Catalan numbers, parking functions, and invariant theory

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Outline

1. Catalan numbers and objects
2. Parking functions and parking space (type A)
3. q-Catalan numbers and cyclic symmetry
4. Reflection group generalization

Catalan numbers

Definition

The Catalan number is

$$\mathrm{Cat}_n := \frac{1}{n+1}\binom{2n}{n}$$

Example

$$\mathrm{Cat}_3 = \frac{1}{4}\binom{6}{3} = 5.$$

It's not even completely obvious it is always an integer.
But it counts many things, at least 205, as of June 6, 2013, according to Richard Stanley's Catalan addendum.

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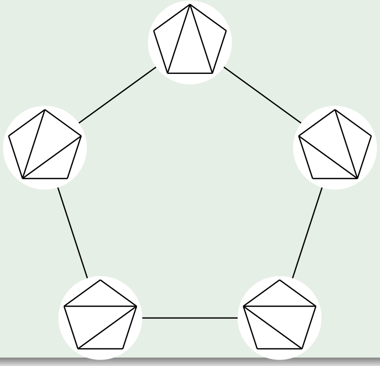
Let's recall a few of them.

Triangulations of an $(n+2)$-gon

Example

There are $5 = Cat_3$ triangulations of a pentagon.

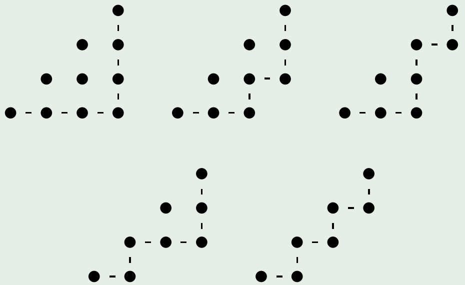


Definition

A Catalan path from $(0,0)$ to (n,n) is a path taking unit north or east steps staying weakly below $y = x$.

Example

The are $5 = \mathrm{Cat}_3$ Catalan paths from $(0,0)$ to $(3,3)$.



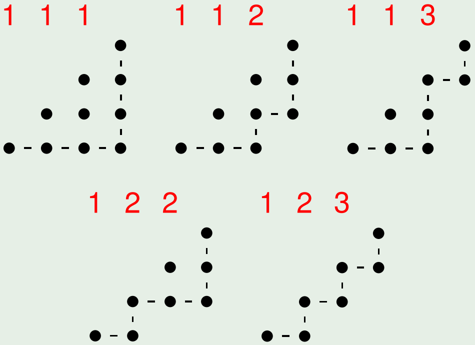
Increasing parking functions

Definition

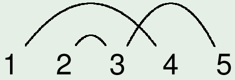
An increasing parking function of size n is
an integer sequence $(a_1, a_2, \ldots, a_n)$ with $1 \leq a_i \leq i$.

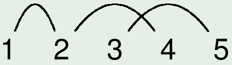
They give the heights of horizontal steps in Catalan paths.

Example



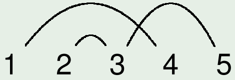
Example

nesting:


nonnesting:


Nonnesting and noncrossing partitions of $\{1, 2, \ldots, n\}$

Example

nesting:


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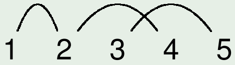
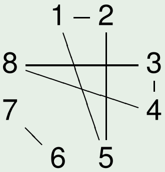
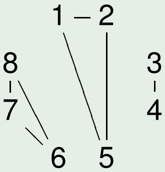


Example

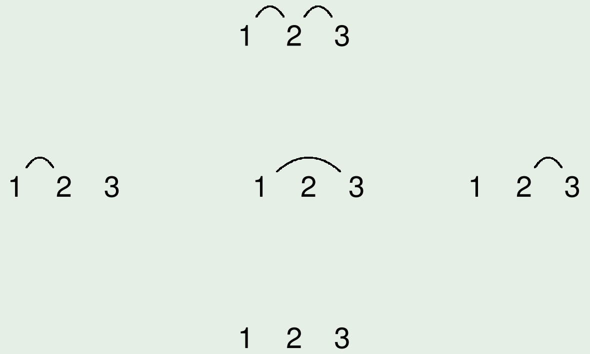
crossing:


noncrossing:


Nonnesting partitions $NN(3)$ of $\{1, 2, 3\}$

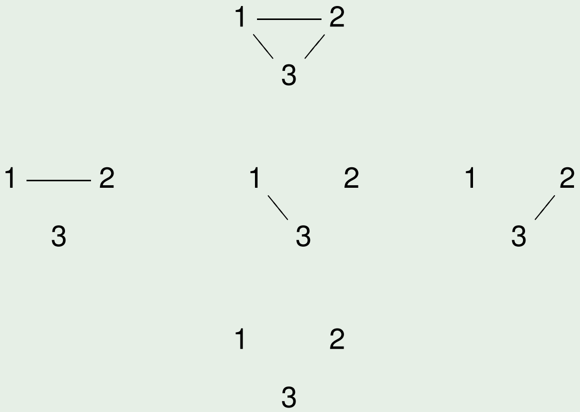
Example

There are $5 = Cat_3$ nonnesting partitions of $\{1, 2, 3\}$.



Example

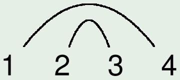
There are $5 = Cat_3$ noncrossing partitions of $\{1,2,3\}$.



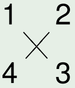
$NN(4)$ versus $NC(4)$ is slightly more interesting

Example

For $n = 4$, among the 15 set partitions of $\{1, 2, 3, 4\}$, exactly one is nesting,



and exactly one is crossing,



leaving $14 = \mathrm{Cat}_4$ nonnesting or noncrossing partitions.

So what are the parking functions?

Definition

Parking functions of length n are sequences $(f(1), \ldots, f(n))$ for which $|f^{-1}(\{1, 2, \ldots, i\})| \geq i$ for $i = 1, 2, \ldots, n$.

Definition (The cheater's version)

Parking functions of length n are sequences $(f(1), \ldots, f(n))$ whose weakly increasing rearrangement is an increasing parking function!

The parking function number $(n+1)^{n-1}$

Theorem (Konheim and Weiss 1966)

There are $(n+1)^{n-1}$ parking functions of length n.

Example

For $n = 3$, the $(3+1)^{3-1} = 16$ parking functions of length 3, grouped by their increasing parking function rearrangement, leftmost:

111					
112	121	211			
113	131	311			
122	212	221			
123	132	213	231	312	321

Parking functions as coset representatives

Proposition (Haiman 1993)

The $(n+1)^{n-1}$ parking functions give coset representatives for

$$\mathbb{Z}^n / \left(\mathbb{Z}[1,1,\ldots,1] + (n+1)\mathbb{Z}^n\right)$$

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$$(\mathbb{Z}_{n+1})^n / \mathbb{Z}_{n+1}[1,1,\ldots,1]$$

or equivalently, by the same isomorphism theorem, for

$$Q/(n+1)Q$$

where here Q is the rank $n-1$ lattice

$$Q := \mathbb{Z}^n / \mathbb{Z}[1,1,\ldots,1] \cong \mathbb{Z}^{n-1}.$$

So what's the parking space?

The parking space is the permutation representation of $W = \mathfrak{S}_n$, acting on the $(n+1)^{n-1}$ parking functions of length n.

Example

For $n = 3$ it is the permutation representation of $W = \mathfrak{S}_3$ on these words, with these orbits:

111					
112	121	211			
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Wondrous!

Just about every natural question about this W-permutation representation Park_n has a beautiful answer.

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As the parking functions give coset representatives for the quotient $Q/(n+1)Q$ where $Q := \mathbb{Z}^n/\mathbb{Z}[1,1,\ldots,1] \cong \mathbb{Z}^{n-1}$, one can deduce this.

Corollary

Each permutation w in $W = \mathfrak{S}_n$ acts on Park_n with character value = trace = number of fixed parking functions

$$\chi_{\mathrm{Park}_n}(w) = (n+1)^{\#(\text{cycles of } w)-1}.$$

Orbit structure?

We've seen the W-orbits in Park_n are parametrized by increasing parking functions, which are Catalan objects. The stabilizer of an orbit is always a Young subgroup

$$\mathfrak{S}_\lambda := \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_\ell}$$

where λ are the multiplicities in any orbit representative.

Example

						λ
111						(3)
112	121	211				(2,1)
113	131	311				(2,1)
122	212	221				(2,1)
123	132	213	231	312	321	(1,1,1)

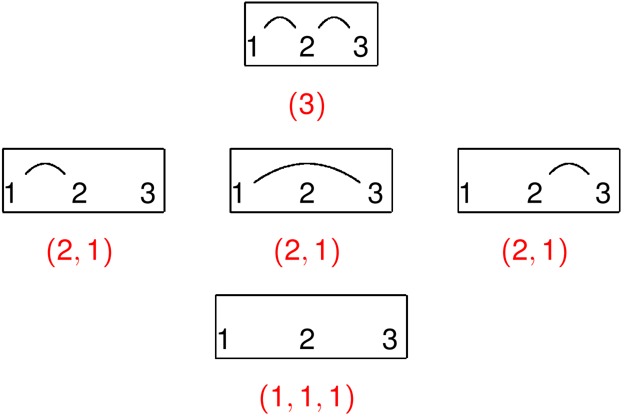
Orbit structure via the nonnesting or noncrossing partitions

That same stabilizer data $\mathfrak{S}_\lambda$ is predicted by the block sizes in

- nonnesting partitions, or
- noncrossing partitions

of $\{1, 2, \ldots, n\}$.

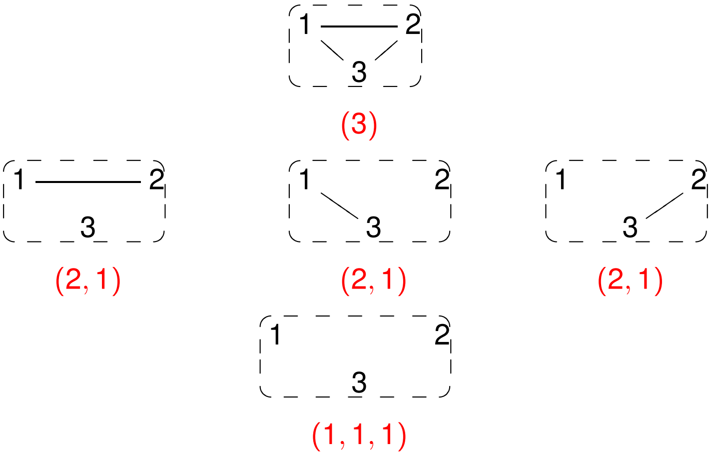
Nonnesting partitions $NN(3)$ of $\{1, 2, 3\}$



Theorem (Shi 1986, Cellini-Papi 2002)

$NN(n)$ bijects to increasing parking functions respecting λ.

Noncrossing partitions $NC(3)$ of $\{1, 2, 3\}$



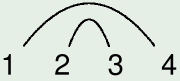
Theorem (Athanasiadis 1998)

There is a bijection $NN(n) \to NC(n)$, respecting λ.

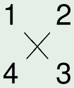
The block size equidistribution for $NN(4)$ versus $NC(4)$

Example

Recall that among the 15 set partitions of $\{1, 2, 3, 4\}$, exactly one was nesting,



and exactly one was crossing,



and note that both correspond to $\lambda = (2, 2)$.

More wonders: Irreducible multiplicities in Park_n

For $W = \mathfrak{S}_n$, the irreducible characters are $\{\chi^\lambda\}$ indexed by partitions λ of n. Haiman gave a product formula for any of the irreducible multiplicities

$$\langle \chi^\lambda, \mathrm{Park}_n \rangle.$$

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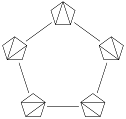
The special case of hook shapes $\lambda = (n-k, 1^k)$ becomes this .

Theorem (Pak-Postnikov 1997)

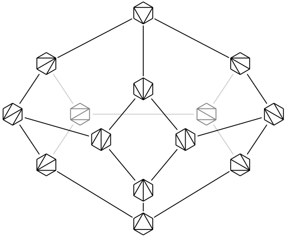
The multiplicity $\langle \chi^{(n-k,1^k)}, \chi_{\mathrm{Park}_n} \rangle_W$ is

- *the number of subdivisions of an $(n+2)$-gon using $n-1-k$ internal diagonals, or*
- *the number of k-dimensional faces in the $(n-1)$-dimensional associahedron.*

Example: n=4



$$\langle \chi^{(3)}, \chi_{\mathrm{Park}_3} \rangle_{\mathfrak{S}_3} = 5$$
$$\langle \chi^{(2,1)}, \chi_{\mathrm{Park}_3} \rangle_{\mathfrak{S}_3} = 5$$
$$\langle \chi^{(1,1,1)}, \chi_{\mathrm{Park}_3} \rangle_{\mathfrak{S}_3} = 1$$



$$\langle \chi^{(4)}, \chi_{\mathrm{Park}_4} \rangle_{\mathfrak{S}_4} = 14$$
$$\langle \chi^{(3,1)}, \chi_{\mathrm{Park}_4} \rangle_{\mathfrak{S}_4} = 21$$
$$\langle \chi^{(2,1,1)}, \chi_{\mathrm{Park}_4} \rangle_{\mathfrak{S}_4} = 9$$
$$\langle \chi^{(1,1,1,1)}, \chi_{\mathrm{Park}_4} \rangle_{\mathfrak{S}_4} = 1$$

q-Catalan numbers

Let's rewrite the Catalan number as

$$\mathrm{Cat}_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(n+2)(n+3)\cdots(2n)}{(2)(3)\cdots(n)}$$

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and consider MacMahon's q-Catalan number

$$\mathrm{Cat}_n(q) = \frac{1}{[n+1]_q}\begin{bmatrix}2n\\n\end{bmatrix}_q := \frac{(1-q^{n+2})(1-q^{n+3})\cdots(1-q^{2n})}{(1-q^2)(1-q^3)\cdots(1-q^n)}.$$

The q-Catalan hides information on cyclic symmetries

The noncrossings $NC(n)$ have a $\mathbb{Z}/n\mathbb{Z}$-action via rotations, whose orbit structure is completely predicted by root-of-unity evaluations of this q-Catalan number.

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Theorem (Stanton-White-R. 2004)

For d dividing n, the number of noncrossing partitions of n with d-fold rotational symmetry is

$$[\mathrm{Cat}_n(q)]_{q=\zeta_d}$$

where ζ_d is any primitive d^{th} root of unity in $\mathbb{C}$.

We called such a set-up a cyclic sieving phenomenon.

Example

Via L'Hôpital's rule, for example, one can evaluate

$$\mathrm{Cat}_4(q) = \frac{(1-q^6)(1-q^7)(1-q^8)}{(1-q^2)(1-q^3)(1-q^4)} = \begin{cases} 14 & \text{if } q = +1 = \zeta_1 \\ 6 & \text{if } q = -1 = \zeta_2 \\ 2 & \text{if } q = \pm i = \zeta_4. \end{cases}$$

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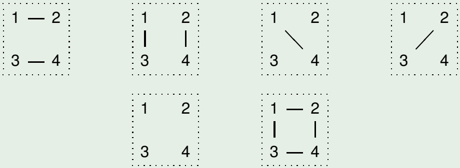
predicting 14 elements of $NC(4)$ total, 6 with 2-fold symmetry,



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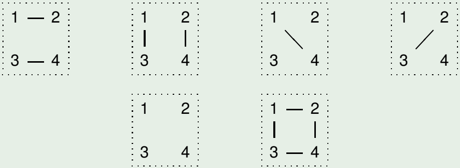


2 of which have 4-fold rotational symmetry.

$\mathrm{Cat}_n(q)$ does double duty hiding cyclic orbit data

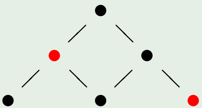
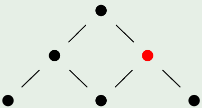
Definition

For a finite poset P, the Duchet-FonDerFlaass (rowmotion) cyclic action maps an antichain $A \longmapsto \Psi(A)$ to the minimal elements $\Psi(A)$ among elements below no element of A. That is,

$$\Psi(A) := \min\{P \setminus P_{\leq A}\}.$$

Example

In P the $(3, 2, 1)$ staircase poset, one has

$A =$  $\longmapsto$ $\Psi(A) =$ 

The Ψ-orbits for the staircase poset $(3, 2, 1)$

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A size 4 orbit (= the rank sets of the poset, plus $A = \varnothing$):

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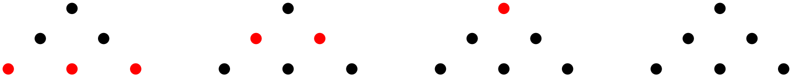
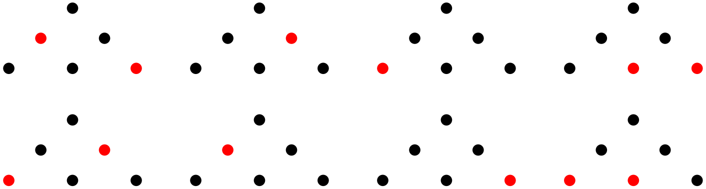


A size 8 orbit:



$\mathrm{Cat}_n(q)$ is doing double duty

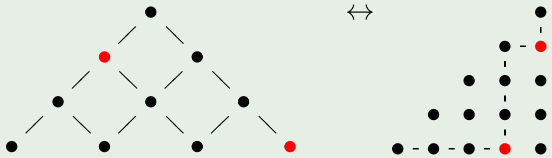
Theorem (part of Armstrong-Stump-Thomas 2011)

For d dividing $2n$ (not n this time), the number of antichains in the $(n-1, n-2, \ldots, 2, 1)$ staircase poset fixed by Ψ^d is

$$[\mathrm{Cat}_n(q)]_{q=\zeta_d}$$

(And these antichains are really disguised Catalan paths.)

Example



How did their theorem predict those orbit sizes?

Example

For $n = 4$ it predicted that, of the 14 $= \mathrm{Cat}_4$ antichains, we'd see

$$\mathrm{Cat}_4(q) = \frac{(1-q^6)(1-q^7)(1-q^8)}{(1-q^2)(1-q^3)(1-q^4)}$$

$$= \begin{cases} 14 \text{ fixed by } \Psi^8 & \text{from setting } q = +1 = \zeta_1 \\ 6 \text{ fixed by } \Psi^4 & \text{from setting } q = -1 = \zeta_2 \\ 2 \text{ fixed by } \Psi^2 & \text{from setting } q = i = \zeta_4 \\ 0 \text{ fixed by } \Psi^1 & \text{from setting } q = e^{\frac{\pi i}{4}} = \zeta_8. \end{cases}$$

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This means there are no singleton orbits, one orbit of size 2, one of size $4 = 6 - 2$, and one orbit of size $8 = 14 - 6$, that is, one free orbit.

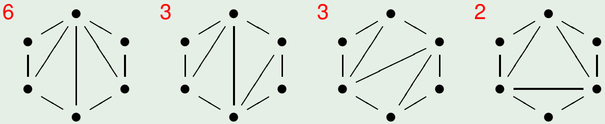
Actually $\mathrm{Cat}_n(q)$ is doing triple duty!

Theorem (Stanton-White-R. 2004)

For d dividing $n+2$, the number of d-fold rotationally symmetric triangulations of an $(n+2)$-gon is $[\mathrm{Cat}_n(q)]_{q=\zeta_d}$

Example

For $n = 4$, these rotation orbit sizes for triangulations of a hexagon



are predicted by

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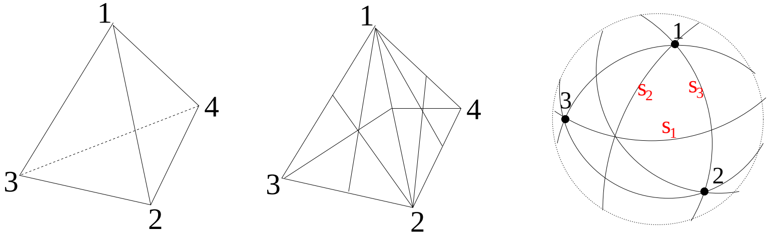
On to the reflection group generalizations

Generalize to irreducible real ref'n groups W acting on $V = \mathbb{R}^n$.

Example

$W = \mathfrak{S}_n$ acts irreducibly on $V = \mathbb{R}^{n-1}$,
realized as $x_1 + x_2 + \cdots + x_n = 0$ within $\mathbb{R}^n$.

It is generated transpositions (i, j),
which are reflections through the hyperplanes $x_i = x_j$.



Invariant theory enters the picture

Theorem (Chevalley, Shephard-Todd 1955)

When W acts on polynomials $S = \mathbb{C}[x_1, \ldots, x_n] = \mathrm{Sym}(V^*)$, its W-invariant subalgebra is again a polynomial algebra

$$S^W = \mathbb{C}[f_1, \ldots, f_n]$$

One can pick $f_1, \ldots, f_n$ homogeneous, with degrees $d_1 \leq d_2 \leq \cdots \leq d_n$, and define $h := d_n$ the Coxeter number.

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One can pick $f_1, \ldots, f_n$ homogeneous, with degrees $d_1 \leq d_2 \leq \cdots \leq d_n$, and define $h := d_n$ the Coxeter number.

Example

For $W = \mathfrak{S}_n$, one has

$$S^W = \mathbb{C}[e_2(\mathbf{x}), \ldots, e_n(\mathbf{x})],$$

so the degrees are $(2, 3, \ldots, n)$, and $h = n$.

Weyl groups and the first W-parking space

When W is a Weyl (crystallographic) real finite reflection group, it preserves a full rank lattice

$$Q \cong \mathbb{Z}^n$$

inside $V = \mathbb{R}^n$. One can choose a root system Φ of normals to the hyperplanes, in such a way that the root lattice $Q := \mathbb{Z}\Phi$ is a W-stable lattice.

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Definition (Haiman 1993)

We should think of the W-permutation representation on the set

$$\mathrm{Park}(W) := Q/(h+1)Q$$

as a W-analogue of parking functions.

Wondrous properties of $\mathrm{Park}(w) = Q/(h+1)Q$

Theorem (Haiman 1993)

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$$\chi_{\mathrm{Park}(W)}(w) = (h+1)^{\dim V^w}.$$

- *The W-orbit count $\#W\backslash Q/(h+1)Q$ is the W-Catalan:*

$$\langle \mathbf{1}_W, \chi_{\mathrm{Park}(W)} \rangle = \prod_{i=1}^{n} \frac{h+d_i}{d_i} =: \mathrm{Cat}(W)$$

W-Catalan example: $W = \mathfrak{S}_n$

Example

Recall that $W = \mathfrak{S}_n$ acts irreducibly on $V = \mathbb{R}^{n-1}$ with degrees $(2, 3, \ldots, n)$ and $h = n$.

One can identify the root lattice $Q \cong \mathbb{Z}^n/(1, 1, \ldots, 1)\mathbb{Z}$.

One has $\#Q/(h+1)Q = (n+1)^{n-1}$, and

$$\begin{aligned}
\mathrm{Cat}(\mathfrak{S}_n) &= \#W\backslash Q/(h+1)Q \\
&= \frac{(n+2)(n+3)\cdots(2n)}{2 \cdot 3 \cdots n} \\
&= \frac{1}{n+1}\binom{2n}{n} \\
&= \mathrm{Cat}_n.
\end{aligned}$$

Exterior powers of V

One can consider multiplicities in $\mathrm{Park}(W)$ not just of

$$\begin{aligned} \mathbf{1}_W &= \wedge^0 V \\ \det{}_W &= \wedge^n V \end{aligned}$$

but all the exterior powers $\wedge^k V$ for $k = 0, 1, 2, \ldots, n$, which are known to all be W-irreducibles (Steinberg).

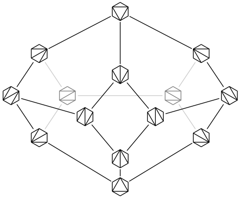
Example

$W = \mathfrak{S}_n$ acts irreducibly on $V = \mathbb{R}^{n-1}$ with character $\chi^{(n-1,1)}$, and on $\wedge^k V$ with character $\chi^{(n-k,1^k)}$.

Theorem (Armstrong-Rhoades-R. 2012)

For Weyl groups W, the multiplicity $\langle \chi_{\wedge^k V}, \chi_{\mathrm{Park}(W)} \rangle$ is

- *the number of $(n-k)$-element sets of compatible cluster variables in a cluster algebra of finite type W,*
- *or the number of k-dimensional faces in the W-associahedron of Chapoton-Fomin-Zelevinsky (2002).*



Two W-Catalan objects: $NN(W)$ and $NC(W)$

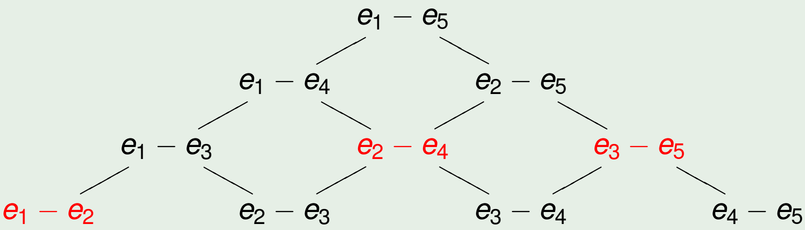
The previous result relies on an amazing coincidence for two W-Catalan counted families generalizing $NN(n), NC(n)$.

Definition (Postnikov 1997)

For Weyl groups W, define W-nonnesting partitions $NN(W)$ to be the antichains in the poset of positive roots Φ_+.

Example

corresponds to this antichain A:



Definition (Bessis 2003, Brady-Watt 2002)

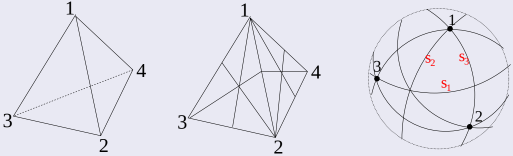
W-noncrossing partitions $NC(W)$ are the interval $[e, c]_{\text{abs}}$ from identity e to any Coxeter element c in absolute order $\leq_{\text{abs}}$ on W:

$$x \leq_{\text{abs}} y \quad \text{if} \quad \ell_T(x) + \ell_T(x^{-1}y) = \ell_T(y)$$

where the absolute (reflection) length is

$$\ell_T(w) = \min\{w = t_1 t_2 \cdots t_\ell : t_i \text{ reflections}\}$$

and a Coxeter element $c = s_1 s_2 \cdots s_n$ is any product of a choice of simple reflections $S = \{s_1, \ldots, s_n\}$.

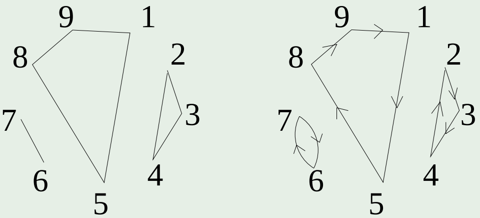


The case $W = \mathfrak{S}_n$

Example

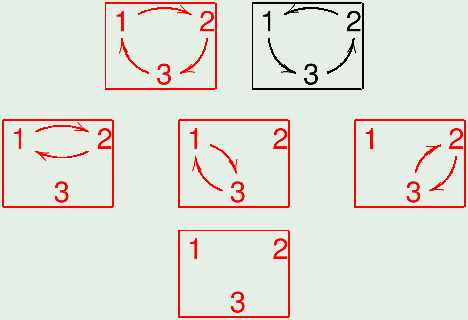
For $W = \mathfrak{S}_n$, the n-cycle $c = (1, 2, \ldots, n)$ is one choice of a Coxeter element.

And permutations w in $NC(W) = [e, c]_{\text{abs}}$ come from orienting clockwise the blocks of the noncrossing partitions $NC(n)$.



The absolute order on $W = \mathfrak{S}_3$ and $NC(\mathfrak{S}_3)$

Example



Generalizing *NN*, *NC* block size coincidence

We understand why $NN(W)$ is counted by $\mathrm{Cat}(W)$.

We do not really understand why the same holds for $NC(W)$.

Worse, we do not really understand why the following holds– it was checked case-by-case.

Theorem (Athanasiadis-R. 2004)

The W-orbit distributions coincide[a] for subspaces arising as

- *intersections* $X = \cap_{\alpha \in A} \alpha^{\perp}$ *for* A *in* $NN(W)$, *and as*
- *fixed spaces* $X = V^w$ *for* w *in* $NC(W)$.

[a] ...and have a nice product formula via Orlik-Solomon exponents.

What about a q-analogue of $\mathrm{Cat}(W)$?

Theorem (Gordon 2002, Berest-Etingof-Ginzburg 2003)

For irreducible real reflection groups W,

$$\mathrm{Cat}(W,q) := \prod_{i=1}^{n} \frac{1-q^{h+d_i}}{1-q^{d_i}}$$

turns out to lie in $\mathbb{N}[q]$, as it is a Hilbert series

$$\mathrm{Cat}(W,q) = \mathrm{Hilb}(\ (S/(\Theta))^W\ ,q)$$

where $\Theta = (\theta_1, \ldots, \theta_n)$ is a magical hsop in $S = \mathbb{C}[x_1, \ldots, x_n]$

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- their $\mathbb{C}$-span carries W-rep'n V^*, like $\{x_1, \ldots, x_n\}$, and
- $S/(\Theta)$ is finite-dim'l (=: the graded W-parking space).

Do you believe in magic?

These magical hsop's do exist, and they're not unique.

Example

For $W = B_n$, the hyperoctahedral group of signed permutation matrices, acting on $V = \mathbb{R}^n$, one has $h = 2n$, and one can take

$$\Theta = (x_1^{2n+1}, \ldots, x_n^{2n+1}).$$

Example

For $W = \mathfrak{S}_n$ they're tricky. A construction by Kraft appears in Haiman (1993), and Dunkl (1998) gave another.

For general real reflection groups, Θ comes from rep theory of the rational Cherednik algebra for W, with parameter $\frac{h+1}{h}$.

$\mathrm{Cat}(W, q)$ and cyclic symmetry

$\mathrm{Cat}(W, q)$ interacts well with a cyclic $\mathbb{Z}/h\mathbb{Z}$-action on $NC(W) = [e, c]_{\mathrm{abs}}$ that comes from conjugation

$$w \mapsto cwc^{-1},$$

generalizing rotation of noncrossing partitions $NC(n)$.

Theorem (Bessis-R. 2004)

For any d dividing h, the number of w in $NC(W)$ that have d-fold symmetry, meaning that $c^{\frac{h}{d}} w c^{-\frac{h}{d}} = w$, is

$$[\mathrm{Cat}(W, q)]_{q=\zeta_d}$$

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But the proof again needed some of the case-by-case facts!

$\mathrm{Cat}(W, q)$ does double duty

Generalizing behavior of $A \longmapsto \Psi(A)$ in the staircase posets, Armstrong, Stump and Thomas (2011) actually proved the following general statement, conjectured in Bessis-R. (2004), suggested by weaker conjectures of Panyushev (2007).

Theorem (Armstrong-Stump-Thomas 2011)

For Weyl group W, and for d dividing $2h$ (not h this time), the number of antichains in the positive root poset Φ_+ fixed by Ψ^d is

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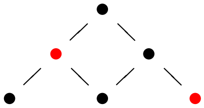
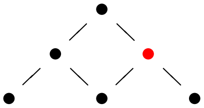
$A =$  $\longmapsto$ $\Psi(A) =$ 

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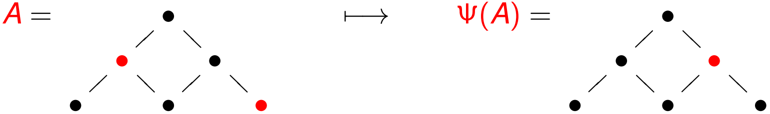

Again, part of the arguments rely on case-by-case verifications.

$\mathrm{Cat}(W, q)$ does triple duty

Generalizing what happens for rotating triangulations of polygons, Eu and Fu proved the following statement that we had conjectured.

Theorem (Eu and Fu 2011)

For Weyl group W, and for d dividing $h+2$ (not h, nor $2h$ this time), the number of clusters having d-fold symmetry under Fomin and Zelevinsky's deformed Coxeter element is

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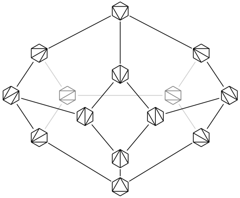

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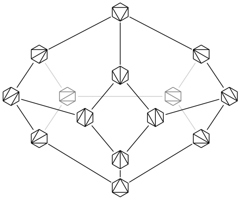

The big question

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Thanks for listening!