EXERCISES FOR CRM-LACIM SPRING SCHOOL LECTURES

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1. Dihedral group $I_2(m)$ is Shephard-Todd $G(m, m, 2)$

Consider the dihedral group $W = I_2(m)$, whose Coxeter presentation is

$$W = \langle s, t : s^2 = t^2 = e = (st)^m \rangle$$

represented as a subgroup of the orthogonal group $O_2(\mathbb{R})$, acting as isometries that preserve a regular m-sided polygon, centered at the origin.

(a) Show that by considering $r = st$ one has

$$W \cong \langle s, r : s^2 = r^m = e, srs = r^{-1} \rangle.$$

(b) Show that if one extends scalars fron $O_2(\mathbb{R}) \subset GL_2(\mathbb{R})$ and considers W as a subgroup of $GL_2(\mathbb{C})$, one can make a change-of-basis so that W becomes the Shephard-Todd group $G(m, m, 2)$, that is, the subgroup of 2×2 monomial matrices whose two nonzero entries ϵ_1, ϵ_2 are m^{th}-roots of unity, with $\epsilon_1 \epsilon_2 = 1$.

(Hint:
Pick a basis for v_1, v_2 that diagonalizes r.
Who are the eigenvalues $\{\lambda_1, \lambda_2\}$ of r?
Where must s send the two lines $\mathbb{C}v_1$ and $\mathbb{C}v_2$?)

2. Reflection groups W determine their degrees uniquely

Prove that a rational function in t of the form

$$f(t) = \frac{1}{(1 - t^{d_1})(1 - t^{d_2}) \cdots (1 - t^{d_n})}$$

with positve integers $d_1 \leq d_2 \leq \cdots \leq d_n$ can have only one such expression, that is, each d_i is uniquely determined.

(Hint:
Consider $1/f(t)$ instead, and try to use unique factorization in $\mathbb{Z}[t]$.
Be careful that $1 - t^d$ is not irreducible– one knows its irreducible factorization.)

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3. $G(de, e, n)$ is not well-generated for $d, e, n \geq 2$

Recall that the Shephard-Todd group $G(de, e, n)$ is the set of all $n \times n$ monomial matrices whose nonzero entries $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ are all $(de)^{th}$ roots-of-unity, and whose product $\epsilon_1 \epsilon_2 \cdots \epsilon_n$ is a d^{th} root-of-unity.

(a) Classify/parametrize all the reflections in $G(de, e, n)$.

(b) Prove that $G(d, 1, n)$ for $d \geq 2, n \geq 1$ is well-generated, that is, it can be generated by n reflections.

(c) Prove similarly that $G(e, e, n)$ for $e \geq 2, n \geq 2$ is well-generated.

(d) Prove that $G(de, e, n)$ for $d, e, n \geq 2$ is **not** well-generated, but that it can be generated by $n + 1$ reflections.

4. Graded version of Nakayama's Lemma

Let R be a commutative $\mathbb{N}$-graded k-algebra over a field k, so there is a k-vector space decomposition

$$R = \bigoplus_{d \geq 0} R_d$$

with $R_0 = k$ and $R_i R_j \subset R_{i+j}$. Define the ideal $R_+ := \oplus_{d>0} R_d$. Let M be an $\mathbb{N}$-graded R-module, so one has a similar decomposition

$$M = \bigoplus_{d \geq 0} M_d$$

with $R_i M_j \subset M_{i+j}$.

(a) Prove that homogeneous elements $\{m_i\}_{i \in I}$ generate M as an R-module if and only if their images $\{\overline{m}_i\}_{i \in I}$ span $\subset M/R_+M$ as a vector space over $k = R/R_+$. (Hint: An induction on degree is helpful in the harder direction.)

(b) Conclude that the homogeneous elements $\{m_i\}_{i \in I} \subset M$ generate M as an R-module minimally (with respect to inclusion) if and only if $\{\overline{m}_i\}_{i \in I}$ form a k-basis for M/R_+M.

(c) Deduce that, while homogeneous minimal R-generating sets for M need not be unique, their *multiset of degrees* are unique. Specifically, show that the number of elements of degree d in any homogeneous minimal R-generating set for M is $\dim_k(M/R_+M)_d$.