$\begin{array}{c|c} & \begin{array}{cccc} x_1 & x_2 & \cdots & x_n \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array} & \begin{bmatrix} \zeta^{-1} & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots \\ & & & & 1 \end{bmatrix} \end{array}$$

t_H on V^*

Hence $f = \sum_{\underline{a} \in \mathbb{N}^n} c_{\underline{a}} \cdot x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \in \mathbb{C}[\underline{x}]^{W,\chi}$

means

$$t_H(f) = \chi(t_H) f = \zeta^{-d_H(\chi)} \cdot f \quad \forall H$$

$$\sum_{\underline{a} \in \mathbb{N}^n} c_{\underline{a}} \zeta^{-a_1} \cdot x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} = \sum_{\underline{a} \in \mathbb{N}^n} c_{\underline{a}} \zeta^{-d_H(\chi)} \cdot x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$$

This forces $a_1 \equiv d_H(\chi) \bmod d_H \quad \forall H$

$$\Rightarrow \quad \underset{\left(= \alpha_H^{d_H(\chi)}\right)}{x_1^{d_H(\chi)}} \text{ divides } f \quad \forall H$$

$$\Rightarrow \quad f_\chi = \prod_H \alpha_H^{d_H(\chi)} \text{ divides } f$$

It only remains to show $f_\chi \in \mathbb{C}[\underline{x}]^{W,\chi}$

since then $f \in \mathbb{C}[\underline{x}]^{W,\chi} \Rightarrow f/f_\chi \in \mathbb{C}[\underline{x}]^W$,

showing $\mathbb{C}[\underline{x}]^{W,\chi} = f_\chi \, \mathbb{C}[\underline{x}]^W$.

To this end, fixing H, one can factor

$$f_\chi = \prod_{H'} \alpha_{H'}^{d_\chi(H')} = \alpha_H^{d_\chi(H)} \prod_{\substack{W_H\text{-orbits } \mathcal{O} \\ \text{of hyperplanes}}} \prod_{H' \in \mathcal{O}} \alpha_{H'}^{d_\chi(H')}$$

$d_\chi(H') = d_\chi(wH')$ since $t_{wH'} = w t_{H'} w^{-1}$

$$= \underbrace{\alpha_H^{d_\chi(H)}}_{\substack{t_H \text{ scales} \\ \text{this by } \zeta^{-d_\chi(H)} \\ = \chi(t_H)}} \cdot \prod_{\mathcal{O}} \Bigg(\underbrace{\prod_{H' \in \mathcal{O}} \alpha_{H'}}_{\substack{W_H\text{-invariant, since one can} \\ \text{choose } \{\alpha_{H'} : H' \in \mathcal{O}\} \text{ to form} \\ \text{a } W_H\text{-orbit, by rescaling}}} \Bigg)^{d_{\mathcal{O}}(H')}$$

$\Rightarrow \; t_H(f_\chi) = \chi(t_H) f_\chi \;\; \forall H$. So $f_\chi \in \mathbb{C}[\underline{x}]^{W,\chi}$ $\square$

REMARK A non-inductive proof of

$$\left([d_1]_q [d_2]_q \cdots [d_n]_q = \right) \mathrm{Hilb}\left(\mathbb{C}[\underline{x}] / (\mathbb{C}[\underline{x}]^W_+), q\right) = \sum_{w \in W} q^{\ell_S(w)}$$

was given by Bernstein-Gelfand-Gelfand, Hiller:
1973, 1981

Pick any $f_{w_0} \in \mathbb{C}[\underline{x}]_{\ell_S(w_0)}$ not in $(\mathbb{C}[\underline{x}]^W_+)$,

and one obtains a basis for $\mathbb{C}[\underline{x}] / (\mathbb{C}[\underline{x}]^W_+)$

$\{f_w\}_{w \in W}$ with $\deg(f_w) = \ell_S(w)$, via

recipe $f_{ws} := \partial_s(f_w)$ if $\ell(ws) = \ell(w) - 1$.

Here $\partial_s(f) := \dfrac{f - s(f)}{\alpha_s}$ are Chevalley's

divided difference operators.

The W-irreducible U_i's fake-degrees $f^{U_i}(q)$ carry equivalent info to the Hilbert series of the S^W-module of U_i-relative invariants

$$S^{W,U_i} := \bigoplus_{d \geq 0} \underbrace{S_d^{W,U_i}}_{\substack{U_i\text{-isotypic} \\ \text{component of } S_d}}$$

PROPOSITION

$$\frac{\mathrm{Hilb}(S^{W,U_i}, q)}{\mathrm{Hilb}(S^W, q)} = \mathrm{Hilb}\left(S/(S_+^W)^{U_i}, q \right)$$

$$= \dim U_i \cdot f^{U_i}(q)$$

$$\frac{\mathrm{Hilb}(S^{W,u_i}, q)}{\mathrm{Hilb}(S^W, q)} = \mathrm{Hilb}\left(S/(S^W_+)^{u_i}, q \right)$$

follows from some commutative algebra...

- S is a free S^W-module

  [because S is integral over S^W
  (every $f \in S$ satisfies a monic polynomial $\prod_{w \in W}(t - w(f))$)
  hence S is a finitely generated S^W-mod,
  so $f_1, \ldots, f_n$ generating S^W are a system of parameters for S,
  so they are also an S-regular sequence,
  because $S = \mathbb{C}[x_1, \ldots, x_n]$ is a Cohen-Macaulay ring.]

- Each S^{W,u_i} is a free S^W-module

  [because the splittings $S \rightleftarrows S^{W,u_i}$
  can be chosen as S^W-module maps]

- One gets a free S^W-module basis for S^{W,U_i} by lifting any $\mathbb{C}$-basis for $S/(S^W_+)^{W,U_i} = S^{W,U_i}/S^W_+ \cdot S^{W,U_i}$

again uses those splittings

because more generally...

LEMMA

For a graded module $M = \bigoplus_{d\geq 0} M_d$ over a graded $\mathbb{C}$-algebra $R = \bigoplus_{d\geq 0} R_d$, a subset $\{m_i\}$ generates M over R $\Leftrightarrow$ $\{\overline{m_i}\}$ $\mathbb{C}$-spans M/R_+M over $R/R_+ = \mathbb{C}$.

(EXERCISE)

Time for generatingfunctionology...

THEOREM (Molien)

W a finite subgroup of $GL(V)$

U an irreducible W-representation

$$\Rightarrow \operatorname{Hilb}(S^{W,U}, q) = \frac{\dim U}{\# W} \sum_{w \in W} \frac{\chi_U(w)}{\det(1-qw)}$$

proof:

$$\operatorname{Hilb}(S^{W,U}, q) = \sum_{d \geq 0} q^d \dim S_d^{W,U}$$

$$= \sum_{d \geq 0} q^d \cdot \dim U \langle \chi_U, S_d \rangle_W$$

$$= \sum_{d \geq 0} q^d \frac{\dim U}{\# W} \sum_{w \in W} \chi_U(w) \cdot \operatorname{Tr}\left(S_d \xrightarrow{w^{-1}} S_d\right)$$

$$= \frac{\dim U}{\# W} \sum_{w \in W} \chi_U(w) \sum_{d \geq 0} q^d \operatorname{Tr}\left(S_d \xrightarrow{w^{-1}} S_d\right)$$

So only need to show

$$\sum_{d\geq 0} q^d \,\mathrm{Tr}\left(S_d \xrightarrow{\omega^{-1}} S_d\right) \overset{?}{=} \frac{1}{\det(1-q\omega)}$$

If ω has eigenvalues $\lambda_1,\dots,\lambda_n$ on V

then $\dfrac{1}{\det(1-q\omega)} = \displaystyle\prod_{i=1}^{n} \frac{1}{1-\lambda_i q}$,

while ω^{-1} has eigenvalues $\lambda_1,\dots,\lambda_n$ on V^*,

eigenvalues $\left\{\lambda_1^{i_1}\cdots\lambda_n^{i_n} : \sum_j i_j = d\right\}$ on $S_d = \mathrm{Sym}^d V^*$

and $\displaystyle\sum_{d\geq 0} q^d \,\mathrm{Tr}\left(S_d \xrightarrow{\omega^{-1}} S_d\right)$

$$= \sum_{d\geq 0} q^d \sum_{\substack{(i_1,\dots,i_n):\\ \sum_j i_j = d}} \lambda_1^{i_1}\cdots\lambda_n^{i_n} = \prod_{i=1}^{n} \frac{1}{1-\lambda_i q} \quad \square$$

COROLLARY W a complex reflection group has $\#W = d_1 d_2 \cdots d_n$

proof:

$$\mathrm{Hilb}(S^W, q)$$

Shephard-Todd Chevalley (a)

$$= \prod_{i=1}^{n} \frac{1}{1-q^{d_i}}$$

Molien with $U = 1\!\!1_W$

$$= \frac{1}{\#W} \sum_{w \in W} \frac{1}{\det(1-qw)}$$

Laurent expansion about $q=1$

$$= \frac{1}{\#W}\Big[\underbrace{\frac{1}{(1-q)^n}}_{\text{from } w=e} + \underbrace{O\big((1-q)^{1-n}\big)}_{\text{from } w\neq e} \Big]$$

multiply by $(1-q)^n$, take $\lim_{q\to 1}$

On left, get $\lim_{q\to 1} \prod_{i=1}^{n} \frac{1-q}{1-q^{d_i}} = \prod_{i=1}^{n} \frac{1}{d_i}$

On right, get $\frac{1}{\#W}$ $\square$

COROLLARY (Shephard-Todd, Chevalley (b))

W a complex reflection group has

$$S/(S^W_+) \cong \mathbb{C}[W] \text{ as } W\text{-rep'ns.}$$

proof:

Since every W-irreducible U has

$\langle \chi_U, \chi_{\mathbb{C}[W]} \rangle_W = \dim U$, only need to show

$$\dim U \overset{?}{=} \langle \chi_U, \chi_{S/(S^W_+)} \rangle_W$$

$$= \lim_{q \to 1} \boxed{\frac{1}{\dim U} \frac{\mathrm{Hilb}(S^{W,U}, q)}{\mathrm{Hilb}(S^W, q)}}$$

the fake-degree polynomial $f^U(q)$

$$\dim U \overset{?}{=} \lim_{q\to 1} \boxed{\frac{1}{\dim U}\frac{\mathrm{Hilb}(S^{W,U},q)}{\mathrm{Hilb}(S^W,q)}}$$

$$= \lim_{q\to 1} \frac{1}{\cancel{\dim U}} \cdot \prod_{i=1}^{n}(1-q^{d_i}) \cdot \frac{\cancel{\dim U}}{\# W} \sum_{w\in W} \frac{\chi_U(w)}{\det(1-qw)}$$

$$= \frac{1}{\# W} \lim_{q\to 1} \prod_{i=1}^{n}(1-q^{d_i}) \Big[\underbrace{\frac{\chi_U(e)}{(1-q)^n}}_{w=e} + \underbrace{O\big((1-q)^{1-n}\big)}_{w\neq e} \Big]$$

$$= \underbrace{\frac{1}{\# W} \prod_{i=1} d_i}_{=1 \text{ by previous COROLLARY}} \cdot \underbrace{\chi_U(e)}_{\dim U} = \dim U$$

$\square$

The other product formulas come from consider W acting on S tensored with exterior algebras on V, V^*:

$$\Lambda^{\bullet} V = \bigoplus_{p=0}^{n} \Lambda^p V \qquad \Big| \qquad \Lambda^{\bullet} V^* = \bigoplus_{p=0}^{n} \Lambda^p V^*$$

$$W \circlearrowright S \otimes \Lambda^{\bullet} V \qquad \Big| \qquad W \circlearrowright S \otimes \Lambda^{\bullet} V^*$$

For either $U = V$ or V^*, use a doubly-graded Hilbert series

$$\mathrm{Hilb}\left((S \otimes \Lambda^{\bullet} U)^W ; q, t \right) := \sum_{p=0}^{n} \sum_{d \geq 0} \dim \left(S_d \otimes \Lambda^p U \right)^W q^d t^p$$

EXTERIOR MOLIEN THEOREM

For W a finite subgroup of $GL(V)$, and $\mathcal{U}$ a W-rep'n

$$\mathrm{Hilb}\left(\left(S \otimes \Lambda^{\bullet} \mathcal{U}\right)^W; q, t\right) = \frac{1}{\# W} \sum_{w \in W} \frac{\det\left(1 + t \cdot w|_{\mathcal{U}^*}\right)}{\det(1 - q \cdot w)}$$

proof:

An exercise very similar to Molien.
Key point is showing

$$\sum_{p=0}^{\dim \mathcal{U}} \sum_{d \geq 0} t^p q^d \operatorname{Tr}\left(S_d \otimes \Lambda^p \mathcal{U} \xrightarrow{w} S_d \otimes \Lambda^p \mathcal{U}\right)$$

$$= \text{the } \bar{w}^{\prime} \text{ summand} \quad \square$$

We again combine this general Molien statement with key structural results on $(S \otimes \Lambda^{\cdot} U)^W$ for refl'n groups W.

Note that $(S \otimes \Lambda^{\cdot} U)^W$ is naturally an S^W-module via $f \in S^W$ multiplying:

$$f\left(\sum h \otimes u_{i_1} \wedge \cdots \wedge u_{i_p}\right) = \sum fh \otimes u_{i_1} \wedge \cdots \wedge u_{i_p}$$

Also one can multiply

$$(S_d \otimes \Lambda^p U)^W \otimes (S_{d'} \otimes \Lambda^{p'} U)^W \longrightarrow (S_{d+d'} \otimes \Lambda^{p+p'} U)^W$$

THEOREM (Solomon 1963, Orlik-Solomon 1980)
Both $(S\otimes\Lambda V)^W$, $(S\otimes\Lambda V^*)^W$ are not only free S^W-modules, but actually **exterior algebras over S^W**:

$$(S\otimes\Lambda V)^W \cong \Lambda_{S^W}\{\Theta_1,\Theta_2,\dots,\Theta_n\}$$

with $\Theta_i = \Theta_i^{(1)}\otimes y_1 + \Theta_i^{(2)}\otimes y_2 + \dots + \Theta_i^{(n)}\otimes y_n \in (S\otimes V)^W$
for $i=1,2,\dots,n$ having
$\deg(\Theta_i^{(j)}) = e_i^*$ (= coexponents)

$$(S\otimes\Lambda V^*)^W \cong \Lambda_{S^W}\{\Theta_1^*,\Theta_2^*,\dots,\Theta_n^*\}$$

with $\Theta_i^* = \frac{\partial f_i}{\partial x_1}\otimes x_1 + \frac{\partial f_i}{\partial x_2}\otimes x_2 + \dots + \frac{\partial f_i}{\partial x_n}\otimes x_n \in (S\otimes V^*)^W$
for $i=1,2,\dots,n$ where $S^W = \mathbb{C}[f_1,f_2,\dots,f_n]$
and hence $\deg(\Theta_i^*) = d_i - 1 = e_i$
(= exponents)

$\Theta_i^* = $ "df_i"

COROLLARY W a complex refletion group with exponents $e_1, e_2, \ldots, e_n$ (so $e_i = d_i - 1$)

$$\Rightarrow \sum_{w \in W} x^{\dim V^w} = (x+e_1)(x+e_2)\cdots(x+e_n)$$

proof: Compare

$$\mathrm{Hilb}\left((S \otimes \Lambda V^*)^W; q, t\right) = \mathrm{Hilb}\left(\Lambda_{S^W}[\theta_1^*, \ldots, \theta_n^*]; q, t\right)$$

exterior Molien

$$\| \qquad\qquad = \prod_{i=1}^{n} \frac{1+tq^{e_i}}{1-q^{d_i}}$$

$$\frac{1}{\# W} \sum_{w \in W} \frac{\det(1+tw)}{\det(1-qw)}$$

$$\| $$

$$\frac{1}{\# W} \sum_{w \in W} \prod_{\substack{\text{eigenvalues} \\ \lambda \neq 1 \text{ of } w}} \frac{1+t\lambda}{1-q\lambda} \cdot \left(\frac{1+t}{1-q}\right)^{\dim V^w}$$

change variables $x = \frac{1+t}{1-q}$

$$\frac{1}{\#W}\sum_{w\in W}\prod_{\lambda\neq 1}\frac{1+t\lambda}{1-q\lambda}\cdot x^{\dim V^w} = \prod_{i=1}^{n}\frac{1+tq^{e_i}}{1-q^{d_i}}$$

$$\boxed{x=\frac{1+t}{1-q}}$$

$$\frac{1}{\#W}\sum_{w\in W}x^{\dim V^w}\prod_{\lambda\neq 1}\frac{1-\lambda+x(1-q)\lambda}{1-q\lambda}$$

$$\prod_{i=1}^{n}\frac{1-q^{e_i}+x(1-q)q^{e_i}}{1-q^{d_i}}$$

$$= \prod_{i=1}^{n}\frac{[e_i]_q+xq^{e_i}}{[d_i]_q}$$

$$\lim_{q\to 1}$$

$$\frac{1}{\cancel{\#W}}\sum_{w\in W}x^{\dim V^w} = \frac{1}{\cancel{\prod_{i=1}^{n} d_i}}\prod_{i=1}^{n}(x+e_i) \qquad \square$$

REMARK

Carter (1972) proved

LEMMA

$$n-\dim V^w = \ell_T(w)$$

for real reflection groups (W,S)

and $T = \{\text{all reflections}\}$

$$\Rightarrow \sum_{w\in W} q^{\ell_T(w)} = \sum_{w\in W} q^{n-\dim V^w}$$

$$= \prod_{i=1}^{n}(1+qe_i) = \prod_{i=1}^{n}(1+q(d_i-1))$$

COROLLARY W a complex reflection group with coexponents $e_1^*, e_2^*, \ldots, e_n^*$

$$\Rightarrow \sum_{w \in W} \det(w) \cdot x^{\dim V^w} = (x - e_1^*)(x - e_2^*) \cdots (x - e_n^*)$$

---

proof: Extremely similar, comparing

$$\mathrm{Hilb}((S \otimes \wedge V)^W; q, t) = \mathrm{Hilb}(\wedge_{S^W}\{\theta_1, \ldots, \theta_n\}; q, t)$$

exterior Molien

$$= \prod_{i=1}^{n} \frac{1 + t q^{e_i^*}}{1 - q^{d_i}}$$

$$\frac{1}{\# W} \sum_{w \in W} \frac{\det(1 + t w^{-1})}{\det(1 - q w)}$$

$$= \frac{1}{\# W} \sum_{w \in W} \prod_{\substack{\text{eigenvalues} \\ \lambda \neq 1 \text{ of } w}} \frac{1 + t \lambda^{-1}}{1 - q \lambda} \cdot \left(\frac{1+t}{1-q} \right)^{\dim V^w}$$

change variables $x = \frac{1+t}{1-q}$

The crucial difference appears because

$$\prod_{\substack{\text{eigenvalues}\\ \lambda\neq 1\\ \text{for } w}} \frac{1-\lambda^{-1}}{1-\lambda} = \prod_{\lambda\neq 1}(-\lambda^{-1}) = (-1)^{n-\dim V^w}\det(w^{-1})$$

for $|\lambda|=1$, $\frac{1-\lambda^{-1}}{1-\lambda} = -\lambda^{-1}$

Previous formula then gives

$$(-1)^n \sum_{w\in W}\det(w)(-x)^{\dim V^w} = \prod_{i=1}^{n}(x+e_i^*)$$

$x \mapsto -x$

$$\sum_{w\in W}\det(w)\, x^{\dim V^w} = \prod_{i=1}^{n}(x-e_i^*)$$

But then why does one have

$$\left(\prod_{i=1}^{n}(q-e_i^*)=\right)\sum_{w\in W}\det(w)\,q^{\dim V^w} \overset{?}{=} \sum_{X\in\mathcal{L}_W}\mu(\hat{0},X)\,q^{\dim X}$$

$$= \sum_{X\in\mathcal{L}_W} q^{\dim X} \sum_{\substack{w\in W:\\ V^w=X}}\det(w)$$

$\boxed{\text{Are these equal?}}$

Yes, because $\mu'(X) := \sum_{\substack{w\in W:\\ V^w=X}}\det(w)$

satisfies the Möbius function recurrence

$$\sum_{\substack{Y\in\mathcal{L}_W:\\ \hat{0}\le Y\le X\\ (\text{i.e. } V\supseteq Y\supseteq X)}}\mu'(Y) = \sum_{\substack{w\in W:\\ V^w\supseteq X}}\det(w) = \sum_{w\in W_X}\det(w) = \begin{cases}1 & \text{if } X=V=\hat{0}\\ 0 & \text{if } X\neq V\end{cases}$$

Steinberg: W_X = reflection group generated by $t\in T$ fixing X

REMARK Terao showed that for any hyperplane arrangement $\mathcal{A} = \{H_1, H_2, \ldots, H_t\}$, if the S-module of $\mathcal{A}$-derivations

$$\mathrm{Der}_{\mathcal{A}}(S) := \left\{ \theta = \sum_{i=1}^{n} \theta^{(i)} \frac{\partial}{\partial x_i} \in S \otimes V : \begin{array}{l} \alpha_H \text{ divides } \theta(\alpha_H) \\ \forall H \in \mathcal{A} \end{array} \right\}$$

$$\updownarrow$$

$$\sum_{i=1}^{n} \theta^{(i)} \otimes y_i$$

is free over S, with S-basis $\{\theta_1, \ldots, \theta_n\}$ having $\deg(\theta_j^{(i)}) = e_j^*(\mathcal{A})$ for $j = 1, \ldots, n$

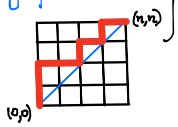
then $$\sum_{X \in \mathcal{L}_{\mathcal{A}}} q^{\dim X} = \prod_{i=1}^{n} \left(q - e_i^*(\mathcal{A}) \right)$$

THEOREM (Terao 1981)

W a complex reflection group has $\mathrm{Der}(\mathcal{A}_W)$ free, with S-basis same as the S^W-basis $\{\theta_1, \ldots, \theta_n\}$ for $(S \otimes V)^W$.

The last product formula...

$\#\left\{\text{Dyck paths } (0,0)\to(n,n)\right\} = \frac{1}{n+1}\binom{2n}{n} = \frac{(n+2)(n+3)\cdots(2n)}{2\cdot 3\cdots n}$



Catalan number

Q = root lattice
$h = \max\{d_1, \dots, d_n\}$

$$\rightsquigarrow \#\left\{\begin{matrix}W\text{-orbits}\\ \text{on}\\ Q/(h+1)Q\end{matrix}\right\} = \frac{1}{\#W}\sum_{w\in W}(h+1)^{\dim V^w} = \frac{(h+d_1)(h+d_2)\cdots(h+d_n)}{d_1 d_2 \cdots d_n} =: \mathrm{Cat}(W)$$

W crystallographic irreducible

...has two assertions, one now easy:

$$\frac{1}{\#W}\sum_{w\in W}(h+1)^{\dim V^w} \overset{?}{=} \frac{(h+d_1)(h+d_2)\cdots(h+d_n)}{d_1 d_2 \cdots d_n}$$

$$\frac{1}{\#W}\left[\sum_{w\in W} q^{\dim V^w}\right]_{q=h+1} = \frac{1}{\#W}\left[\prod_{i=1}^{n}(q+d_i-1)\right]_{q=h+1}$$

The other one ...

$$\#\left\{\begin{array}{c}W\text{-orbits}\\ \text{on}\\ Q/(h+1)Q\end{array}\right\} \overset{?}{=} \frac{1}{\#W}\sum_{w\in W}(h+1)^{\dim V^w} \quad \left(=: \mathrm{Cat}(W)\right)$$

W crystallographic irreducible

follows from Haiman's (1993) lemma that when $w \in W$ permutes $Q \cong \mathbb{Z}^n$

$$mQ \cong \mathbb{Z}^n$$

$$Q/mQ \cong (\mathbb{Z}/m\mathbb{Z})^n,$$

if one has $\gcd(m,h)=1$ (e.g. $m=h+1$)

then $\chi_{Q/mQ}(w) := \#\{\underline{x} \in Q/mQ : w(\underline{x}) = \underline{x}\}$

$$= m^{\dim V^w}$$

REMARK

The product $Cat(W) := \prod_{i=1}^{n} \frac{h+d_i}{d_i}$ is not always an integer for complex W.

But for W which are well-generated, that is, generated by n reflections, $Cat(W)$ has other interpretations, e.g. counting Bessis's W-noncrossing partitions.

For well-generated W, one can even interpret

$$Cat(W,q) := \prod_{i=1}^{n} \frac{[h+d_i]_q}{[d_i]_q}$$

via representations of rational Cherednik algebras.

The rational Cherednik algebra $H_{t,c}(W)$ with $t=1$ will only have some finite-dimensional simple modules for certain special values of the parameters $c : T \longrightarrow \mathbb{C}$.

Taking $c(t) = 1 + \frac{1}{h}$ $\forall t$ is such a choice, and then ...

THEOREM (Berest-Etingof-Ginzburg 2003, Gordon 2003)

$$\mathrm{Cat}(W,q) := \prod_{i=1}^{n} \frac{[h+d_i]_q}{[d_i]_q} = \mathrm{Hilb}\left(L_{1,c}(\mathbb{1}_W)^W, q\right)$$

where $L_{1,c}(\mathbb{1}_W)$ is the $H_{1,c}$-simple with $\mathbb{1}_W$ at its top, and $L_{1,c}(\mathbb{1}_W)^W$ its (graded) W-fixed subspace.

REMARK A mysterious fact:

W well-generated (i.e. generated by $n = \dim V$ reflections)

well-understood (Bessis 2001)

$\Updownarrow$

case-by-case check!

$$e_i^* + e_{n+1-i} = h$$

for $i = 1, 2, \ldots, n$

("exponent-coexponent duality")

How does one prove results like

Solomon?

$$\left(S \otimes \wedge^{\bullet} V^*\right)^W = \wedge_{S^W}\{\underbrace{\theta_1^*, \ldots, \theta_n^*}_{S^W\text{-basis of } (S \otimes V^*)^W}\}$$

$\deg(\theta_i^*) = e_i$

(and $\theta_i^* = $ "df_i" so $e_i = d_i - 1$)

Orlik-Solomon?

$$\left(S \otimes \wedge^{\bullet} V\right)^W = \wedge_{S^W}\{\underbrace{\theta_1, \ldots, \theta_n}_{S^W\text{-basis of } (S \otimes V)^W}\}$$

$\deg(\theta_i) = e_i^*$

- Not hard to check that lifting the n copies of V^* for $i=1,\dots,n$ found in $S/(S_+^W)$

$$\begin{aligned} x_1 &\longmapsto \Theta_i^{(1)} \\ x_2 &\longmapsto \Theta_i^{(2)} \\ &\vdots \\ x_n &\longmapsto \Theta_i^{(n)} \end{aligned}$$

exactly correspond to an S^W-basis $\{\Theta_1,\dots,\Theta_n\}$ for $(S\otimes V)^W$

via $\Theta_i = \sum_{j=1}^{n} \Theta_i^{(j)} \otimes y_j$

explaining why $\deg \Theta_i^{(j)} = e_i^*$

(and similar for $\deg \Theta_i^{(j)*} = e_i$)

- Also easy to show their wedges
  $\{\theta_{i_1} \wedge \cdots \wedge \theta_{i_k}\} \subset (S \otimes \Lambda^k V)^W$
  are S^W-linearly independent, by extending scalars to the W-invariant rational functions
  $\mathrm{Frac}(S^W)$ $\left(= \mathrm{Frac}(S)^W = \mathbb{Q}(x_1, \ldots, x_n)^W \right)$
- One then checks that these wedges have the appropriate sum of their degrees to be an S^W-basis of $(S \otimes \Lambda^k V)^W$ using ...

LEMMA (Gutkin, 1973, Opdam 1998)

Any W-rep'n U has the sum of degrees for an S^W-basis of $(S \otimes U)^W$ given by this local data

$$\sum_{\text{hyperplanes } H} \; \sum_{j=0}^{\#W_H - 1} j \cdot \langle U\big\downarrow^{W}_{W_H}, \det^j \rangle_{W_H}$$

These predicted degree sums can be easily computed for $U = \wedge^k V$ or $\wedge^k V^*$

Once one has S^W-linearly independent elements inside $(S \otimes \wedge^k V)^W$ with this degree sum, they must be an S^W-basis

Proof sketch of Gutkin-Opdam Lemma is calculation of degree sum as

$$\left[\frac{\partial}{\partial q}\frac{\mathrm{Hilb}((S\otimes U)^W,q)}{\mathrm{Hilb}(S^W,q)}\right]_{q=1}$$

$$\overset{\text{Molien}}{=}\frac{1}{\#W}\sum_{w\in W}\chi_U(w^{-1})\underbrace{\left[\frac{\partial}{\partial q}\frac{\prod_{i=1}^{n}(1-q^{d_i})}{\det(1-qw)}\right]_{q=1}}_{\substack{\text{vanishes unless}\\ w=e \text{ or } w=t \\ \text{a reflection}}}$$

leading to the local calculation $\square$

- Finally, why do $df_i = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} \otimes x_i$ give an S^W-basis for $(S \otimes V^*)^W$, so that $\theta_i^* = df_i$ works?

Their wedges $\{df_{i_1} \wedge \cdots \wedge df_{i_k}\}$ are $\mathrm{Frac}(S^W)$-linearly independent since their Jacobian $J = \det\left(\frac{\partial f_i}{\partial x_j}\right) \neq 0$ (as $f_1, \ldots, f_n$ are algebraically independent)

$$J = f_{\det} = \prod_H \alpha_H^{\#W_H - 1}$$

(Steinberg 1964)

helps show wedges have the correct degree sum.

Thanks for your attention,

and thanks CRM & LaCIM!